Programming in Prolog: Integers

- Represent Non-Negative Integers as

  0, s(0), s(s(0)), s(s(s(0))), ...

- Constant: `0`
  Unary function symbol: `s`

(Just for Pedagogical Reasons!)

Addition

- *Prolog* uses *predicates*; not *functions*.
  Use

  $$\texttt{sum(X, Y, Z)}$$

  which holds *iff* $\texttt{X} + \texttt{Y} = \texttt{Z}$

  [not "`sum(X,Y)`" returns "`Z`"]

- Axioms for Addition:

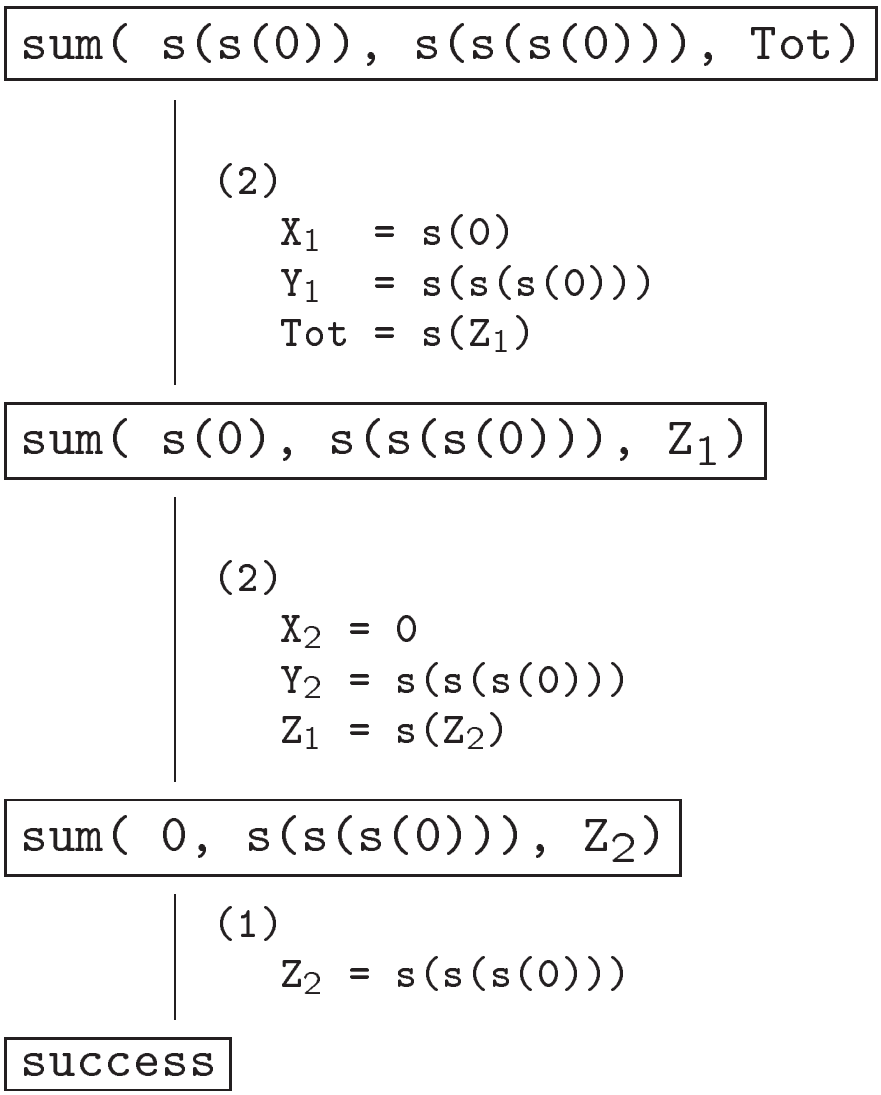
  `sum( 0, Y, Y).` *(1)*
  `sum(s(X), Y, s(Z))` :- `sum(X,Y,Z).` *(2)*

- Notice: If first arg is constant,
  $\Rightarrow$ only 1 head will unify.

