Approximation algorithm for minimum λ-edge-connected k-subgraph with metric costs

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August, 2008

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- **Input:** Given a weighted undirected graph G and integer parameters k and λ
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Generalizes many classical problems:

- k-MST $\equiv (k, 1)-$subgraph problem.
  Approximation factor: $\sqrt{k}$ [13], $O(\log^2 k)$ [1], $O(\log n)$ [12], constant [3, 8] and 2 [9].
- min-cost λ-edge-connected spanning graph
  $\equiv (|V(G)|, \lambda)-$subgraph problem

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- For arbitrary λ: as hard as the k-densest subgraph problem [11]; best known approximation factor is $O(n^{\frac{1}{3}-\epsilon})$ for some $\epsilon > 0$.
- Chekuri and Korula [5]: $O(\log^2 n)$-approx for $(k, 2)$-subgraph problem with *node-connectivity constraint.*

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- Note that the constant factor approximation algorithms for k-MST and k-TSP are on graphs with metric cost function.
- The constant in the $O(1)$ term is between 400-500.
- Our algorithm is inspired by the work of Cheriyan and Vetta [4] for subset-node-connectivity problem.

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Our algorithm presents a solution whose cost is bounded within an $O(1)$-factor of these two bounds.

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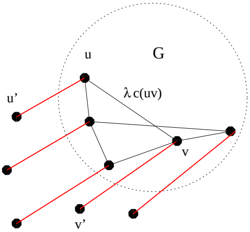
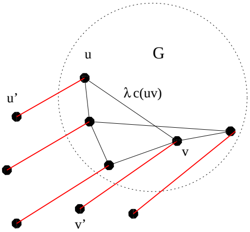
- In Phase 1, we obtain a $(k - \lambda/7, \lambda)$-subgraph, call it H, which has cost $O(\text{OPT})$.
- In Phase 2, we show how to expand H to a (k, λ)-subgraph, while keeping the cost within $O(\text{OPT})$.



To find a low cost $(k - O(\lambda), \lambda)$-subgraph:

- Create a new graph $G'(V \cup V', E')$ from G by creating a new vertex u' for each $u \in G$ and $E' = E \cup \{uu' | u \in V\}$, $c(uu') = s_u$.



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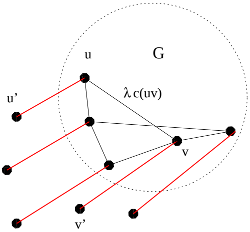

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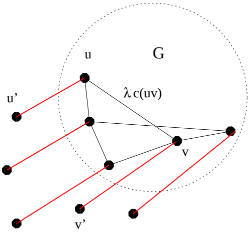

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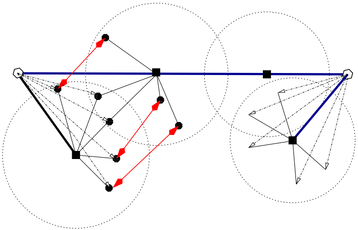
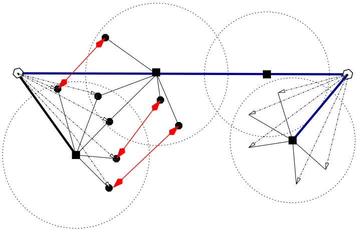
- Using a ρ-approx alg, say ST-alg, for k-Steiner tree, find a k-Steiner tree T' in G' on terminal set V'.
- Note that T' minimizes (approximately) $\sum_{u \in T'} s_u + \lambda \sum_{e \in T'} c_e$, and has at least k nodes of G.



- Using the two lower bounds mentioned, it is easy to show that:
  Claim:$c(T') = 4\rho\mathrm{OPT}$.
  Let $T_0 \subseteq G$ be the tree obtained by deleting dummy vertices of T'.



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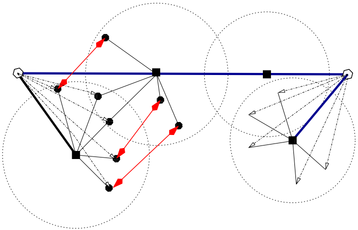

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- For every $(u, v) \in T_0$ put a matching between the vertices in $S_u - S_v$ and $S_v - S_u$ to obtain λ-edge-connectivity.

