RN, Chapter
7.4 - 7.8

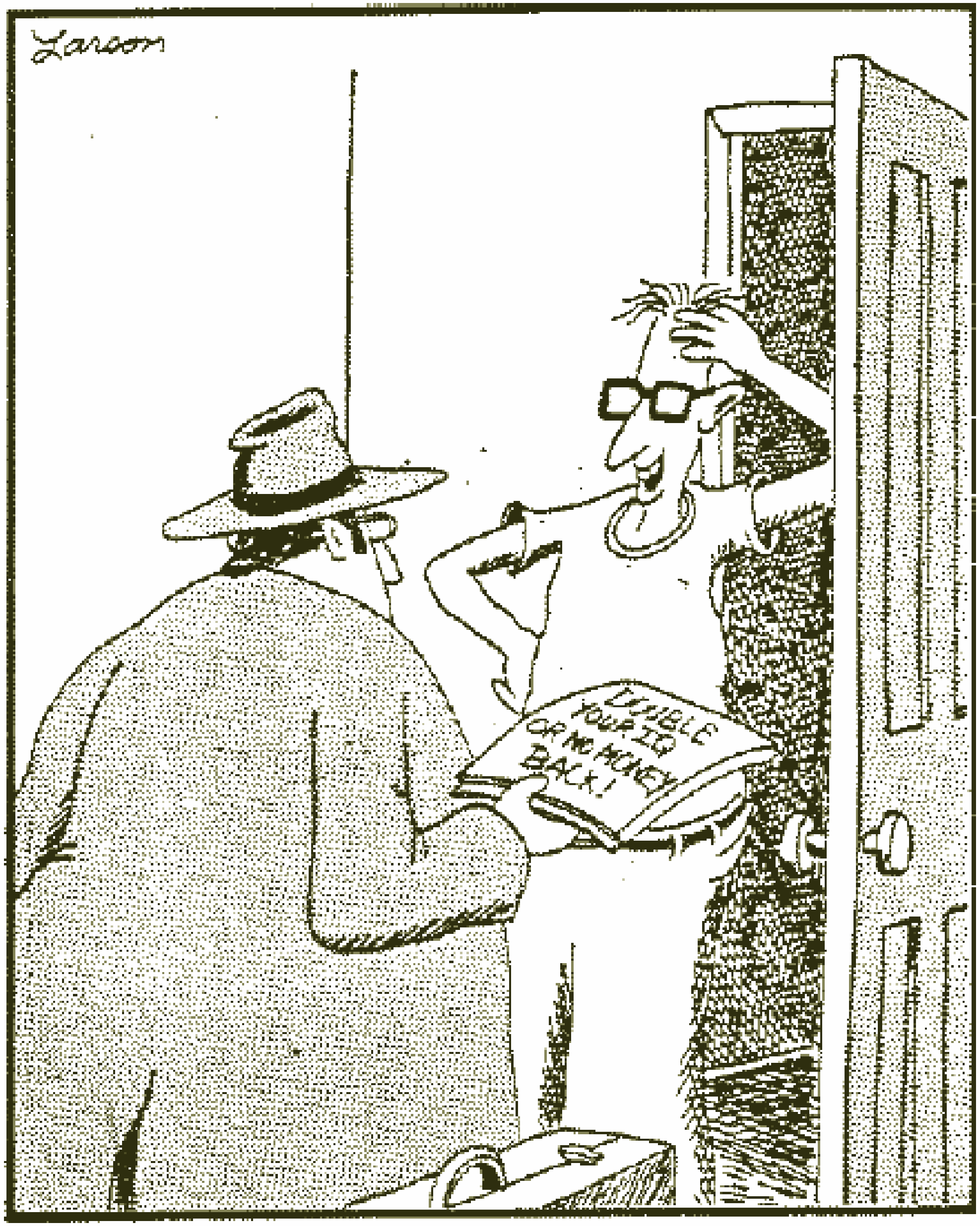
Propositional Logic

Logical Agents

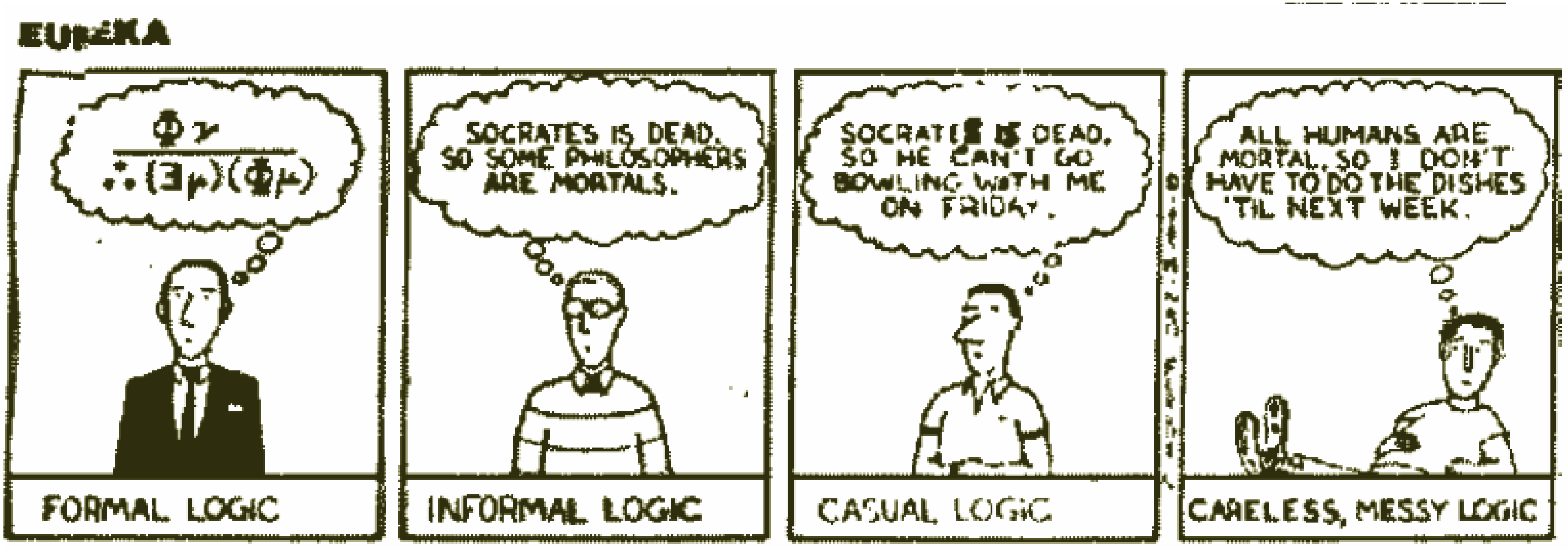
- Reasoning [Ch 7 – 7.3]
- Propositional Logic [Ch 7.4 - 7.8]
  - Syntax
  - Semantics
    - Models
    - Entailment
  - Proof Process
    - Forward / Backward chaining
    - Resolution
- Predicate Calculus
  - Representation [Ch 8]
  - Inference [Ch 9]
- Implemented Systems [Ch 10]
- Applications [Ch 8.4,10]
- Planning [Ch 11]

Logic in General

Logics are formal languages
for representing information
such that conclusions can be drawn



"Well, I dunno… Okay, sounds good to me."


EUREKA
SOCRATES IS DEAD, SO SOME PHILOSOPHERS ARE MORTALS.
SOCRATES IS DEAD, SO HE CAN'T GO BOWLING WITH ME ON FRIDAY.
ALL HUMANS ARE MORTAL, SO I DON'T HAVE TO DO THE DISHES 'TIL NEXT WEEK.
FORMAL LOGIC
INFORMAL LOGIC
CASUAL LOGIC
CARELESS, MESSY LOGIC

Components of a Logic

- **Syntax** defines the sentences in the language
  ... what does it look like?
- **Semantics** define "meaning" of sentences;
  i.e., define truth of a sentence in a world.
  How is it linked to the world?
- **Proof Process** "new facts from old"
  find implicit information... "pushing symbols"

- Eg, wrt arithmetic
  - $x+2 \geq y$ is sentence; ~~x2+y >~~ is not
  - $x+2 \geq y$ is true iff
    the number $x+2$ is no less than the number y
  - $x+2 \geq y$ is ***true*** in a world where $x = 7; y = 1$
  - $x+2 \geq y$ is ***false*** in a world where $x = 0; y = 6$

Propositional Logic: Syntax

- Atomic Propositions…   "basic statements about world"
  - $W_{3,4}$: Wumpus at location [3, 4]
  - $S_{1,1}$: Stench at location [1, 1]
  - …
- Build sentences from atomic propositions using connectives

$\wedge$	$\vee$	$\neg$	$\Rightarrow$	$\Leftrightarrow$
and	or	not	implies	equivalence
			if. . . then	(biconditional)

- Eg:

$$
\begin{aligned}
\neg S_{1,1} &\Rightarrow \neg W_{1,1} \wedge \neg W_{1,2} \wedge \neg W_{2,1} \\
\neg S_{2,1} &\Rightarrow \neg W_{1,1} \wedge \neg W_{2,1} \wedge \neg W_{2,2} \wedge \neg W_{3,1} \\
\neg S_{1,2} &\Rightarrow \neg W_{1,1} \wedge \neg W_{1,2} \wedge \neg W_{2,2} \wedge \neg W_{1,3} \\
S_{1,2} &\Rightarrow W_{1,1} \vee W_{1,2} \vee W_{2,2} \vee W_{1,3}
\end{aligned}
$$

Semantics… based on Models

- "**Model**" ≡ "completely specified possible world"
  Every claim is either true or false

  $m \models \varphi$ means
  formula φ holds in model m
  Otherwise $m \not\models \varphi$

- **Propositional case:** Complete assignment
- Eg,

  $m_1 \models A$

	A	B	C	D
m_1	+	0	0	+

  "A is true in m_1" … "m_1 is a model of A"

- Also… $m_1 \models D$ $m_1 \not\models C$
- What about… $\neg B$? $A \vee B$? … $A \,\&\, \neg C \,\&\, D$?