`sum( 2, 3, 5 )`
$\equiv$ `sum( s(1), 3, s(4) )` $\overset{(2)}{\rightsquigarrow}$ `sum( 1, 3, 4 )`
$\equiv$ `sum( s(0), 3, s(3) )` $\overset{(2)}{\rightsquigarrow}$ `sum( 0, 3, 3 )`
$\overset{(1)}{\rightsquigarrow}$ SUCCESS

Computing Integer Sums

Find Tot s.t. 2+3= Tot.

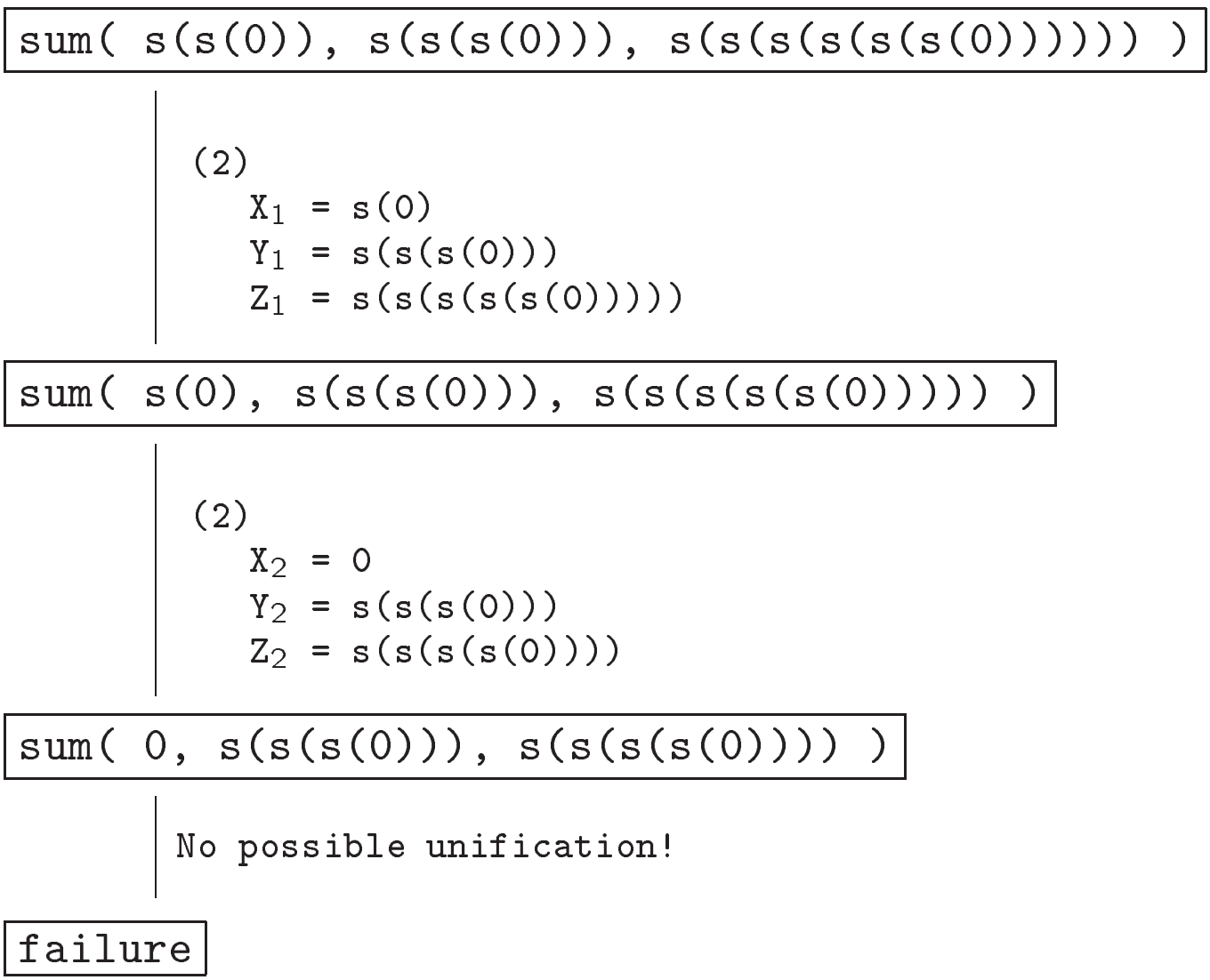


so $Z_1 = s(Z_2) = s(s(s(s(0))))$

$Tot = s(Z_1) = s(s(s(s(s(0)))))$

Verifying Integer Sums

Does "2" + "3" = "6"?