More details on how to do it...

- Let $v_1, \ldots, v_k$ be an ordering of vertices of T_0 s.t. $s_{v_1} \leq s_{v_2} \leq \ldots s_{v_k}$.

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  - Active vs. Inactive Balls: Every vertex is active unless it is close to an active ball with smaller s_u value.
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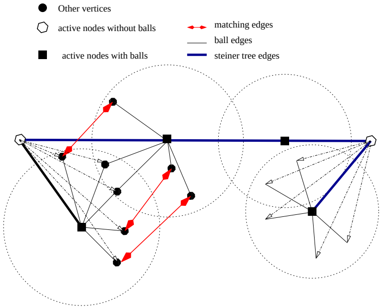
Algorithm for Phase 1



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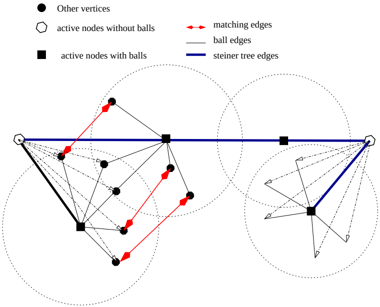

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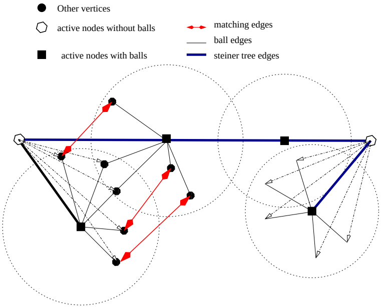

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2. For every $(u, v) \in T_1$ put a matching between the vertices in $S_u - S_v$ and $S_v - S_u$ to obtain λ-edge-connectivity.
3. It can be shown that the resulting graph H is λ-edge-connected, has at least $k - \lambda/7$ nodes and has cost at most $28\rho\text{OPT}$.

Phase 2:

How to expand H to have size k.

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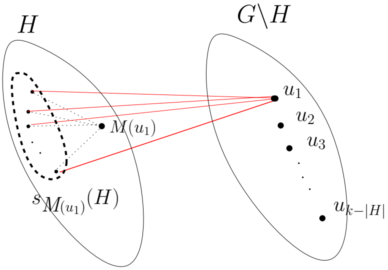
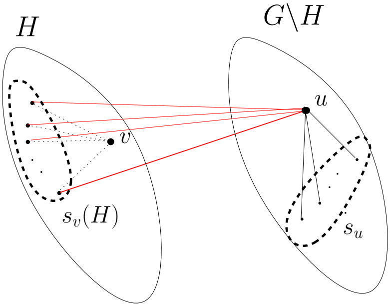


- connect each u_i to $S_{M(u_i)}$ to obtain λ-edge-connectivity. Total cost added: $\lambda c(M) + 2c(H)$
- Show if $|G^* \backslash H| \leq \lambda/3$, then this can be done with $c(M) \leq \frac{6\text{OPT}}{\lambda}$

Case 2

If there is a vertex $u \in G \backslash H$ s.t. $s_u + d(u, H)$ is small and S_u contains at least $\lambda/7$ vertices in $G \backslash H$ then we can augment H with small cost.



- It can be shown that if $|G^* \backslash H| > \lambda/3$, i.e. Case 1 does not happen, then this happens
- The cost of augmenting H in this case is $\leq 12\text{OPT} + 3c(H)$.

Conclusion

- So we can extend H to a (k, λ)-subgraph by spending a total of at most $12\text{OPT} + 3c(H)$.
- Recalling that $c(H) = O(\text{OPT})$, the total approximation ratio is $18 + 108\rho$ with $\rho \leq 4$ being the ratio for k-Steiner tree.
- Getting a small constant factor approximation seems challenging, for general values of λ.
- For general cost functions, even for the special case of $\lambda = 3$, there is no known non-trivial approximation algorithm or lower bound.

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