Propositional logic: Semantics

- Each model specifies { true, false } for each proposition symbol
- Eg,

	A	B	C	D
m_1	0	+	0	+

- Rules for evaluating truth wrt model *m*:

$\neg S$	is true iff	S	is false		
$S_1 \wedge S_2$	is true iff	S_1	is true and	S_2	is true
$S_1 \vee S_2$	is true iff	S_1	is true or	S_2	is true
$S_1 \Rightarrow S_2$	is true iff	S_1	is false or	S_2	is true
"	is false iff	S_1	is true and	S_2	is false

$$m \models \neg S \equiv m \not\models S$$
$$m \models S_1 \wedge S_2 \equiv m \models S_1 \text{ and } m \models S_2$$
$$m \models S_1 \vee S_2 \equiv m \models S_1 \text{ or } m \models S_2$$
$$m \models S_1 \Rightarrow S_2 \equiv m \not\models S_1 \text{ or } m \models S_2$$

Propositional logic: Semantics

	A	B	C	D
m_1	0	0	0	+

$\neg S$	is true iff	S	is false		
$S_1 \wedge S_2$	is true iff	S_1	is true and	S_2	is true
$S_1 \vee S_2$	is true iff	S_1	is true or	S_2	is true
$S_1 \Rightarrow S_2$	is true iff	S_1	is false or	S_2	is true
"	is false iff	S_1	is true and	S_2	is false

$m \models \neg S \equiv m \not\models S$

$m \models S_1 \wedge S_2 \equiv m \models S_1$ and $m \models S_2$

$m \models S_1 \vee S_2 \equiv m \models S_1$ or $m \models S_2$

$m \models S_1 \Rightarrow S_2 \equiv m \not\models S_1$ or $m \models S_2$

m ⊨? A v (~B & C)

True if either m ⊨ A or m ⊨ ~B & C

- But m ⊭ A

  So need m ⊨ ~B & C

- True if m ⊨ ~B and m ⊨ C
  - m ⊨ ~B holds if m ⊭ B — True…
  - So need only m ⊨ C
  - Fails…

Semantics of Connectives

P	Q	¬P	P&Q	P v Q	P⇒Q	P⇔ Q
0	0	+	0	0	+	+
0	+	+	0	+	+	0
+	0	0	0	+	0	0
+	+	0	+	+	+	+

- Just need &, ¬ :
  - P v Q means ¬(¬ P & ¬ Q)
  - P ⇒ Q means ¬P v Q

    ... counterintuitive: truth value of "5 is even ⇒ Sam is smart" ?
- P⇔ Q means (P ⇒ Q) & (Q ⇒ P)
- "&" relatively easy, as *complete knowledge*
- "v", "¬" more difficult, as *partial information*

Models of a Formula

- Initially all 2^n models are possible

	A	B	C	D
m_1	0	0	0	0
m_2	0	0	0	+
m_3	0	0	+	0
⋮	⋮	⋮	⋮	⋮
m_{2^n-1}	+	+	+	0
m_{2^n}	+	+	+	+

- Assertion α ELIMINATES possible worlds

  Eg, $\neg A$ eliminates models m where $m \models A$

- $M(\alpha) = \{ m \mid m \models \alpha \}$ is set of all models of α
- $M(\neg A) =$

	A	B	C	D
m_1	0	0	0	0
m_2	0	0	0	+
m_3	0	0	+	0
⋮	⋮	⋮	⋮	⋮
m_{2^n-1}	0	+	+	+

Example of Entailment

- Initially:
- Background knowledge:

  Tell(KB, "$S_{12} \Rightarrow W_{11} \vee W_{13}$")

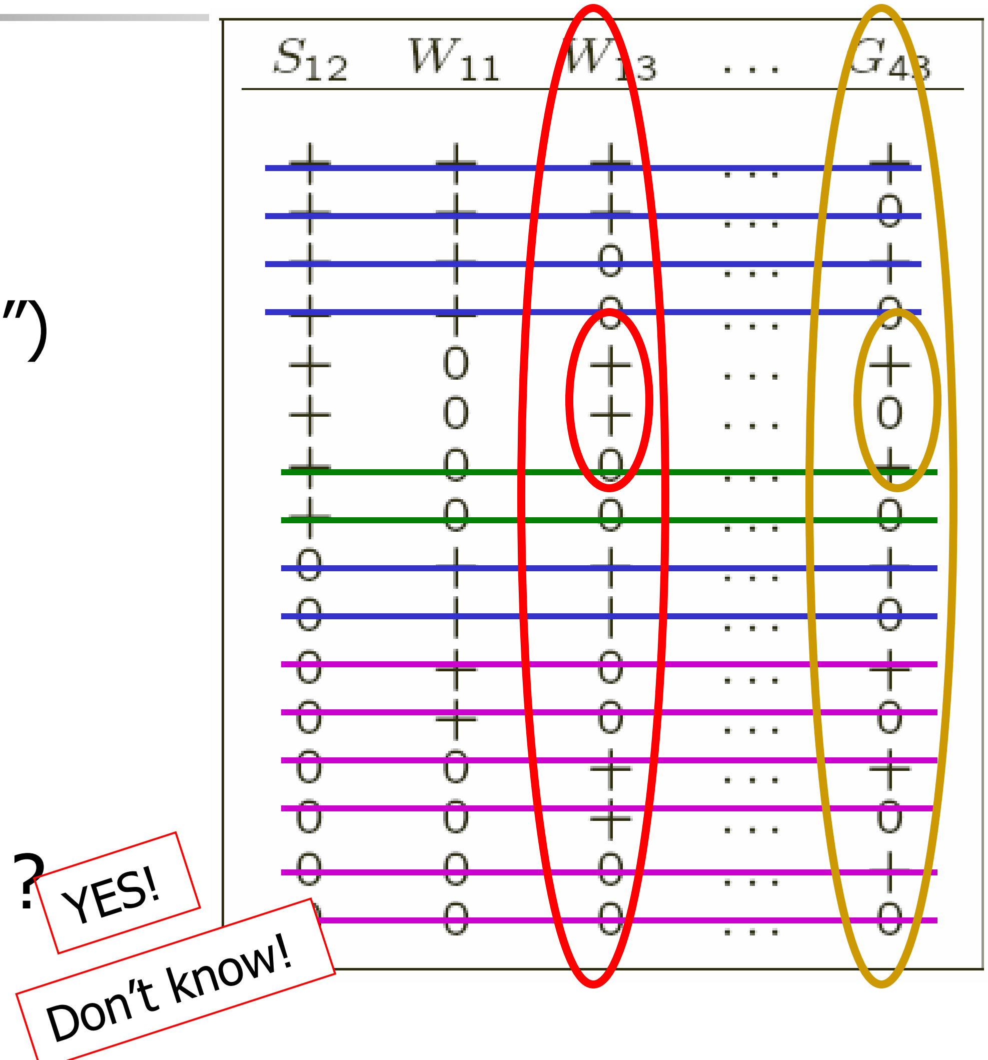
- Alive at start…

  Tell(KB, " $\neg W_{11}$")

- Smell something. . .

  Tell(KB, "S_{12}")

- Is Wumpus @ [1, 3] ?
- Is Gold @ [4, 3] ?

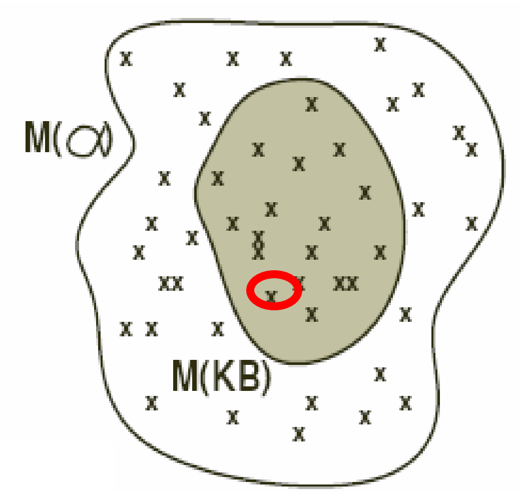


What to believe?