What User Really Types (III)

```
obed% sicstus
C-Prolog version 1.4
| ?- [user].          % User now in ``Assert-fact'' mode.
| sum(0, Y, Y).
| sum( s(X), Y, s(Z)) :- sum(X,Y,Z).
| ^D user con...     % Typing ``^D'' exits ``assert'' mode.

yes

| ?- sum( s(s(0)), s(s(s(0))), Tot).

Tot = s(s(s(s(s(0))))) _
                     %  Prolog returns this binding list.
                     %  User types CR ⇒ answer is accepted

yes                  % ≡ ``Prolog found accepted answer''

| ?- sum( s(s(0)), s(s(s(0))), s(s(s(s(s(0))))) ).

no                   % ≡ Prolog found no answers

| ?-
```

Comments wrt `sum`

- Notation: $\bar{n}$ means $\underbrace{\texttt{s(s(...(s(}\texttt{0))...)}}_{n\ times}$

Eg: $\begin{cases} \bar{2} = \texttt{s(s(0))} \\ \bar{0} = \texttt{0} \end{cases}$

- Derived:

  `sum(`$\bar{2}$`, `$\bar{3}$`, Tot)`

  succeeds with `Tot =` $\bar{5}$.

- Verified:

  `sum(`$\bar{2}$`, `$\bar{3}$`, `$\bar{5}$`)` succeeds

  `sum(`$\bar{2}$`, `$\bar{3}$`, `$\bar{6}$`)` fails

Subtracting using sum

Find W s.t. $\bar{1} + W = \bar{3}$

sum(s(0), W, s(s(s(0))))

(2)
$X_1 = 0$
$Y_1 = W$
$Z_1 = s(s(0))$

sum(0, Y_1, s(s(0)))

(1)
$Y_1 = s(s(0))$

success

$\Rightarrow W = Y_1 = s(s(0))$; ie, $W = \bar{2}$

---

sum(D, $\bar{3}$, $\bar{8}$) succeeds with D = $\bar{5}$.

