Level Set Methods

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What is Level Set Methods?

- A numerical technique for front propagation
- Level set methods represent a contour / front geometrically
- Consider a function $\phi(x, y)$ over the rectangular image domain; intersection of the $x$-$y$ plane and $\phi$ represents a contour
A plane and a function $\phi(x, y)$ can together represent a front: Eulerian or implicit representation.
Let \((x(t), y(t))\) represent coordinates of a front at time \(t\), then

\[ \varphi(x(t), y(t), t) = 0 \]

Using chain rule of differentiation:

\[ \frac{\partial \varphi}{\partial t} + (\nabla \varphi) \cdot \left( \frac{dx}{dt}, \frac{dy}{dt} \right) = 0 \]

Note that front velocity is speed \((F)\) times the direction of flow (normal to curve):

\[ \left( \frac{dx}{dt}, \frac{dy}{dt} \right) = F \frac{\nabla \varphi}{|\nabla \varphi|} \]

Substituting we have front propagation equation:

\[ \frac{\partial \varphi}{\partial t} + F |\nabla \varphi| = 0 \]
Matlab Demo

• Matlab demo (lev_demo.m)
An Example

• Segmentation video
Derivation of Yezzi et al.’s Speed Function

\[ E(X, Y) = -\frac{1}{2} \int_C (u - v)^2 \, dx \, dy \]

\((X(s), Y(s))\) denotes a closed curve \(s \in [0, L]\), \(L\) is the length of the curve.

\(C\) is the region enclosed by the closed curve.

\(u\) and \(v\) are defined as:

\[ u = \frac{S_u}{A_u}, \quad v = \frac{T - S_u}{hw - A_u} \]

\(T\) is the sum of image intensity values all over the image domain.

\(h\) and \(w\) are image height and width. Note that \(T, h, w\) does not depend on \((X, Y)\).

\(S_u\) and \(A_u\) are defined as:

\[ S_u = \int_C I(x, y) \, dx \, dy, \quad A_u = \int_C \, dx \, dy. \]
Derivation…

We need to compute first variation of the functional $E$:

$$
\delta E = -(u - v)(\delta u - \delta v) = \\
- (u - v) \left( \frac{A_u \delta S_u - S_u \delta A_u}{A_u^2} - \frac{(hw - A_u) \delta S_u + (T - S_u) \delta A_u}{(hw - A_u)^2} \right)
$$

So, we need to compute the first variations of $S_u$ and $A_u$.
Derivation...

At this point we need to invoke Green's theorem:

\[ V = \int_c \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy = \int_0^L \left[ f(X,Y) \dot{X} + g(X,Y) \dot{Y} \right] ds, \]

where \( \dot{X} = \frac{dX}{ds}, \dot{Y} = \frac{dY}{ds} \)

We can now use the rule of calculus of variations to obtain the first variation of \( V \):

\[ \frac{\delta V}{\delta X} = \frac{\partial}{\partial X} \left[ f(X,Y) \dot{X} + g(X,Y) \dot{Y} \right] - \frac{d}{ds} \left\{ \frac{\partial}{\partial X} \left[ f(X,Y) \dot{X} + g(X,Y) \dot{Y} \right] \right\} = \frac{\partial g}{\partial X} \dot{Y} \]

\[ \frac{\delta V}{\delta Y} = \frac{\partial}{\partial Y} \left[ f(X,Y) \dot{X} + g(X,Y) \dot{Y} \right] - \frac{d}{ds} \left\{ \frac{\partial}{\partial Y} \left[ f(X,Y) \dot{X} + g(X,Y) \dot{Y} \right] \right\} = \frac{\partial f}{\partial Y} \dot{X} \]

Combined, we can write: \( \delta V = (\frac{\delta V}{\delta X}, \frac{\delta V}{\delta Y}) = (\frac{\partial g}{\partial X} \dot{Y}, \frac{\partial f}{\partial Y} \dot{X}) \)
Derivation...

Let's turn our attention to $S_u$. Let us define,

\[
g(x, y) = \frac{1}{2} \int_0^x I(t, y)dt
\]

\[
f(x, y) = -\frac{1}{2} \int_0^y I(x, t)dt
\]

Note that using Leibnitz's rule, we can write:

\[
\frac{\partial g}{\partial x} = \frac{1}{2} I(x, y) \quad \text{and} \quad \frac{\partial f}{\partial x} = -\frac{1}{2} I(x, y)
\]

Then,

\[
\int_0^L \left[ f(X, Y) \dot{X} + g(X, Y) \dot{Y} \right] ds = \int_c \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy = \int_c I(x, y) dx dy = S_u
\]

Therefore using the previous slide, we can write:

\[
\delta S_u = \left( \frac{\partial g}{\partial X}, \frac{\partial f}{\partial Y} \right) \dot{X} = \frac{1}{2} I(X, Y)(\dot{Y}, -\dot{X}) = \frac{1}{2} I\dot{N}
\]
Derivation…

Using similar tricks, we can write: \( \delta A_u = \frac{1}{2} \vec{N} \)

\[
\delta E = -(u - v) \left( \frac{A_u \delta S_u - S_u \delta A_u}{A_u^2} - \frac{(hw - A_u) \delta S_u + (T - S_u) \delta A_u}{(hw - A_u)^2} \right)
\]

So,

\[
= -(u - v) \left( \frac{I - u}{A_u} + \frac{I - v}{hw - A_u} \right) \vec{N}
\]

The curve evolution equation is given by gradient descent rule:

\[
\frac{dC}{dt} = -\delta E = (u - v) \left( \frac{I - u}{A_u} + \frac{I - v}{hw - A_u} \right) \vec{N}
\]

Finally, the speed function for level set is given by: \( F = (u - v) \left( \frac{I - u}{A_u} + \frac{I - v}{hw - A_u} \right) \)