- Suppose you believe KB, and KB $\models \alpha$
  Then you should believe α !
- Why?
  1. "Believe KB"
     $\Rightarrow$ Real world m_{RW} in $M(KB)$
  2. KB $\models \alpha$ means $M(KB) \subseteq M(\alpha)$
     $\Rightarrow$ $m_{RW} \in M(\alpha)$

  . . . $m_{RW} \models \alpha$

  Ie, α holds in the Real World,
  so you should believe it!



Translate Knowledge into Action

- Include LOTS of rules like

  $A_{1,1}$ & $East_A$ & $W_{2,1} \Rightarrow \neg$Forward
- Observations re World
- Try to prove one of…

  { Forward, Turn Left, …, Shoot }
- After proof

  KB $\vdash$ Action

  perform Action

Comments on Logic

1. Why reason?
2. Entailment $\models$ vs Inference $\vdash$
3. Relation to world...
4. Succinct Representation ?

Issue#2:Entailment vs Derivation

- **Entailment** KB $\models \alpha$
  Semantic Relation:
  α MUST hold whenever KB holds
- **Derivation** KB $\vdash_i \alpha$
  Computational (Syntactic) Process:
  Maps $\langle$KB, α $\rangle$ to { Yes, No }
- $\vdash_i$ can be arbitrary but...
  want $\vdash_i$ that corresponds to $\models$!
- GOAL: $\vdash_{SC}$ that returns all+only entailments:
  For any KB, ,

  KB $\vdash_{SC} \alpha$ if-and-only-if KB $\models \alpha$

$\vdash_N$(KB, α) = No

$\vdash_A$(KB, α) =

Yes iff | α | = 1

$\vdash_{1S}$(KB, α) =

Yes iff
1-step derivation

Properties of Derivation Process

- Only 1 $\models$, but many possible proof procedures $\vdash_i$
- $\vdash_i$ is **Sound** iff
  - $\vdash_i$ ONLY returns facts that must be true
  - $\forall KB, \rho \quad KB \vdash_i \rho \Rightarrow KB \models \rho$
- $\vdash_i$ is **Complete** iff
  - $\vdash_i$ returns every fact that must be true
  - $\forall KB, \rho \quad KB \models \rho \Rightarrow KB \vdash_i \rho$

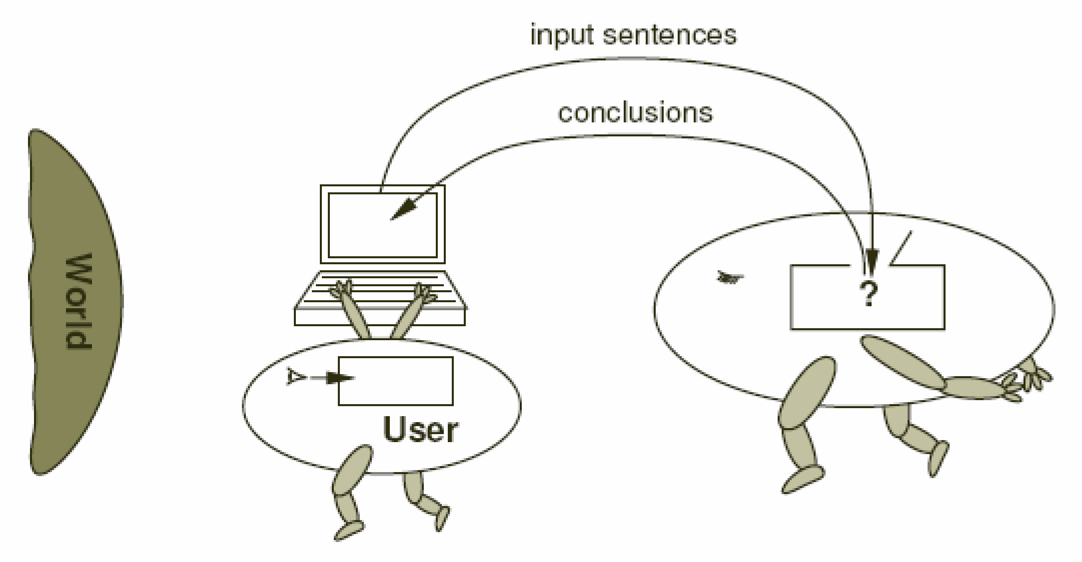
If $\vdash$ is SOUND+COMPLETE,

$\Rightarrow \quad \vdash \equiv \models$

$\Rightarrow$ Computer can IGNORE SEMANTICS and just push symbols!

Tenuous Link to Real World

- Challenge: “world” is not in computer
  . . . only a “representation" of world



- Computer only has sentences
  (hopefully about world)
  ... sensors can provide some grounding

Proof Process

- $KB = \{\varphi_j\}$ ...= SET of information "pieces"

  ... called "propositions" φ_j

  - Any rep'n will only explicitly include SOME of the true propositions

- Proof process specifies which other propositions to believe

  Agent that believes KB,
  will also believe DERIVED propositions

  written $KB \vdash_i \rho$

  ... called "derives" (deduces, proves, ...)

- Eg:

$$\left\{ \begin{array}{l} \text{Socrates is man} \\ \text{All men are mortal} \end{array} \right\} \vdash_i \text{Socrates is mortal}$$

Proof Methods

- **Model checking** … "truth table enumeration"

  (sound and complete for propositional)

  Compute complete truth table over *k* variables

  $S_{1,1}, S_{1,2}, \ldots, W_{1,1}, \ldots, B_{1,1}, \ldots$

  Here, $\geq$ 12 variables $\Rightarrow$ $\geq 2^{12} = 4096$ rows

  Find subset where KB holds; see if α holds in all

- **Application of inference rules**

  Generate "legitimate" (sound) new sentences from old

  <u>Proof</u> = a sequence of inference rule applications

  Can use inference rules as operators in a standard search alg

Example of Model Checking

- $\alpha \equiv A \vee B$
  $KB = (A \vee C) \,\&\, (B \vee \neg C)$
- $KB \models^{?} \alpha$?
- Check all possible models:
- $KB \models \alpha$ means
  α must be true wherever KB is true

As α is True every time **KB** is true, conclude $KB \models \alpha$!

A	B	C	$A \vee C$	$B \vee \neg C$	KB	α
False	False	False	False	True	False	False
False	False	True	True	False	False	False
False	True	False	False	True	False	True
False	True	True	True	True	**True**	True
True	False	False	True	True	**True**	True
True	False	True	True	False	False	True
True	True	False	True	True	**True**	True
True	True	True	True	True	**True**	True

Challenges

- Model checking is very expensive!

  ... needs to consider 2^k models... or ∞ !
- Decision about No pit in [1, 2]

  does not depend on anything dealing at [3, 4], ...

  but $\vdash_{MC}$ still needs to consider combinatorial set of complete models

- Other inference processes can be more "local"

#2: Applying Inference Rules

- **Proof Process** is mechanic process

  Implemented by . . .

  Applying sequence of individual Inference Rules
  to initial set of propositions, to find new propositions
- Each rule is *sound*. . .

  (Ie, if believe "antecedent", must believe conclusion)
- Uses MONOTONICITY:

  If $KB1 \models \alpha$, then $KB1 \cup KB2 \models \alpha$

  Can just deal with subset of propositions

- Search issues. . .
  - which inference rule ?
  - which propositions ?

New Facts from Old: Using Inference Rules

$$\begin{array}{lll} \text{If} & \text{``}P \Rightarrow Q\text{''} & \in KB \\ & \text{``}P\text{''} & \in KB \\ \hline \text{Then} & \text{can add in ``}Q\text{'' to } KB & \end{array}$$

Called "Modus Ponens"

Written:

$$[MP] \quad \begin{array}{l} P \Rightarrow Q \\ P \\ \hline Q \end{array}$$

Verify Soundness

- Modus Ponens:

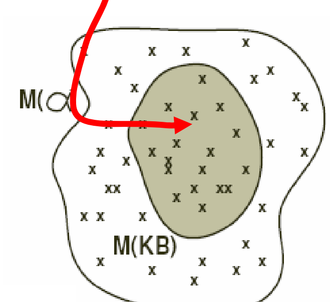
$$\frac{\alpha \Rightarrow \beta \qquad \alpha}{\beta}$$

- Truth table:

α	β	$\alpha \Rightarrow \beta$
+	+	+
~~+~~	~~0~~	~~0~~
~~0~~	~~+~~	~~+~~
~~0~~	~~0~~	~~+~~

  - Consider all worlds where $\{\alpha,\ \alpha \Rightarrow \beta\}$ hold
  - Observe: β holds here as well!