To do subtraction directly:

```
diff(X,Y,Z) :- sum(Y,Z,X)
```

Other uses of sum

- sum($\bar{3}$, $\bar{4}$, s(s(W))) succeeds with W = $\bar{5}$

  W s.t. $3 + 4 = (W + 1) + 1$

- sum($\bar{3}$, s(W), $\bar{6}$) succeeds with W = $\bar{2}$

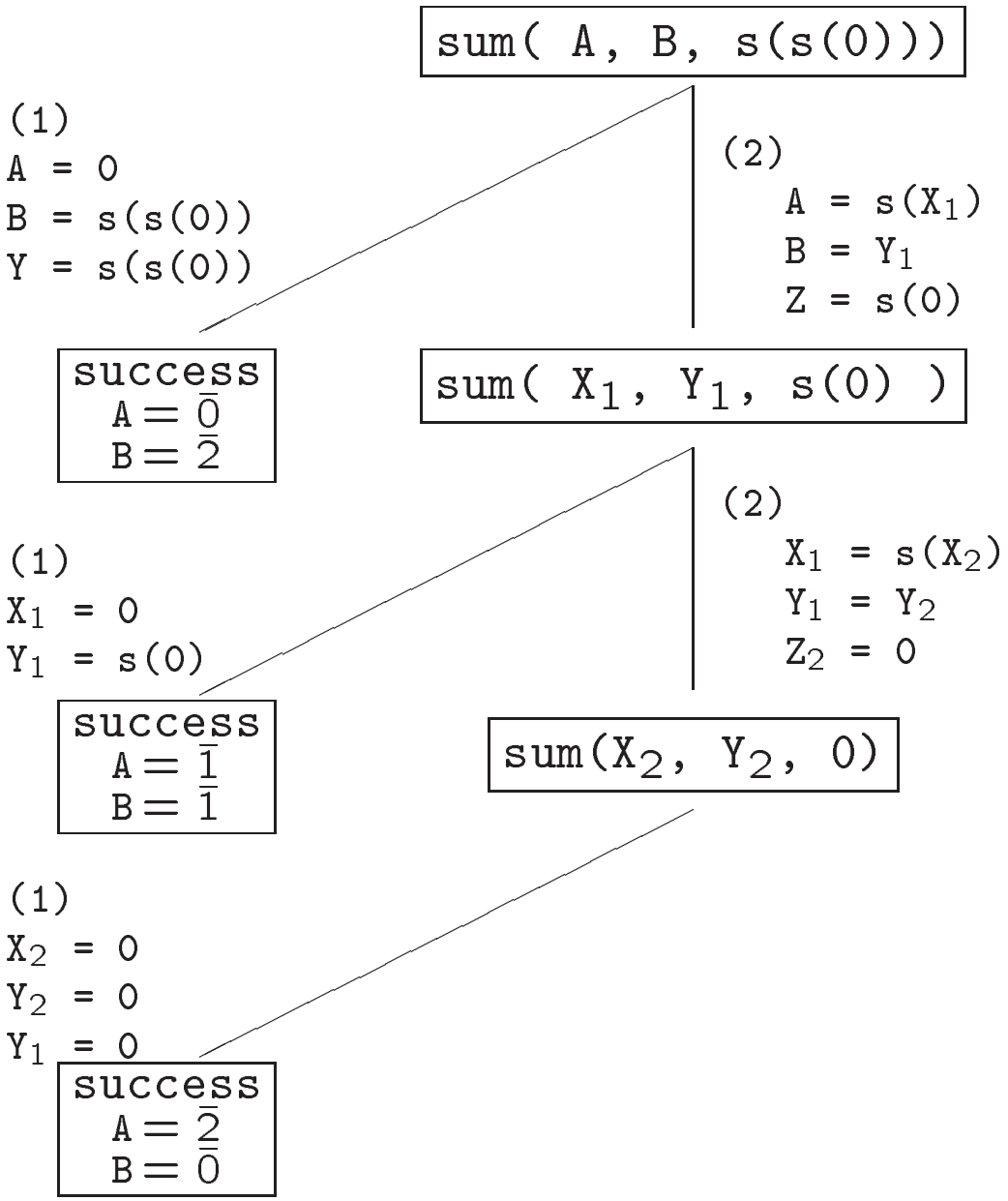
  W s.t. $3 + (W + 1) = 6$

- What about sum(A, B, $\bar{2}$)?

What User Really Types (IV)

```
| ?- [sumf].                    % File holds 2 sum facts.
yes
| ?- sum( s(0), W, s(s(0)) ).
                                % Find W s.t.  1̄ + W = 3̄.
W = s(s(0)) ;                   % Found W = 2̄
                                % User asks for another answer:
no                              % ≡ ¬∃ other answers
| ?- sum( D, s(s(0)), s(s(s(s(s(s(s(0))))))) ).
                                % Find D: D + 3̄ = 8̄.
D = s(s(s(s(0)))) _             % Found D = 5̄
yes
| ?- sum( s(s(0)), s(s(s(0))), s(W) ).
                                % Find W: 3̄ + 4̄ = (W + 1) + 1
W = s(s(s(s(0)))) _             % Found W = 5̄
yes
| ?- sum( s(s(0)), s(W), s(s(s(s(0)))) ).
                                % Find W: 3̄ + (W + 1) = 6̄
W = s(0) _                      % Found W = 2̄
yes
| ?- sum( A, B, s(0) ).
     ...
```

sum(A, B, s(s(0))) ?


sum(A, B, s(s(0)))
(1)
A = 0
B = s(s(0))
Y = s(s(0))
success
A = 0
B = 2
(2)
A = s(X1)
B = Y1
Z = s(0)
sum(X1, Y1, s(0))
(1)
X1 = 0
Y1 = s(0)
success
A = 1
B = 1
(2)
X1 = s(X2)
Y1 = Y2
Z2 = 0
sum(X2, Y2, 0)
(1)
X2 = 0
Y2 = 0
Y1 = 0
success
A = 2
B = 0

