

Simulation of Meccano Differential Analyzers

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Meccano n. *Trademark.* A construction set of miniature metal or plastic parts from which mechanical models can be made.

The Collins Paperback English Dictionary.

Introduction

The first differential analyzer was built in 1931 by Vannevar Bush at the Massachusetts Institute of Technology. This work soon came to the attention of Douglas Hartree of the University of Manchester who, assisted by his research student Arthur Porter, built a model consisting mostly of Meccano parts to judge its suitability for calculations arising in atomic physics. With the successful completion of the model, funds were obtained for the construction of a full-scale machine which was built commercially. During World War II this machine and several other similar ones including one at the University of Cambridge were used for military purposes. With the development of electronic digital computers beginning in the late 1940s these machines were no longer used. However at present there is considerable interest in the construction of the Meccano models and their use for educational purposes

This paper describes a simulation, written in the **J** programming language, of a five-integrator Meccano model and gives a few examples of its use. The pioneering Manchester work is described in Porter (2003). Two papers, Robinson (2005a) and Robinson (2005b) give, respectively, a history of the Meccano models and a description of the contemporary construction of one such machine. A brief description of the **J** language and a comparison with conventional programming languages is given in Smillie (2000).

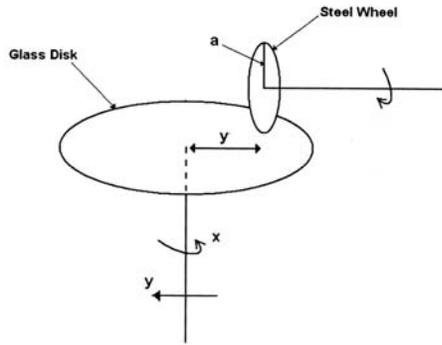
Integrators

Integration with a differential analyzer is performed by one or more integrators, identical in design and differing only in their interconnections with other components of the machine. As shown in the figure on the next page an integrator consists of a small vertical wheel which turns on a horizontal axis as it rolls on the surface of a horizontal disk which rotates on a vertical axis. The displacement of the wheel from the centre of the disk varies as the disk rotates and is proportional to the value of the dependent variable. The rotation of the disk gives the change in the independent variable.

Suppose that the disk rotates through a small angle Δx and the wheel is at a radial distance y from the centre of the disk. Therefore the wheel turns an amount

$$\Delta z = (y\Delta x) / a ,$$

where a is the radius of the wheel. Thus after a given period of operation the total rotation of the wheel is proportional to $\int y dx$. Since the wheel rests lightly on the disk so that it may slide freely along a radius as the value of the dependent variable changes, its torque must be amplified before its motion may be transmitted to another part of the analyzer.



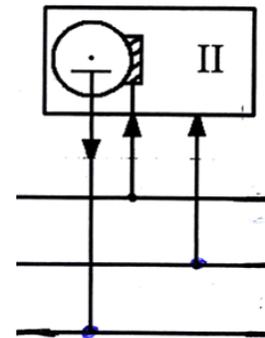
As this simulation models only the logic of the differential analyzer, the design and construction of torque amplifiers may be ignored although they are an important feature in the Meccano models.

Let us assume now that the disk rotates in discrete steps of size Δx and that the rotation of the disk begins when x has the value x_0 . Also assume that the initial radial displacement of the wheel on the disk is y_0 and that subsequent displacements are y_1, y_2, \dots . Then the successive rotations of the wheel are $(\Delta x) y_1/a, (\Delta x) y_2/a, \dots$. Therefore, the total rotation of the wheel after n steps is proportional to the sum

$$(\Delta x) y_0 + (\Delta x) y_1 + (\Delta x) y_2 + \dots + (\Delta x) y_n .$$

As Δx becomes smaller, this sum will correspond more closely to the rotations of the physical disk and wheel, and thus become a better approximation to the value of the desired integral.