$M(\alpha,\ \alpha \Rightarrow \beta, \beta)$



$$M\begin{pmatrix} \alpha \\ \alpha \Rightarrow \beta \end{pmatrix} = M\begin{pmatrix} \alpha \\ \alpha \Rightarrow \beta \\ \beta \end{pmatrix}$$

(Sound) Inference Rules

$$[MP] \quad \frac{\alpha \Rightarrow \beta \quad \alpha}{\beta}$$

$$[\vee D] \quad \frac{\alpha \vee \beta \quad \neg\alpha}{\beta}$$

$$[\& I] \quad \frac{\alpha \quad \beta}{\alpha \wedge \beta}$$

$$[MT] \quad \frac{\alpha \Rightarrow \beta \quad \neg\beta}{\neg\alpha}$$

$$[RC] \quad \frac{\alpha \Rightarrow \beta \quad \beta \Rightarrow \gamma}{\alpha \Rightarrow \gamma}$$

$$[\& E] \quad \frac{\alpha \wedge \beta}{\alpha}$$

$$[MG] \quad \frac{\alpha \Rightarrow \beta \quad \neg\alpha \Rightarrow \beta}{\beta}$$

$$[\vee I] \quad \frac{\alpha}{\alpha \vee \beta}$$

$$[Rsln] \quad \frac{\alpha \vee \beta \quad \gamma \vee \neg\alpha}{\gamma \vee \beta}$$

...

Sequence of Inference Steps

1. $\alpha \,\&\, \beta$
2. $\alpha \Rightarrow \gamma$
3. $\beta \,\&\, \gamma \Rightarrow \delta$

&E 1 $\Rightarrow$

1. $\alpha \,\&\, \beta$
2. $\alpha \Rightarrow \gamma$
3. $\beta \,\&\, \gamma \Rightarrow \delta$
4. α
5. β

MP 4,2 $\Rightarrow$

1. $\alpha \,\&\, \beta$
2. $\alpha \Rightarrow \gamma$
3. $\beta \,\&\, \gamma \Rightarrow \delta$
4. α
5. β
6. γ

&I 5,6 $\Rightarrow$

1. $\alpha \,\&\, \beta$
2. $\alpha \Rightarrow \gamma$
3. $\beta \,\&\, \gamma \Rightarrow \delta$
4. α
5. β
6. γ
7. $\beta \,\&\, \gamma$

MP 7,3 $\Rightarrow$

1. $\alpha \,\&\, \beta$
2. $\alpha \Rightarrow \gamma$
3. $\beta \,\&\, \gamma \Rightarrow \delta$
4. α
5. β
6. γ
7. $\beta \,\&\, \gamma$
8. δ

Sequence of Inference Steps

M
$$\left(\begin{array}{l} \alpha \,\&\, \beta \\ \alpha \Rightarrow \gamma \\ \beta \,\&\, \gamma \Rightarrow \delta \end{array} \right)$$

Exactly the same worlds!!
So if believe FIRST,
must believe SECOND!

M
$$\left(\begin{array}{l} \alpha \,\&\, \beta \\ \alpha \Rightarrow \gamma \\ \beta \,\&\, \gamma \Rightarrow \delta \\ \alpha \\ \beta \\ \gamma \\ \beta \,\&\, \gamma \\ \delta \end{array} \right)$$

Answering Queries

- Adding Truths (Forward Chaining)

  Given KB_0, find KB_N s.t.

  $$\boxed{KB_0} \overset{ri1}{\leadsto} \boxed{KB_1} \cdots \overset{riN}{\leadsto} \boxed{KB_N}$$

  (If $\{ ri_j \}_j$ sound, then $KB_0 \models KB_N$)

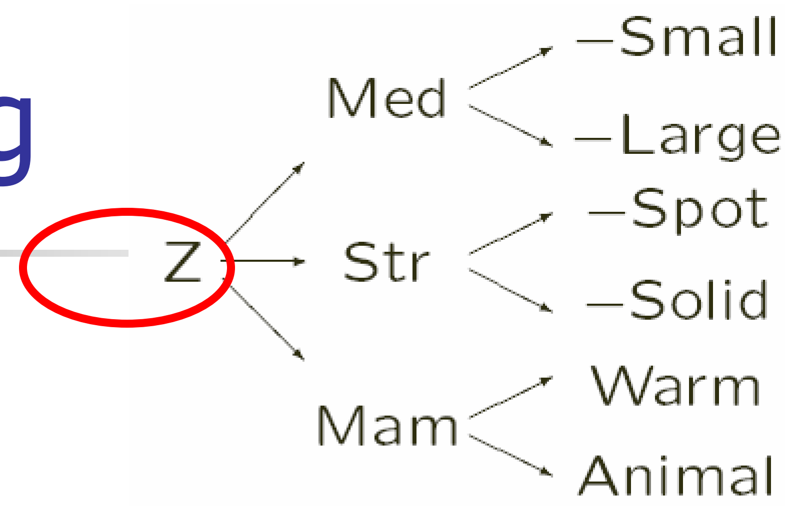
- Answering Questions (Backward Chaining)

  Given KB, σ determine if $KB_0 \models^? \sigma$

  Requires sound $\{ ri_j \}_j$ s.t.

  - $$\boxed{KB_0} \overset{ri1}{\leadsto} \boxed{KB_1} \cdots \overset{riN}{\leadsto} \boxed{KB_N}$$
  - $\sigma \in KB_N$

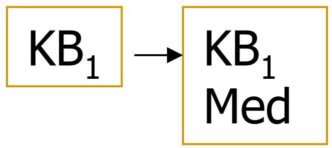
Forward Chaining



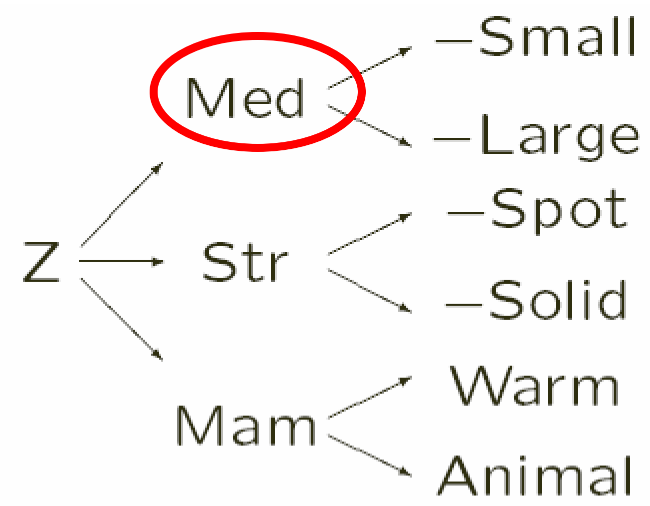
Query: Animal ?

KB$_1$

- Zebra
- Zebra ⇒ Medium
- Zebra ⇒ Striped
- Zebra ⇒ Mammal
- Medium ⇒ NonSmall
- Medium ⇒ NonLarge
- Striped ⇒ NonSolid
- Striped ⇒ NonSpot
- Mammal ⇒ Animal
- Mammal ⇒ Warm
- ...



Forward Chaining



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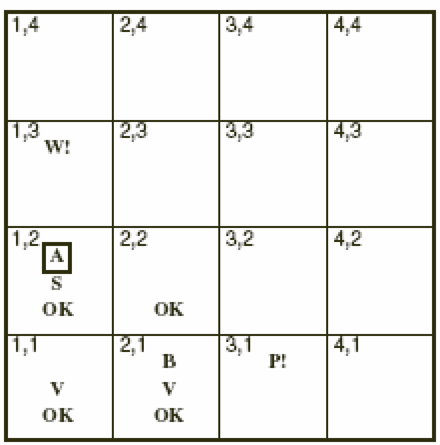
- Medium ⇒ NonLarge
- Striped ⇒ NonSolid
- Striped ⇒ NonSpot
- Mammal ⇒ Animal
- Mammal ⇒ Warm
- ...

KB_1 → KB_1 Med → KB_1 Med -Smal

Example: Is Wumpus at [1, 3] ?

$R_1: \quad \neg S_{1,1} \Rightarrow \neg W_{1,1} \wedge \neg W_{1,2} \wedge \neg W_{2,1}$
$R_2: \quad \neg S_{2,1} \Rightarrow \neg W_{1,1} \wedge \neg W_{2,1} \wedge \neg W_{2,2} \wedge \neg W_{3,1}$
$R_3: \quad S_{1,2} \Rightarrow W_{1,1} \vee W_{1,2} \vee W_{2,2} \vee W_{1,3}$

$F_1: \quad \neg S_{1,1}$
$F_2: \quad \neg S_{2,1}$
$F_3: \quad S_{1,2}$



1. MP: $\neg S_{1,1}, \quad \neg S_{1,1} \Rightarrow \neg W_{1,1} \wedge \neg W_{1,2} \wedge \neg W_{2,1}$
   $\Longrightarrow \quad \neg W_{1,1} \wedge \neg W_{1,2} \wedge \neg W_{2,1}$
2. And-Elimination: $\ldots \Longrightarrow \quad \neg W_{1,1}, \neg W_{1,2}, \neg W_{2,1}$
3. MP+AE: $\neg S_{2,1}, \quad \neg S_{2,1} \Rightarrow \neg W_{1,1} \wedge \neg W_{2,1} \wedge \neg W_{2,2} \wedge \neg W_{3,1}$
   $\Longrightarrow \quad \neg W_{1,1}, \quad \neg W_{2,1}, \quad \neg W_{2,2}, \quad \neg W_{3,1}$
4. MP: $S_{1,2}, \quad S_{1,2} \Rightarrow W_{1,1} \vee W_{1,2} \vee W_{2,2} \vee W_{1,3}$
   $\Longrightarrow \quad W_{1,1} \vee W_{1,2} \vee W_{2,2} \vee W_{1,3}$
5. Resln: $\neg W_{1,1}, \quad W_{1,1} \vee W_{1,2} \vee W_{2,2} \vee W_{1,3}$
   $\Longrightarrow \quad W_{1,2} \vee W_{2,2} \vee W_{1,3}$
6. Resln: $\neg W_{2,2}, \quad W_{1,2} \vee W_{2,2} \vee W_{1,3}$
   $\Longrightarrow \quad W_{1,2} \vee W_{1,3}$
7. Resln: $\neg W_{1,2}, \quad W_{1,2} \vee W_{1,3}$
   $\Longrightarrow \quad \boxed{W_{1,3}}$

How to Reason?