`sum` and Variables

- `sum(A, B, 5̄)` succeeds with

`A` $= \bar{0}$	`B` $= \bar{5}$
`A` $= \bar{1}$	`B` $= \bar{4}$
`A` $= \bar{2}$	`B` $= \bar{3}$
`A` $= \bar{3}$	`B` $= \bar{2}$
`A` $= \bar{4}$	`B` $= \bar{1}$
`A` $= \bar{5}$	`B` $= \bar{0}$

- What about `sum(A, B, C)` ?

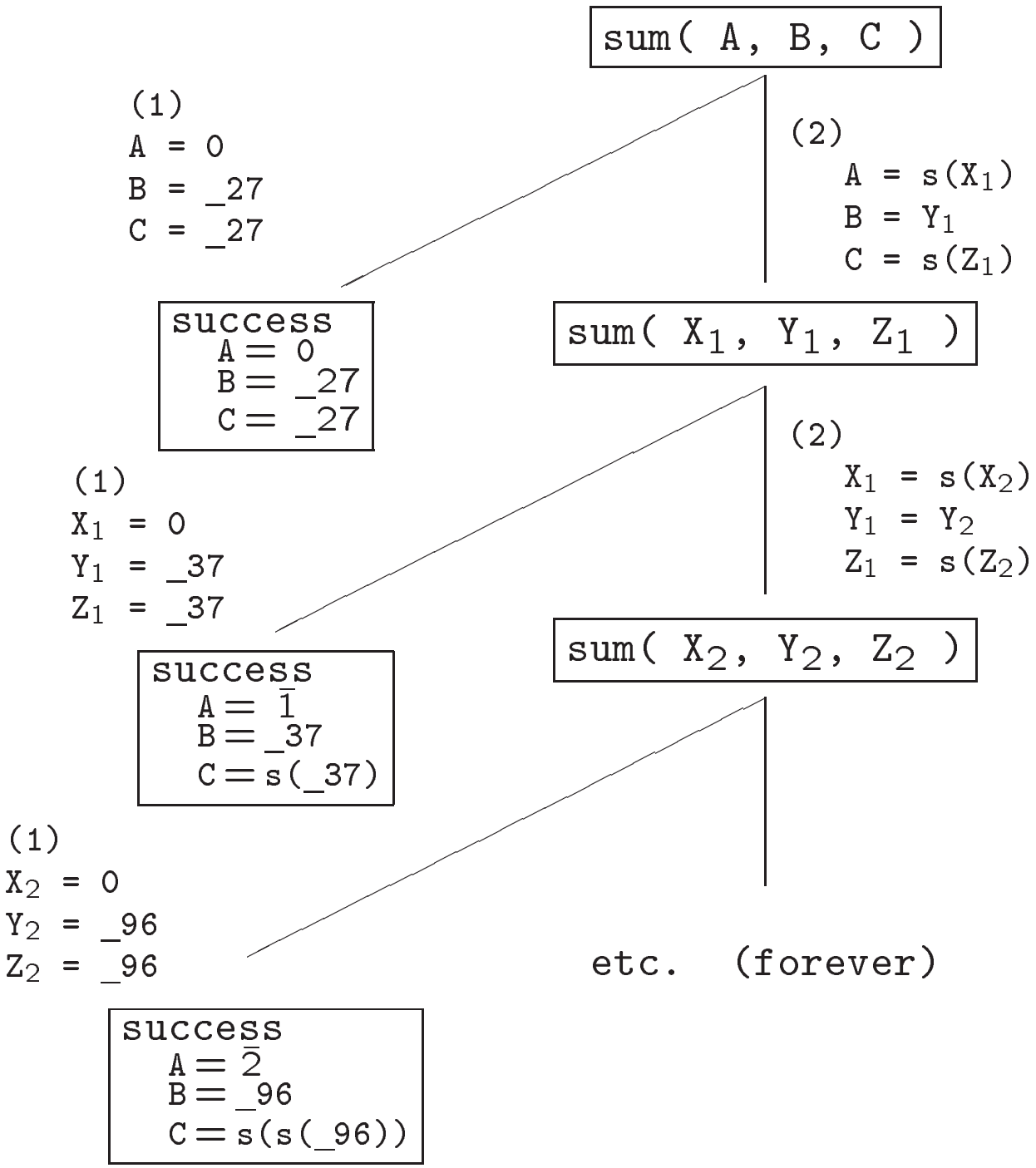
  succeeds with `A=`$\bar{0}$ `B=_27` `C=_27`

