Heat equation and image inpainting

1D heat equation:

\[ u_t (x,t) = k \frac{\partial^2 u}{\partial x^2} \quad -\infty < x < \infty \]
\[ 0 < t < \infty \]
\[ u(x,0) = g(x) \]

is called "initial condition"

The solution is given by

\[ u(x,t) = \int \phi(x-y,t)g(y) \, dy \]

\[ \phi(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp \left( - \frac{x^2}{4kt} \right) \]

is called heat kernel

Similar result exists in 2D.
2D heat equation (isotropic):

\[ \frac{\partial u}{\partial t} = K \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \]

2D Laplacian or simply Laplacian

\[ u(x, y, 0) = g(x, y) \]

Initial condition.

\[ u(x, y, t) = \iint \phi(x-x', y-y', t) g(x', y') \, dx' \, dy' \]

This is the solution.

\[ \phi(x, y, t) = \frac{1}{4\pi t} \exp\left(-\frac{x^2+ty^2}{4Kt}\right) \]

2D heat kernel

For image inpainting, we think of the inpainting process as "intensity diffusion" into the damaged areas from the intact portions of the image.
We want to reach the solution of the heat eqn:

\[
\frac{\partial u(x,y,t)}{\partial t} = k \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\]

\[
\rightarrow u(x,y,t) = I(x,y,t) \quad \text{in the good area}
\]

\[
U(x,y,0) = 0 \quad \text{in the bad area}
\]

This is called Dirichlet Boundary Condition.

Let's think of an iterative solution.

Let's also think of a non-iterative solution.
\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

This is known as the Laplace's equation with Dirichlet boundary conditions.

We can solve this equation by

\[ \begin{cases} L u = 0 \\ u(x,y) = I(x,y) \text{ for some pixels} \end{cases} \]