Superabundant Numbers and the Riemann Hypothesis

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The function $\sigma(n) = \sum_{d|n} d$ is the *sum of divisors function*, so for example $\sigma(12) = 28$. In 1913 Gronwall proved that

$$\limsup_{n\to\infty} \frac{\sigma(n)}{e^{\gamma}\ n \log\log n} = 1,$$

where $\gamma \approx 0.57721$ is Euler's constant (see [4, Theorem 323]). This says that the maximal size of $\sigma(n)$ is roughly $e^{\gamma} n \log\log n$. The following theorem of Robin [7, Theorem 2] gives a more refined version of this upper bound.

Theorem 1 *For $n \geq 3$ we have*

$$\frac{\sigma(n)}{e^{\gamma}\ n \log\log n} < 1 + \frac{0.6483}{e^{\gamma}(\log\log n)^2}. \tag{1}$$

In [7] Robin also proved the following striking result.

Theorem 2 *The Riemann Hypothesis is true if and only if*

$$\frac{\sigma(n)}{e^{\gamma}\ n \log\log n} < 1, \quad \text{for} \quad n \geq 5041. \tag{2}$$

For a lively exposition of this theorem and its connection to the Riemann Hypothesis see [5]. In this note, we propose a method that will establish explicit upper bounds for $\sigma(n)/e^{\gamma} n \log\log n$. Our main observation is that the least number violating the inequality (2) should be a superabundant number.

A positive integer n is said to be *superabundant* if $\sigma(m)/m < \sigma(n)/n$ for all $m < n$. The first 20 superabundant numbers are 1, 2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360, 720, 840, 1260, 1680, 2520, 5040, 10080. The sequence of superabundant numbers is the sequence A004394 in Sloane's Encyclopedia [8]. The list of the first 500 superabundant numbers is available at [6], where

$$M = 254842478774746236945594692013150330453594741501619230768504865767603607680000$$

is the largest superabundant number in the list.

Theorem 3 *If there is any counterexample to Robin's inequality* (2)*, then the least such counterexample is a superabundant number.*

Proof It is known that there is no counterexample n to Robin's inequality (2) with $5040 < n \leq 10080$. Note that 10080 is a superabundant number.

Now, assume that $n > 10080$ is the minimal counterexample to the inequality and also assume that n is not superabundant. Then there is a $k < n$ such that $\sigma(k)/k \geq \sigma(n)/n$. Without loss of generality we may assume that $k \geq 10080$. This is true since if $k < 10080$, then $\sigma(k)/k < \sigma(10080)/10080$ (as 10080 is superabundant) and so we could choose instead $k = 10080$.

We next note that since

$$\frac{\sigma(k)}{k} \geq \frac{\sigma(n)}{n} \geq e^{\gamma} \log\log n > e^{\gamma} \log\log k$$

then k ($\geq 10080 > 5040$) is a counterexample to Robin's inequality (2). However, we assumed that n was the minimal counterexample, so this is a contradiction. Therefore, if a counterexample > 5040 exists, then the least counterexample must be superabundant. $\square$

We point out that superabundant numbers form a very thin subset of the natural numbers. More precisely, let $S(x)$ be the number of superabundant numbers not exceeding x. Then in [1], Alaoglu and Erdös state that

$$\frac{\log(S(x))}{(\log\log x)^2}$$

is bounded. It is an open problem whether

$$\frac{\log(S(x))}{\log\log x} \tag{3}$$

is bounded or not. In [2], Briggs reports that the ratio (3) gives about 1.2 at $\log\log x = 5$. Moreover, Erdös and Nicolas [3] proved that

$$\liminf_{x\to\infty} \frac{\log(S(x))}{\log\log x} \geq 1 + \frac{5}{48} \approx 1.1042.$$

As a consequence of Theorem 3, one can attempt to disprove the Riemann Hypothesis computationally by testing the inequality (2) for superabundant numbers. See [2] for information regarding this approach and methods for generating superabundant numbers. The author of [2] also mentions that, at the moment, there is no (guaranteed, fast) algorithmic method known for finding all superabundant numbers below a given bound.

Finally we can numerically verify that the only numbers in the list of the first 500 superabundant numbers in [6] that do not satisfy Robin's inequality (2) are 2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360, 720, 840, 2520, 5040. So as a direct corollary of the above theorems we have

$$\frac{\sigma(n)}{e^{\gamma}\ n\log\log n} < 1 + \frac{0.6483}{e^{\gamma}(\log\log M)^2} = 1.013617,$$

for $n \geq 5041$, where M is the superabundant number recorded above.

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