Q: Given KB, q, how to determine if $KB \models^{?} q$?

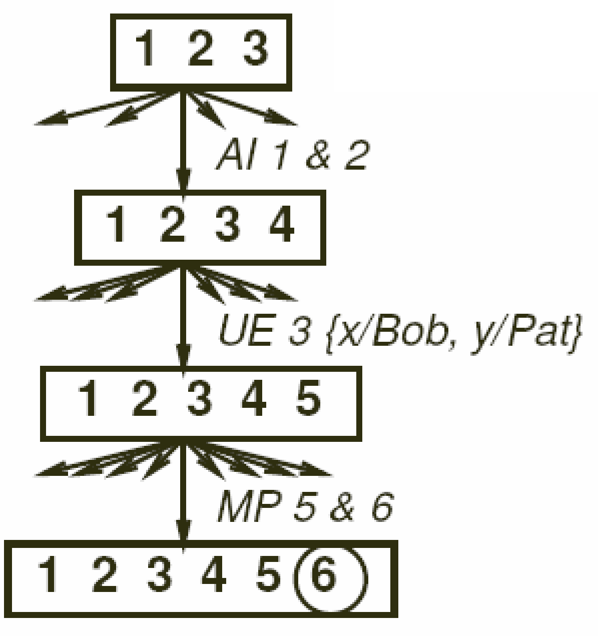
A: Select Inference Rule IR
Select fact(s) $\{F_i\}$ from KB
Apply rule IR to facts $\{F_i\}$ … to get new fact γ
. . . Add γ to KB
Repeat until find $\gamma = q$

Issues:

1. Lots of Inference Rules
   Which one to use, when?
2. Is overall system "complete"?
   If ∃ answer, guaranteed to find it?

Inference ≈ Search

- Operators ≈ inference rules
- States ≈ sets of sentences
- Goal test ≈
  does state contain query sentence?



- Problem:
  - huge branching factor!
  - large depth??

Resolution Rule (Propositional)

- Most Simple:

$$\frac{\alpha \vee \beta \qquad \neg\beta}{\alpha}$$

$$\frac{\textbf{Man} \vee \textbf{Mouse} \qquad \neg\textbf{Mouse}}{\textbf{Man}}$$

- Almost as Simple:

$$\frac{\textbf{Man} \vee \textbf{Mouse} \qquad \neg\textbf{Mouse} \vee \textbf{CatFood}}{\textbf{Man} \vee \textbf{CatFood}}$$

- General:

$$\frac{\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n \qquad \neg\alpha_n \vee \beta_2 \vee \ldots \vee \beta_m}{\alpha_1 \vee \alpha_2 \vee \ldots \vee \beta_2 \vee \ldots \vee \beta_m}$$

(View as set: So A v B v A → A v B)

Conjunctive Normal Form

- Every theory can be written in **Conjunctive Normal Form** (CNF)
  - conjunction of disjunctions of literals

clauses

$$(A \vee \neg B) \ \& \ (B \vee \neg C \vee \neg D) \ \& \ (B \vee E \vee \neg A)$$

- Can write as sets:

$$\{ \{A, \neg B\}, \ \{B, \neg C, \neg D\}, \ \{B, E, \neg A\} \}$$

- Note: {} = Falsity!

Conversion to Conjunctive Normal Form

$P \Rightarrow \neg(Q \Rightarrow R)$

- Eliminate implication, iff, ...

  $\neg P \vee \neg(\neg Q \vee R)$

- Move $\neg$ inwards

  $\neg P \vee (Q \,\&\, \neg R)$

- Distribute & over v

  $(\neg P \vee Q) \,\&\, (\neg P \vee \neg R)$

- Change to SET notation

$$\left\{ \begin{array}{l} \neg P \vee Q \\ \neg P \vee \neg R \end{array} \right\}$$

$(\alpha \Rightarrow \beta) \mapsto \neg\alpha \vee \beta$

$(\alpha \Leftrightarrow \beta) \mapsto (\neg\alpha \vee \beta)$

$\qquad \& \, (\alpha \vee \neg\beta)$

$\neg\neg\alpha \mapsto \alpha$

$\neg(\alpha \vee \beta) \mapsto \neg\alpha \,\&\, \neg\beta$

$\neg(\alpha \,\&\, \beta) \mapsto \neg\alpha \vee \neg\beta$

Can be EXPONENTIALLY larger than original formula

DNF $\Rightarrow$ CNF

Is Resolution Sufficient?

- Subsumes:

$$[MP] \qquad \frac{p \Rightarrow q \quad p}{q} \qquad \frac{\neg p \vee q \quad p}{q}$$

$$[MT] \qquad \frac{p \Rightarrow q \quad \neg q}{\neg p} \qquad \frac{\neg p \vee q \quad \neg q}{\neg p}$$

$$[RC] \qquad \frac{p \Rightarrow q \quad q \Rightarrow r}{p \Rightarrow r} \qquad \frac{\neg p \vee q \quad \neg q \vee r}{\neg p \vee r}$$

$$[MG] \qquad \frac{p \Rightarrow q \quad \neg p \Rightarrow q}{q} \qquad \frac{\neg p \vee q \quad p \vee q}{q}$$

$$[\bowtie] \qquad \frac{\neg p \quad p}{\{\}}$$

$$\ldots$$

- Is Resolution *sufficient*?
  Complete inference process?

Resolution $\vdash_R$ Process

Given theory KB, and query σ:

1. Find two clauses in KB

   $\alpha \equiv A \vee Q \vee B$

   $\beta \equiv C \vee \neg Q \vee D$

   that have complementary ...

   If none, return No... else...

2. Smash them!

   $\gamma \equiv A \vee B \vee C \vee D$

3. Does this new γ match σ ?
   - If so, return YES
   - If not, add to KB $\leftarrow$ KB + γ
     Go to 1.

Is this process COMPLETE?
Can it answer EVERY KB+query??

Resolution is NOT Complete

- Resolution $\vdash_R$ smashes together clauses

  Eg... $\{ \ldots, \alpha \vee A, \ldots, \neg A \vee \beta, \ldots\} \vdash_R \alpha \vee \beta$

- But if $KB = \{\}$, $\vdash_R$ cannot derive *anything*
- Tautologies $p \vee \neg p$ always entailed

  $$\{\} \models (p \vee \neg p)$$

  But

  $$\{\} \not\vdash_R (p \vee \neg p)$$

- Also... $\{p\} \models (p \vee p)$ but $\{p\} \not\vdash_R (p \vee p)$

  ...

Is this process COMPLETE?
Can it answer EVERY KB+query??

NO!

Refutation

$R \subseteq S$, then $R \cap \sim S = \{\}$

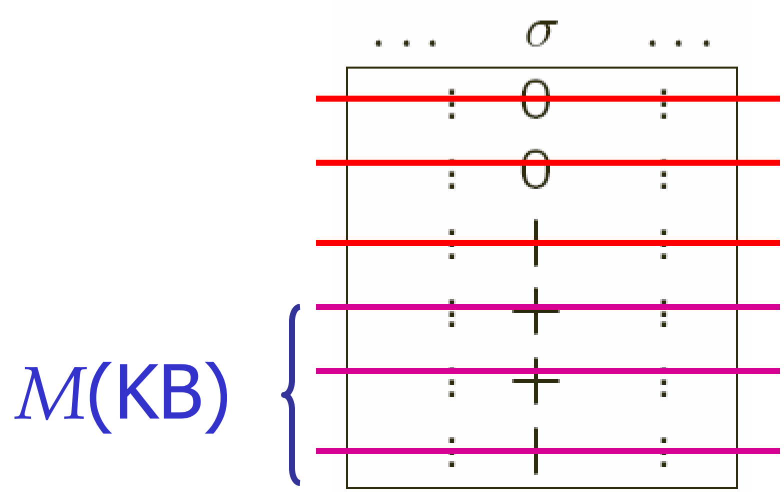
- Resolution can still be used for entailment! Using Refutation Proof :
- $KB \models \sigma$ means σ is true in all models of KB
- Now assert $\neg\sigma$
  - … ie, $KB \cup \neg\sigma$