The figure on the right shows the conventional representation of an integrator. The horizontal lines represent the rotating shafts or buses and the vertical lines represent the connections to the integrator represented by the circular figure in the rectangle. The Roman numeral II specifies the second integrator. The middle vertical line indicates that the first bus corresponds to the independent variable which controls the rotation of the horizontal disk, and the right vertical line indicates that the second bus corresponds to the dependent variable given by the displacement of the vertical wheel on the disk. The left vertical line indicates that the output of the integrator, i.e., the rotation of the wheel, controls the rotation of the third bus.



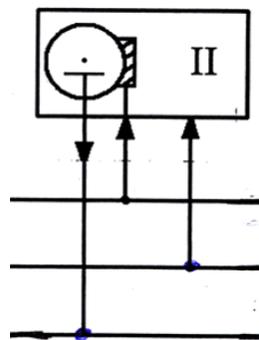
Simulator

The simulator has five integrators and eleven buses. The first bus corresponds to the independent variable in the differential equation, the next five buses to the five integrators, and the remaining five to

auxiliary buses which will be introduced later. The buses are numbered 0, 1, 2, ..., 10, and the values representing their cumulative rotations by the variables $z_0, z_1, z_2, \dots, z_{10}$. (The variable z_0 has the synonyms x and t .) The connections of the integrators with the buses are defined in the **J** language by dyadic functions, one for each integrator, and named I, II, III, IV and V. The left argument of a function gives the number of the bus associated with the rotation of the horizontal disk for the corresponding integrator, the right argument gives the displacement of the vertical wheel for that integrator, and the result the total rotation of the wheel which gives an approximation to the desired integral up to that stage of the integration. Successive pairs of values of the independent and dependent variables are stored as the rows of a two-column table associated with the integrator, and named TABLE1, TABLE2, TABLE3, TABLE4 and TABLE5.

As an example consider the integrator shown in the schematic on the previous page and repeated here for convenience. Since the second integrator is involved, the name of the corresponding **J** function is II.

The first three buses are used with the first bus numbered 0 connected to the horizontal wheel, the second numbered 1 connected to the displacement of the vertical disk, and the third bus numbered 2 connected to the rotation of the vertical wheel. Therefore the bus connections to the integrator may be represented in the simulator by the expression

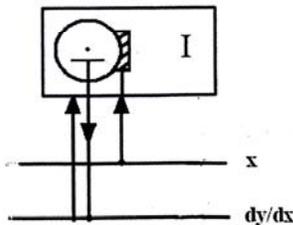


$$z_2 =: 0 \text{ II } z_1$$

The successive values of the cumulative incremental changes in the rotation of the disk and wheel are given in the two-column table TABLE2.

The Windows form given at the end of the paper shows the format used for the input of the integrator definitions and their initial settings and for the graphical display of the results of a simulation.

A first-order equation



As a simple example we shall discuss the use of the simulator to solve the first-order differential equation