  then `A=`$\bar{1}$ `B=_32` `C=s(_32)`

  then `A=`$\bar{2}$ `B=_97` `C=s(s(_97))`

  . . . for ever . . .

sum(A, B, C) ?


sum(A, B, C)
(1)
A = 0
B = _27
C = _27
success
A = 0
B = _27
C = _27
(2)
A = s(X1)
B = Y1
C = s(Z1)
sum(X1, Y1, Z1)
(1)
X1 = 0
Y1 = _37
Z1 = _37
success
A = 1
B = _37
C = s(_37)
(2)
X1 = s(X2)
Y1 = Y2
Z1 = s(Z2)
sum(X2, Y2, Z2)
(1)
X2 = 0
Y2 = _96
Z2 = _96
success
A = 2
B = _96
C = s(s(_96))
etc. (forever)

What User Really Types (V)

```
| ?- [sumf].          % File holds 2 sum facts.
yes
| ?- sum( A, B, s(s(0)) ).
                      % Find A, B s.t.  A + B = 2̄.
A = 0
B = s(s(0)) ;         % Found A = 0̄, B = 2̄
                      % User asks for another answer:
A = s(0)
B = s(0) ;            % Found A = 1̄, B = 1̄

A = s(s(0))
B = 0 ;               % Found A = 2̄, B = 0̄

no                    % ≡ There are no other answers

| ?- sum( s(s(A)), s(B), s(s(s(s(0))))) ).
                      % Find A, B s.t.
                      %     ((A + 1) + 1) + (B + 1)  =  4̄.
A = 0
B = s(0) ;            % Found A = 0̄, B = 1̄

A = s(0)
B = 0 _               % Found A = 1̄, B = 0̄
                      % User accepts this answer.
yes

| ?-
```

User's Typing — (con't)

```
| ?- sum( A, B, C).
                        % Find A, B, C   s.t.  A + B = C.

A = 0
B = _1
C = _1 ;                % Found   A = 0,  B = C = anything

A = s(0)
B = _1
C = s(_1) ;             % Found   A = 1,  B = anything
                        %   C = 1 + B

A = s(s(0))
B = _1
C = s(s(_1)) ;          % Found   A = 2,  B = anything
                        %   C = 2 + B

A = s(s(s(0)))
B = _1
C = s(s(s(_1))) _       % Found   A = 3,  B = anything
                        %   C = 3 + B
                        % User accepts this answer.

yes

| ?-
```

`sum` and `half`?

- What about Halving?

  $\boxed{\texttt{sum(H,H,}\bar{1}\texttt{4)}}$ succeeds with $\texttt{H=}\bar{7}$.

- Consider

```
half(X,H) :- sum(H,H,X).
```

. . . but

$$
\begin{aligned}
\text{half}(\bar{5},\text{H}) &\mapsto \text{sum}(\text{H},\ \text{H},\ \bar{5}) \\
&\mapsto \text{sum}(\text{H},\ \text{s}(\text{H}),\ \bar{4}) \\
&\mapsto \text{sum}(\text{H},\ \text{s}(\text{s}(\text{H})),\ \bar{3}) \\
&\mapsto \quad \ldots \\
&\mapsto \text{sum}(\text{H},\ \text{s}(\text{s}(\text{s}(\text{s}(\text{s}(\text{H}))))),\ \bar{0}) \\
&\mapsto \times
\end{aligned}
$$

fails, after 6 steps.