  This removes each model where σ is true

$\Rightarrow$ it has NO models

- $M(KB \cup \neg\sigma) = \{\}$

$\Rightarrow$ $KB \cup \neg\sigma \models$ False



Refutation Proof

- Deduction Theory

$KB \models \sigma$ *iff*

$KB \cup \neg\sigma$ is inconsistent *iff*

$KB \cup \neg\sigma \models$ False

- To prove σ

Add $\neg\sigma$ to KB

If can prove a Contradiction, False, then $KB \models \sigma$

Refutation Complete

- $\vdash$ is Complete *iff*

  $\forall$ KB, σ: KB $\models \sigma \Rightarrow$ KB $\vdash \sigma$

- $\vdash$ is REFUTATION Complete *iff*

  $\forall$ KB: KB $\models \{\} \Rightarrow$ KB $\vdash \{\}$

- **Resolution $\vdash_R$ is REFUTATION COMPLETE**
- If KB $\models \sigma$ then

  $\exists$ resolution proof of False from KB $\cup \neg\sigma$

Proof…

If KB $\models \sigma$ then

$\exists$ resolution proof of False from KB $\cup \neg\sigma$

Proof: Let $RC(\Gamma)$ be deductive closure of Γ using Resl'n

Need only show: if $\{\} \notin RC(\Gamma)$,
then Γ is consistent … *i.e.*, Γ has model.

Build model over variables $v_1, \ldots, v_k$:
For $i = 1..k$
- $\star$ if $\exists c_j \in RC(\Gamma)$ s.t.
  $\neg v_i \in c_j$ and assg'n to $v_1, \ldots, v_{i-1}$ false
  then $v_i \leftarrow$ false
- $\star$ otherwise $v_i \leftarrow$ true

This assignment $\{\pm v_1, \ldots, \pm v_k\}$ is model for Γ !

Using Refutation Resolution

- Given KB, σ

  Let $\Gamma = KB \cup \neg\sigma$:

  Try to prove False {}, using $\vdash_R$

  $\Gamma \vdash_R^? False$

  If succeed, then $KB \models \sigma$

  If fail, then $KB \not\models \sigma$

- Problem:
  - Resolution works by smashing CLAUSES!

  $\Rightarrow$ Need to encode KB, σ as clauses
- Can always be done!

Example of Resolution Process

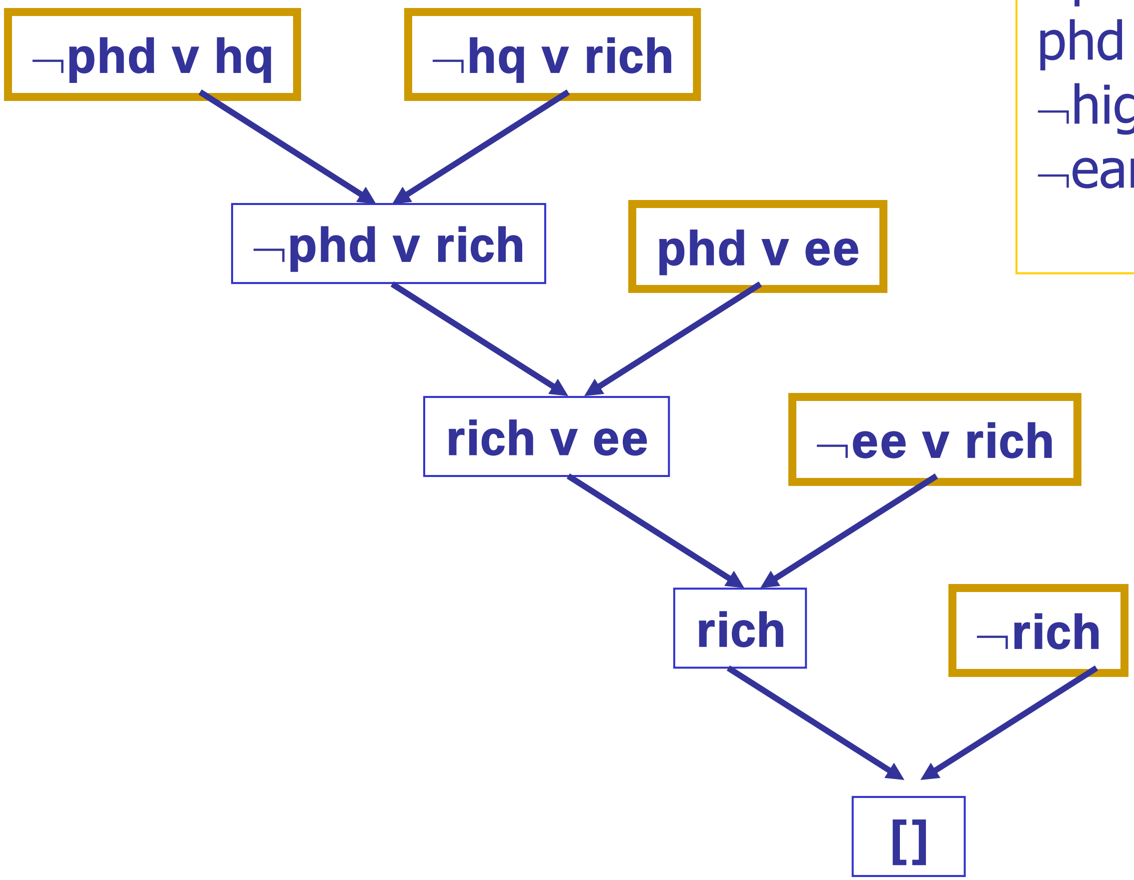
- Knowledge base

phd	$\Rightarrow$ highlyQualified
¬phd	$\Rightarrow$ earlyEarnings
highlyQualified	$\Rightarrow$ rich
earlyEarnings	$\Rightarrow$ rich

- Goal: rich ?
- NOTE: simple RuleChaining will NOT work!
- Now what?
  - REFUTATION PROOF!
  - Convert to "CNF Form" (including ¬ of goal)
  - Resolve, seeking []
  - (Return solution)

Resolution Proof

- To prove rich from KB, add ¬rich



¬phd v highlyQualified
phd v earlyEarnings
¬highlyQualified v rich
¬earlyEarnings v rich

Inference Using Resolution

Given KB, σ

- 1. Convert KB to CNF: *CNF(KB)*
- 2. Convert $\neg\sigma$ to CNF: *CNF($\neg\sigma$)*
- 3. CNF(KB) $\cup$ CNF($\neg\sigma$) $\vdash_R$ {} ?

- If succeed, then KB $\models \sigma$
- If fail, then KB $\not\models \sigma$

- For propositional logic:
  - sound
  - complete
  - decidable
- But
  - Exponential time in general (not "just" NP-hard)
  - Linear time for Horn clauses
  - Linear time for 2-CNF clauses

Length of Resolution Proof?

- Can Resolution be FORCED to take exponentially many steps?
  - Posed [Cook / Karp, 1971/72]... related to NP vs. co-NP
  - Resolved [Haken 1985]

- Pigeon-Hole (PH) problem:

  Cannot place *n+1* pigeons in *n* holes (1/hole)
- PH takes exponentially many steps (for Resolution) no matter what order, strategy, . . .

- Important:
  - PH hidden in many practical problems
  - Makes theorem proving/ reasoning expensive
  - Contributed to recent move to model-based methods

Pigeon-Hole Principle

- $P_{i,j}$ for Pigeon i in hole j.

- Every pigeon is in some hole:

$$P_{1,1} \vee P_{1,2} \vee P_{1,3} \vee \ldots \vee P_{1,n}$$
$$P_{2,1} \vee P_{2,2} \vee P_{2,3} \vee \ldots \vee P_{2,n}$$
$$\vdots$$
$$P_{(n+1),1} \vee P_{(n+1),2} \vee P_{(n+1),3} \vee \ldots \vee P_{(n+1),n}$$

- Every pigeon is in at most one hole:

$$(\neg P_{1,1} \vee \neg P_{1,2}),\ (\neg P_{1,1} \vee \neg P_{1,3}),\ \ldots (\neg P_{1,(n-1)} \vee \neg P_{1,n})$$
$$\vdots$$
$$(\neg P_{2,1} \vee \neg P_{2,2}), \ldots, (\neg P_{2,(n-1)} \vee \neg P_{2,n})$$

- Every hole has at most one pigeon:

$$(\neg P_{1,1} \vee \neg P_{2,1}),\ (\neg P_{1,1} \vee \neg P_{3,1}),\ \ldots$$
$$(\neg P_{1,2} \vee \neg P_{2,2}),\ (\neg P_{1,2} \vee \neg P_{3,2}),\ \ldots$$
$$\vdots$$

Result

- Requires $O(n^3)$ clauses
- Resolution proof that

  PH is inconsistent

  requires dealing with at least exponential # of clauses, no matter how clauses are resolved!