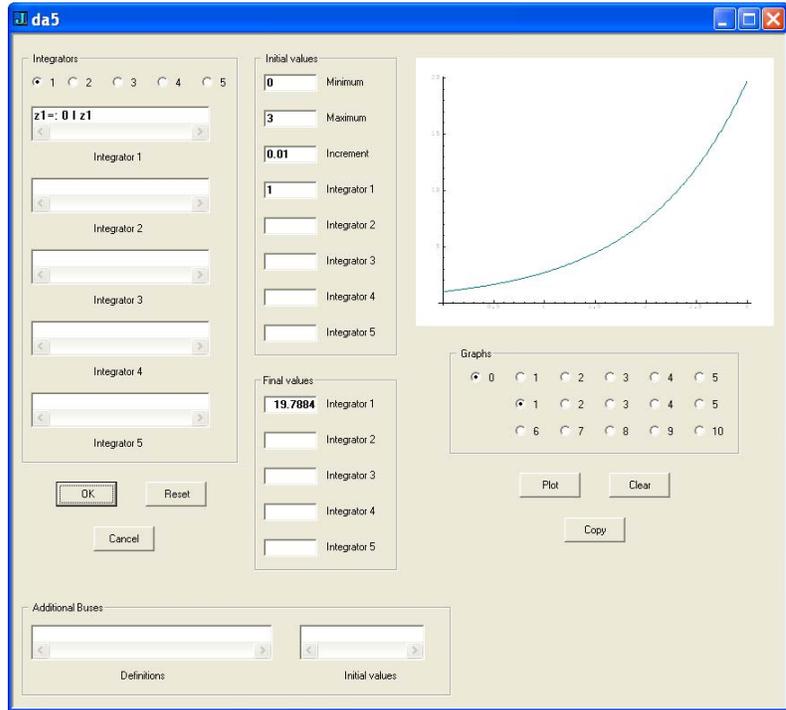
$$dy/dx = y$$

with the initial condition $y(0) = 1$. The solution is $y = e^x$ where e is the exponential function. We shall assume that the integration is from $x = 0$ to $x = 3$ so that the final value of the dependent variable

y is e^3 or 20.0855. The schematic on the left shows the integrator connections which we may use to define the integrator in the simulator by the expression

z1=: 0 I z1

with the following initial conditions: minimum x is 0, maximum x is 3, and dy/dx is 1. The Windows form after the simulation has been run with an increment size of 0.01 is shown on the right. The radiobutton for the number of integrators is set at 1, its default value. The final value of the function y is given as 19.7884. (Changing the x increment to 0.001, for example, gives a more accurate final value of 20.0755.) The solution is stored in the two-column table TABLE1 which has



301 rows of which the first and last three are as follows:

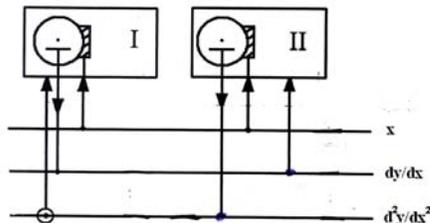
```

0.00  1.0000
0.01  1.0100
0.02  1.0201
. . . . .
2.98  19.3986
2.99  19.5925
3.00  19.7885

```

A second-order equation

In this section we shall give an example of the use of the simulator to solve the second-order



ordinary differential equation for simple harmonic motion

$$d^2y/dx^2 = -y$$

with the initial conditions $y(0) = 0$ and $dy/dx(0) = 1$

which has the solution

$$y = \sin x .$$

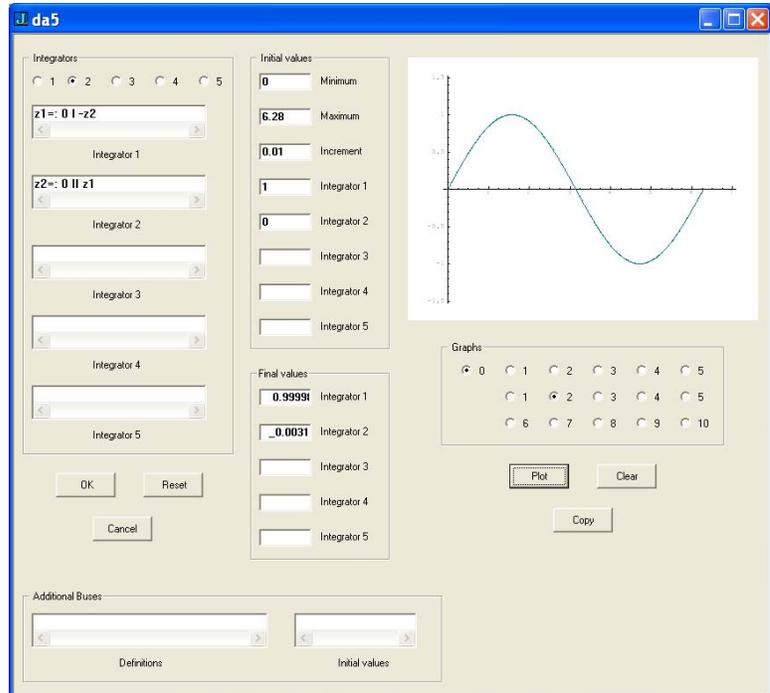
In the schematic the small circle at the lower end of the connection from bus number 2 indicates that the value passed to the integrator is negated. From this schematic we may derive the expressions

