CMPUT 675: Approximation Algorithms **Fall 2014**

Lecture 4 (Sep 10): The Traveling Salesman Problem

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4.1 Traveling Salesman Problem

The following algorithm describes a $\frac{3}{2}$-approximation for the Traveling Salesman Problem, defined in the previous lecture. The shortcutting step used by this algorithm is performed as described in the proof of Theorem 1 from Lecture 3. The algorithm is one of the oldest approximations, from Christofides in 1976 [C76].

Definition 1 *Given a graph $G = (V, E)$, a* matching *M is a subset of edges so each $v \in V$ is the endpoint of at most one edge in M. The matching M is said to be* perfect *if every vertex in v is an endpoint of some edge in M, equivalently $|M| = |V|/2$.*

We will use the following fact without proof.

Theorem 1 *There is a polynomial-time algorithm that determines if a graph has a perfect matching. Furthermore, if such a matching exists and if the edges have costs then we can also find a minimum-cost perfect matching in polynomial time.*

The main idea behind this improved approximation is that we can fix the odd-degree nodes in the minimum spanning tree by being more clever than simply doubling every edge. Instead, we will just match them up in the cheapest way possible.

Algorithm 1 Traveling Salesman $\frac{3}{2}$-approximation

Input: A metric on nodes V with costs $c(u, v)$,$u, v \in V$.
Output: A Hamiltonian cycle H.
  $T \leftarrow$ a minimum spanning tree of the metric
  $D \leftarrow$ odd degree nodes in T
  $M \leftarrow$ minimum cost perfect matching of D
  $\mathcal{C} \leftarrow$ Eulerian circuit of the graph $(V, T + M)$ (the + means keep both copies of an edge if it lies in T and M)
  shortcut $\mathcal{C}$ to get a Hamiltonian cycle H
  return H

Theorem 2 *Algorithm 1 is a $\frac{3}{2}$-approximation*

Proof. First, note that $T + M$ is an Eulerian graph since it is connected (as it contains T) and clearly every $v \in V$ has even degree in $T + M$.

Let OPT denote the cost of an optimum solution.

As we know that the shortcutting step will not increase the size of a Hamiltonian cycle, we can affirm that

$$\begin{aligned}\text{cost}(H) &\leq \text{cost}(\mathcal{C}) \\ &= \text{cost}(T) + \text{cost}(M) \\ &\leq OPT + \text{cost}(M).\end{aligned}$$

The last inequality is justified because T is the cheapest spanning tree and the optimum solution, being a Hamiltonian cycle, contains some spanning tree.

We need to demonstrate that the following is true:

$$\text{cost}(M) \leq \frac{OPT}{2}.$$

Let H^* be an optimum solution, so $\text{cost}(H)^* = OPT$.

Shortcut H^* past nodes in $V - D$ to get a cycle H_D spanning D. Since shortcutting does not increase the cost,

$$\text{cost}(H_D) \leq \text{cost}(H^*) = OPT.$$

Say the cycle H_D follows nodes $v_1, v_2, \ldots, v_{|D|}, v_1$. Let M_1 and M_2 be the two perfect matchings on D obtained by taking the edges of H_D alternatively, such that

$$M_1 = \{(v_1 v_2), (v_3 v_4), ..., (v_{|D-1|} v_{|D|})\}$$

and

$$M_2 = \{(v_2 v_3), (v_4 v_5), ..., (v_{|D|} v_1)\}$$

Then $\text{cost}(M_1) + \text{cost}(M_2) = \text{cost}(H_D)$ so

$$\min(\text{cost}(M_1), \text{cost}(M_2)) \leq \frac{\text{cost}(H_D)}{2} \leq \frac{OPT}{2}.$$

Since M is the minimum-cost perfect matching of D, then

$$\text{cost}(M) \leq \min(\text{cost}(M_1), \text{cost}(M_2)) \leq \frac{OPT}{2}.$$

Thus, $\text{cost}(H) \leq \frac{3}{2} \cdot OPT$

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References

C76 N. CHRISTOFIDES, Worst-case analysis of a new heuristic for the travelling salesman problem, Report 388, Graduate School of Industrial Administration, CMU, 1976.