  [Haken85]

⇒ "Method can't count"

- Can word in Predicate Calculus ... same problem

Generality; Choice Points

- As any theory can be translated to CNF
  and as resolution is []-complete,
  All deduction in terms of Resolution.
- Only decision is…
  Which (two literals in which) two clauses to (try to) Resolve?

- Eg:
  - Insist on using an atomic literal:
    Unit Resolution (F or B)
    - Only positive atomic literal: Forward reasoning
    - Only negative atomic literal: Backward reasoning
  - Set of support
  - Ancestry filtering, ordered(lock)
  - ...

Resolution Strategy I: Unit Preference

Goal: to find {} (clause w/ 0 literals)

- For $R = Resolve(P, Q)$
  $|R| = |P| + |Q| - 2$

- If |P| = 4 and |Q| = 3, then |R| = 5
  … so |R| > |P|, |Q|
  Is this progress?

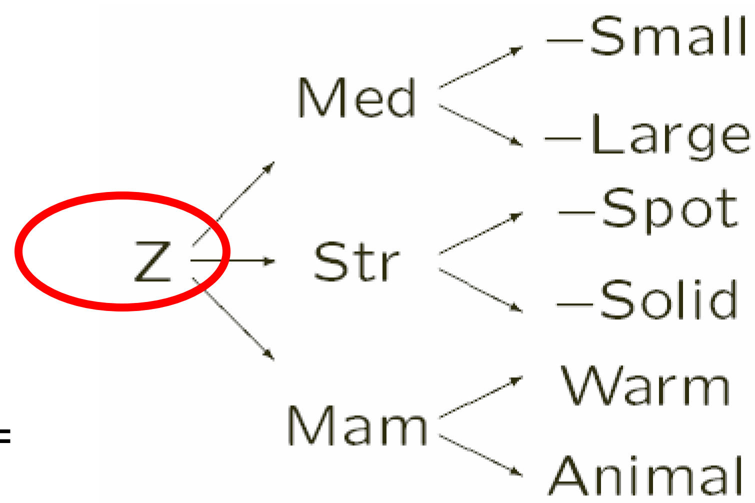
- But if |P| = 1, then |Resolve(P,Q)| = |Q| – 1
  PROGRESS towards 0 !
- Unit Preference:
  Given KB, may resolve P and Q only if
  P is single literal ("unit clause")
- Does it work?

Unit Propagation≈ Forward/Backward Reasoning

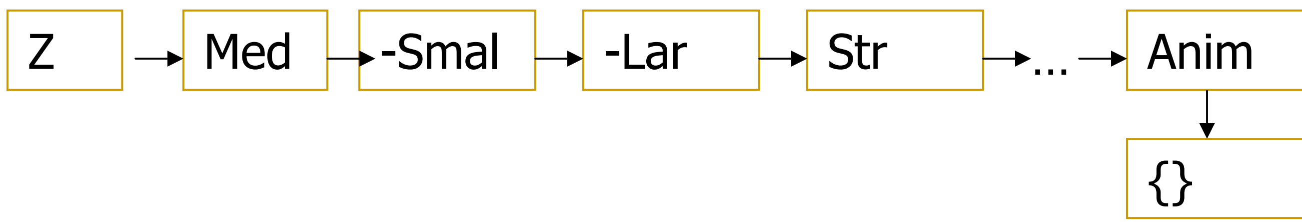
Query: Animal ?

KB_1

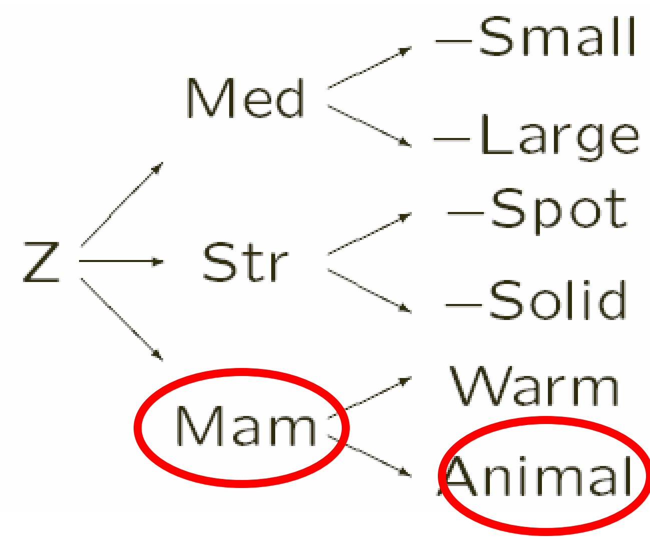
- Zebra
- ¬Zebra v Medium
- ¬Zebra v Striped
- ¬Zebra v Mammal
- ¬Medium v NonSmall
- ¬Medium v NonLarg
- ¬Striped v NonSolid
- ¬Striped v NonSpot
- ¬Mammal v Animal
- ¬Mammal v Warm
- ¬Animal



Forward Reasoning ==
use POSITIVE literal



Backward Chaining



Query: Animal ?

KB$_1$

- Zebra
- ¬Zebra v Medium
- ¬Zebra v Striped
- ¬Zebra v Mammal
- ¬Medium v NonSmall
- ¬Medium v NonLarg
- ¬Striped v NonSolid
- ¬Striped v NonSpot
- ¬Mammal v Animal
- ¬Mammal v Warm
- ¬Animal

Backward Reasoning ==
use NEGATIVE literal

¬Animal → ¬Mam → ¬Zebra → {}

- **Forward chaining**:
  - Start with known facts; add in other correct facts
  - Required 9 steps
- **Backward chaining**:
  - Go from Goal to subGoal to subsubGoal to ...
  - Required 2 steps

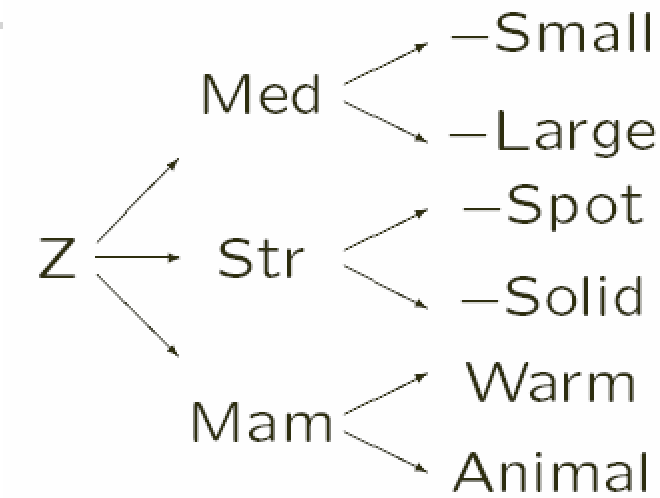
Comparing Forward vs Backward

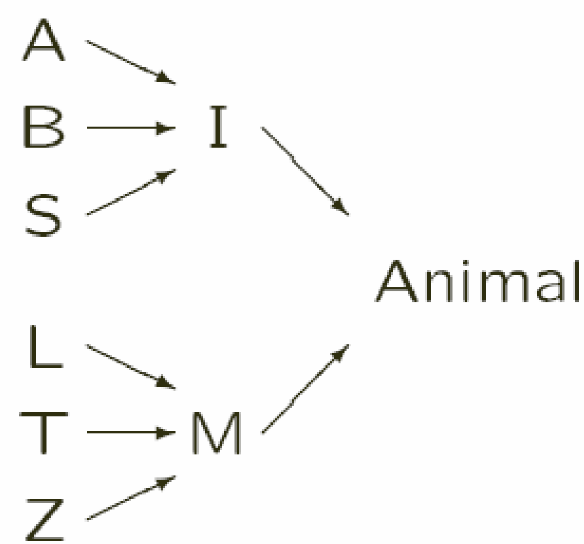
Query: Animal ?

	FC	BC
KB_1	9	**2**
KB_2	**2**	8

KB_1
- Zebra
- Zebra ⇒ Medium
- Zebra ⇒ Striped
- Zebra ⇒ Mammal
- Medium ⇒ NonSmall
- Medium ⇒ NonLarge
- Striped ⇒ NonSolid
- Striped ⇒ NonSpot
- Mammal ⇒ Animal
- Mammal ⇒ Warm
- …

KB_2
- Zebra
- Ant ⇒ Insect
- Bee ⇒ Insect
- Spider ⇒ Insect
- Insect ⇒ Animal
- Lion ⇒ Mammal
- Tiger ⇒ Mammal
- Zebra ⇒ Mammal
- Mammal ⇒ Animal…





Horn Clauses … aka Rules

- Every theory can be written in **Conjunctive Normal Form** (CNF)
  - conjunction of disjunctions of literals

clauses

$$(A \lor \neg B) \,\&\, (B \lor \neg C \lor \neg D) \,\&\, (B \lor E \lor \neg A)$$

- Some theories are **Horn**
  - conjunction of Horn clauses (clauses with ≤ 1 positive literal)

$$(A \lor \neg B) \,\&\, (B \lor \neg C \lor \neg D)$$

  - Often written as set of implications:

$$(B \Rightarrow A) \,\&\, (C \,\&\, D) \Rightarrow B$$

Normal Forms

- Information needs to be in specific form to use inference rules. . .