$$z1 =: 0 \text{ I } -z2$$

and

$$z2 =: 0 \text{ II } z1$$

for the integrators. In the Windows form we note that we have plotted the solution against the independent variable x by selecting radiobutton 0 in the first row and radiobutton 2 in the second row in the box below the graph. This may be done either initially and then clicking the OK button to run the simulation, or after the simulation has been done and then clicking the Plot button. (The



default selection of the radiobuttons is 0 in the first row and 1 in the second which in this example will give the graph of the cosine function.) Also by selecting either radiobuttons 1 and 2 in the first and second rows, or radiobuttons 2 and 1, we obtain a circular plot.

Exponential distribution

The exponential distribution arises in the statistical study of the strengths of certain materials and in the analysis of waiting times between random events. The probability density function is

$$p(x) = \lambda e^{-\lambda x}$$

where x is in the range $0 \leq x \leq \infty$ and $\lambda \geq 0$ is a parameter. We shall use the simulator to find the cumulative distribution function of this distribution for the arbitrary value of 0.5 for λ .

From the first example we saw that the exponential function $y = e^x$ was given by the solution of the differential equation $dy/dx = y$ with appropriate initial conditions and that the simulator definition for the integrator was

$$z1 =: 0 \text{ I } z1$$

Therefore the expression

$$z1 =: 0 \text{ I } -z1$$

corresponds to the differential equation $dy/dx = -y$ which has the solution $y = e^{-x}$. Now we could write the expression

$$z1 =: 0 \text{ I } -0.5 * z1$$

for the solution $y = e^{-0.5x}$. However it is more realistic to have the multiplication performed by one of the auxiliary buses as it would be in an actual differential analyzer so that the definition of the first simulator would become

$$z1 =: 0 \text{ I } -z6$$

where $z6 =: 0.5 * z1$. Finally we introduce the second integrator defined by

$$z2 =: 0 \text{ II } z7$$

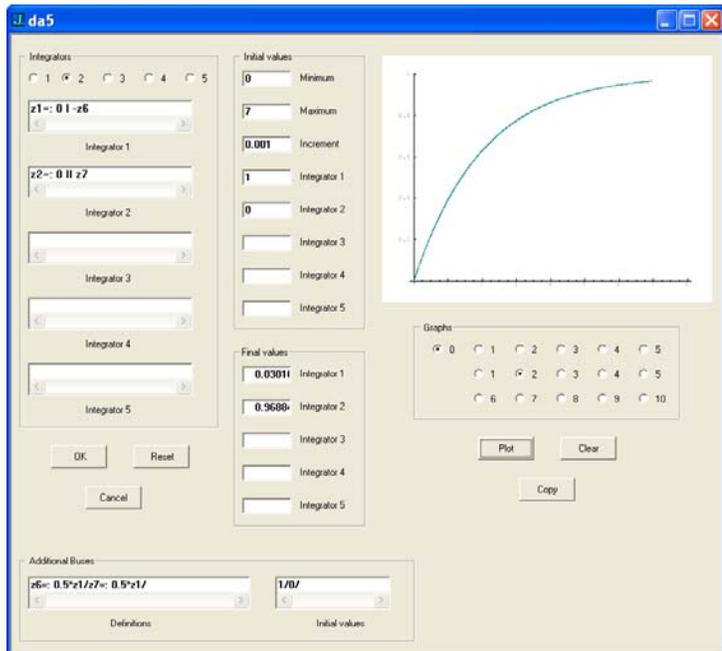
where $z7 =: 0.5 * z1$ which may be used to give the cumulative distribution function for the geometric distribution. We should note that the definitions of the auxiliary buses and their initial values are entered in the Windows form in the Auxiliary buses boxes as the lists

$$z6 =: 0.5 * z1 / z7 =: 0.5 * z1 /$$

and

$$1/0/ ,$$

respectively, with each definition or initial value terminated by a forward slash /. The graph displayed in the Windows form is that of the cumulative density function of the exponential distribution.