$$\boxed{\text{Need to worry about } \left\{ \begin{array}{c} \textit{Success} \\ \textit{Failure} \end{array} \right\} \text{ behavior!}}$$

Better Definition of `half`

- Consider

```
half(      0,  0).
half(s(s(X)),  s(H)) :- half(X,H).
```

Eg 1

$$\boxed{\texttt{half}(\bar{6},\texttt{F})} \mapsto \text{half}(\bar{4}, H_1) \quad [F = s(H_1)]$$
$$\mapsto \text{half}(\bar{2}, H_2) \quad [H_1 = s(H_2)]$$
$$\mapsto \text{half}(\bar{0}, H_3) \quad [H_2 = s(H_3)]$$
$$\mapsto \quad H_3 = 0 \quad \Rightarrow F = s(s(s(H_3))) = \bar{3}$$

which succeeds after *only 3 steps*!

Eg 2

$$\boxed{\texttt{half}(\bar{5},\texttt{F})} \mapsto \text{half}(\bar{3}, H_1) \quad [F = s(H_1)\ ]$$
$$\mapsto \text{half}(\bar{1}, H_2) \quad [H_1 = s(H_2)]$$
$$\mapsto \times$$

which fails after *only 2 steps*!

- This def'n is

  Faster on BOTH success and failure.

Another Integer Example: Multiplication

```
| ?- [user].
| prod( 0, Y, 0 ).
| prod( s(X), Y, Z ) :- prod(X,Y,W), sum(Y,W,Z).
| ^D user con...
yes
| ?- prod( 0, s(s(s(s(0)))), P1)
                        % Find P1:  0̄ × 4̄ = P1
P1 = 0 _                % succeeds with P1 = 0̄
yes
| ?- prod( s(0), s(s(s(s(0)))), P2)
                        % Find P2:  1̄ × 4̄ = P2
P2 = s(s(s(s(0)))) _
                        % succeeds with P2 = 4̄
yes                     % (requires 1 addition)
| ?- prod( s(s(s(0))), s(s(s(s(0)))), P3)
                        % Find P3:  3̄ × 4̄ = P3
P3 = s(s(s(s(s(s(s(s(s(s(s(s(0))))))))))) _
                        % succeeds with P3 = 12̄
yes                     % (requires 3 additions)
| ?-
```

Division using prod?

- Consider:

prod($\bar{2}$, Q, $\bar{6}$)	$\rightsquigarrow$	prod($\bar{1}$,Q,W),sum(Q,W,$\bar{6}$)
prod($\bar{1}$, Q, W)	$\rightsquigarrow$	prod(0,Q,W'),sum(Q,W',W)
prod(0, Q, W')	$\rightsquigarrow$	{W'/0}
sum(Q,W',W){W'/0}	$\rightsquigarrow$	sum(Q,0,W)

Need to solve (correct) subgoal:

$$\boxed{\texttt{sum(Q,0,W), sum(Q,W,}\bar{6}\texttt{)}}$$

tries	$\texttt{Q} = \bar{0},\ \texttt{W} = \bar{0}$	fails
	$\texttt{Q} = \bar{1},\ \texttt{W} = \bar{1}$	fails
	$\texttt{Q} = \bar{2},\ \texttt{W} = \bar{2}$	fails
	$\texttt{Q} = \bar{3},\ \texttt{W} = \bar{3}$	succeeds

A terrible way to do division!

- Worse yet:

  $\boxed{\texttt{prod(}\bar{2}\texttt{,Q,}\bar{5}\texttt{)}}$ runs forever!

  (can fix by re-ordering 2nd axiom.)

How not to Program Using `prod` for Square Root

- `prod(S,S,4̄9)`

  $\Rightarrow$ { `S=s(X)`, `S=Y`, `Z=4̄9` }

  $\Rightarrow$ { `S/s(X)`, `s(X)/Y`, `Z/4̄9` }

  `prod(X,s(X),W), sum(s(X),W,4̄9)`

- Eventually succeeds with `X=6̄`, `W=4̄2` and so `S=7̄`.

- Atrocious way of finding square roots!

Lessons

- No substitute for algorithm development
- Need to know how axioms can be used
- Should consider failure behavior