- **Conjunctive Normal Form** (CNF - universal)
  - conjunction of disjunctions of literals

clauses

(A v ¬B) & (B v ¬C v D)

- **Disjunctive Normal Form** (DNF - universal)
  - disjunction of conjunctions of literals

terms

(A & B) v (A v ¬C) & (A & ¬D)

- **Horn Form** (restricted)
  - conjunction of Horn clauses (clauses with $\leq$1 positive literal)
  - Often written as set of implications:

(A v ¬B) & (B v ¬C v ¬D)

$(B \Rightarrow A)$ & $(C \& D) \Rightarrow B$

"Chaining" Inference Processes

- In general, need to consider $A \& B \Rightarrow C$, not just $A \Rightarrow C$
- Here, both F- and B- reasoning were also

  Unit Preferences

  Typical, but can be generalized …

- **If Horn theory:**
  - **Forward Chaining** …
    is COMPLETE for ATOMIC Queries
    DataDriven – could be done by *Tell*
  - **Backward Chaining**
    is COMPLETE for ATOMIC Queries
  - Each is worst-case *O(n)*…
    but different actual run-times ...depends on Branching Factor...
- But NOT everything is HORN!
  - $S_{1,2} \Rightarrow W_{1,1} \vee W_{1,2} \vee W_{2,2} \vee W_{1,3}$
  - $\neg S_{1,2} \vee W_{1,1} \vee W_{1,2} \vee W_{2,2} \vee W_{1,3}$

Unit Resolution

Can resolve P and Q only if . . .

- **Unit Preference**: |P| = 1

  STATUS: Not complete

  $$\left\{\begin{matrix} A \vee B & \neg A \vee B \\ A \vee \neg B & \neg A \vee \neg B \end{matrix}\right\}$$

  But ... *Refutation Complete* for Horn clauses.

- Horn , each clause has ≤ 1 positive literal

  Horn: A A v ¬B ¬B ¬A v ¬B ...

  NotHorn: A v B A v ¬Q v W

Ordered Resolution

Ordered Resolution: $\left\{ \begin{array}{ll} A \vee B & \neg A \vee B \\ A \vee \neg B & \neg A \vee \neg B \end{array} \right\}$

- Literals in each clause are ordered:

  $P = \langle p_1 \vee p_2 \vee \ldots \rangle, \quad Q = \langle q_1 \vee q_2 \vee \ldots \rangle$

- Can resolve P and Q only if p_1 corresponds to $\neg q_1$

STATUS: Refutation complete for Horn

Resolution Strategies, II

- **Set of Support**: Resolve P, Q only if P ∈ S where S KB is "set of support".
  . . . then add resolvent to S.

  Complete if *Consistent(KB- S)*
- **Backward Reasoning**:
  Initial Support: S = negated query ¬σ
- **Forward Reasoning**:
  Initial Support: S = original KB

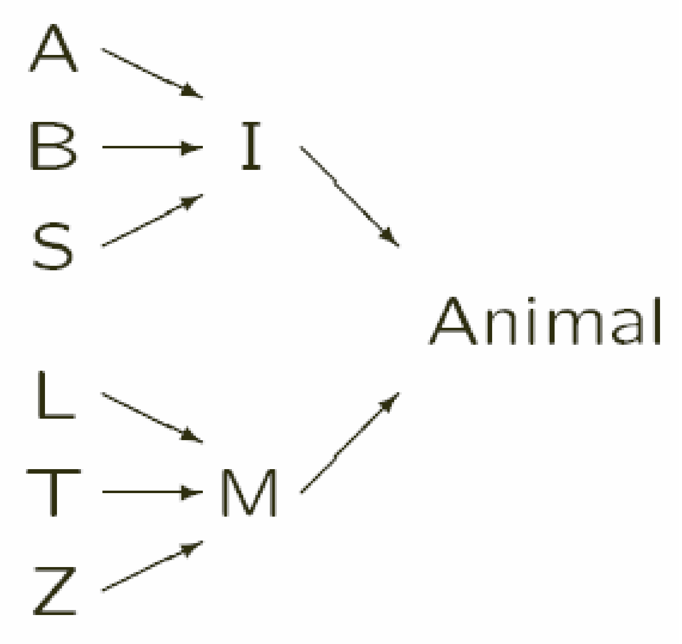
- Q: Which is better?
- A: Depends on branching factor!

Set-of-Support: Backward Reasoning

Zebra
¬Ant v Insect
¬Bee v Insect
¬Spider v Insect
¬Insect v Animal
¬Lion v Mammal
¬Tiger v Mammal
¬Zebra v Mammal
¬Mammal v Animal

¬Animal

Set of Support

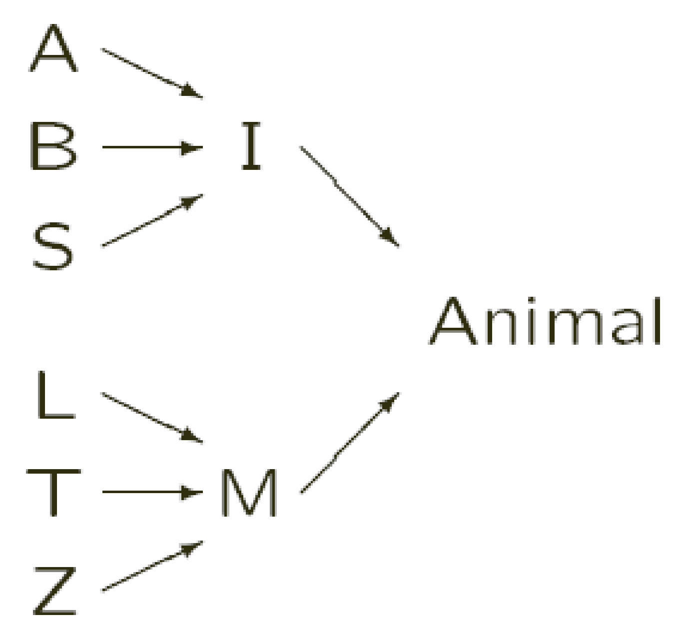


Set-of-Support: Forward Reasoning

Set of Support

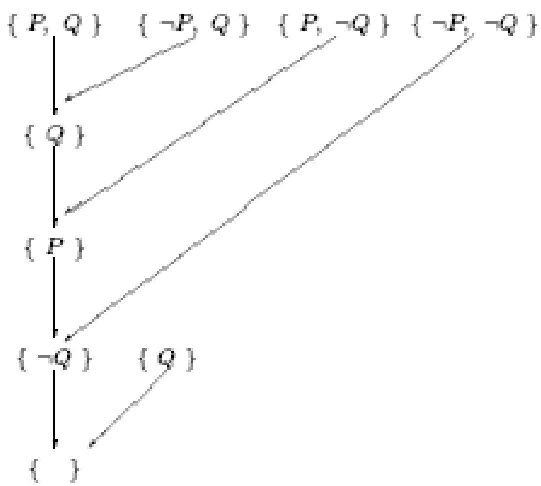
¬Animal

Zebra
¬Ant v Insect
¬Bee v Insect
¬Spider v Insect
¬Insect v Animal
¬Lion v Mammal
¬Tiger v Mammal
¬Zebra v Mammal
¬Mammal v Animal



Resolution Strategies, III

- **Input Resolution:** only if P in original KB
  STATUS: Not complete.
- **Linear resolution**: only if P in original KB
  or P is ancestor of Q in proof tree



- **STATUS**: Refutation complete
  (if KB consistent, then KB u ~σ inconsistent
  iff LinRes, starting with σ, reaches {})

Deduction Theorem, Validity, Satisfiability

- Sentence is valid iff true in all models

  Eg, A v ~A, $A \Rightarrow A$, $(A \,\&\, (A \Rightarrow B)) \Rightarrow B$

  - ... related to inference via **Deduction Theorem**:

$$KB \models \alpha \quad \text{iff} \quad (KB \Rightarrow \alpha) \text{ is valid}$$

- Sentence is satisfiable iff true in <u>some</u> model

  Eg, A v B, C

- Sentence is unsatisfiable iff true in <u>no</u> models

  Eg, $A \,\&\, \neg A$

- Satisfiability related to inference via ...

$$KB \models \alpha \quad \text{iff} \quad (KB \,\&\, \neg\alpha) \text{ is unsatisfiable}$$

.... prove α by reductio ad absurdum

Basic concepts of logic

- syntax: formal structure of sentences
- semantics: truth of sentences wrt models
- entailment: necessary truth of one sentence given another
- derivation: deriving sentences from other sentences
- soundness: derivations produce only entailed sentences
- completeness: derivations can produce all entailed sentences

Summary

- Logical agents make inferences from a knowledge base

  to derive new information (used to make decisions)
- Even simple tasks (Wumpus World) require ability to
  - represent *partial* and *negated* information,
  - reason by cases, etc.
- Propositional logic often sufficient
- Resolution is sound + complete … if exponential time