Damped oscillations

A second-order equation for damped oscillations is given by

$$d^2y/dt^2 + 2dy/dt + 50y = 0$$

with initial conditions $y(0) = -0.1$ and $dy/dt(0) = 0$. We shall omit the schematic for the integrators and give only the definitions of the integrators and auxiliary buses and their initial values. The two integrators I and II may be defined by the expressions

$$z1 =: 0 \text{ I } z2$$

and

$$z2 =: 0 \text{ II } -z8$$

with initial conditions 0 and 5 for the minimum and maximum values of the independent variable, 0.01 for the increment, and -0.1 and 0 for the initial values of the dependent variable and the first derivative. The auxiliary buses are defined by the expressions

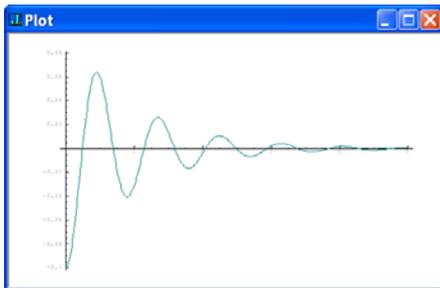
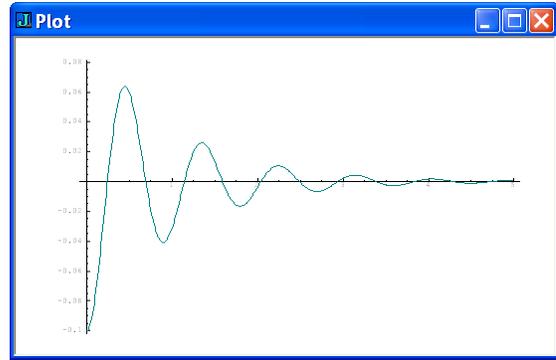
```
z6=: 50 * z1 ,
z7=: 2 * z2
```

and

```
z8=: z6 + z7
```

with initial values of `_5`, `0` and `_5`.

Instead of displaying the Windows form showing the graphical solution of the equation we shall show only the graph.. This may be accomplished by running the simulation, and then clicking the Copy button



which will give the currently displayed graph which may then be copied into the Clipboard and inserted into the required document as has been done here.

The differential equation in this example may be shown to have the explicit solution

$$y = \exp(-t) [(-1/70) \times \sin(7t) + (-1/10) \times \cos(7t)] .$$

Using the graphing facilities of **J**, the details of which need not concern us, we may obtain the graph of the explicit solution shown here which agrees with that obtained numerically by the simulator:

Large-amplitude pendulum

As a final example we will give the solution of the following non-linear second-order differential equation for a viscously damped large-amplitude pendulum:

$$d^2y/dx^2 + \sin x + 0.1 \times dy/dx = 0 .$$

The restoring force is proportional to the sine of the angle of displacement and there is a velocity-dependent damping term. With integrators I and II the integration is with respect to the independent variable with constant increments while with integrators III and IV the variable increments are determined by the output of integrator I. The schematic is given in Appendix 2 from which may derive the integrator and auxiliary bus definitions which are given here together with their initial values:

```
z1=: 0 I z2      1
z2=: 0 II -z7    0
z3=: 1 III z4    0.14
z4=: 1 IV -z3    0.99
z6=: 0.1 * z2    0
z7=: z3 + z6     0.14
```

The Windows form for the solution is also shown in Appendix 2.

Acknowledgements

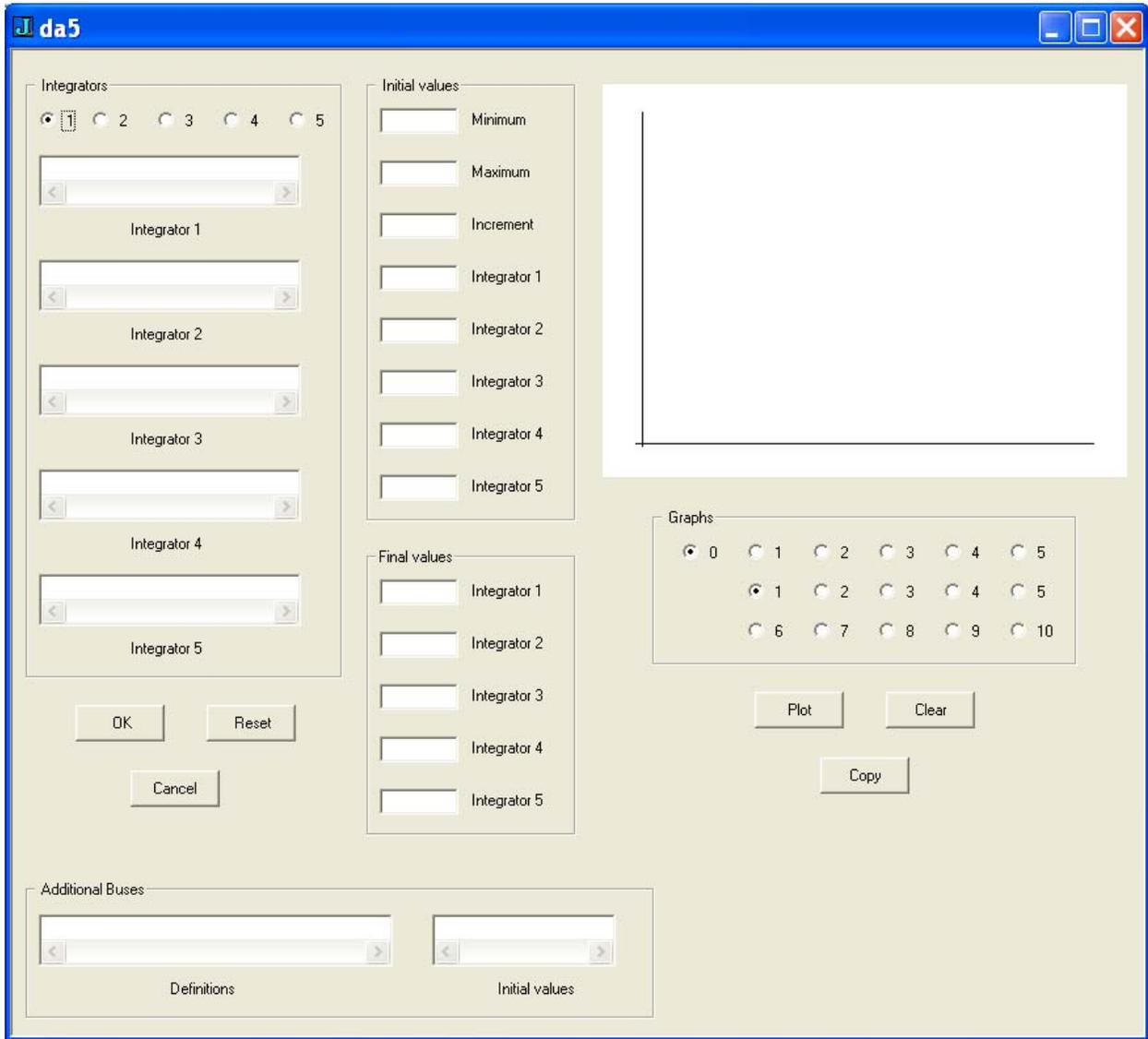
I wish to thank Arthur Porter for stimulating my interest in Meccano differential analyzers and for his continuing interest in this work, and Tim Robinson for his many helpful comments, continuing encouragement and patience.

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Appendix 1. Windows form for simulator



Appendix 2. Viscously damped large-amplitude pendulum

