Approximability of the Unique Coverage Problem

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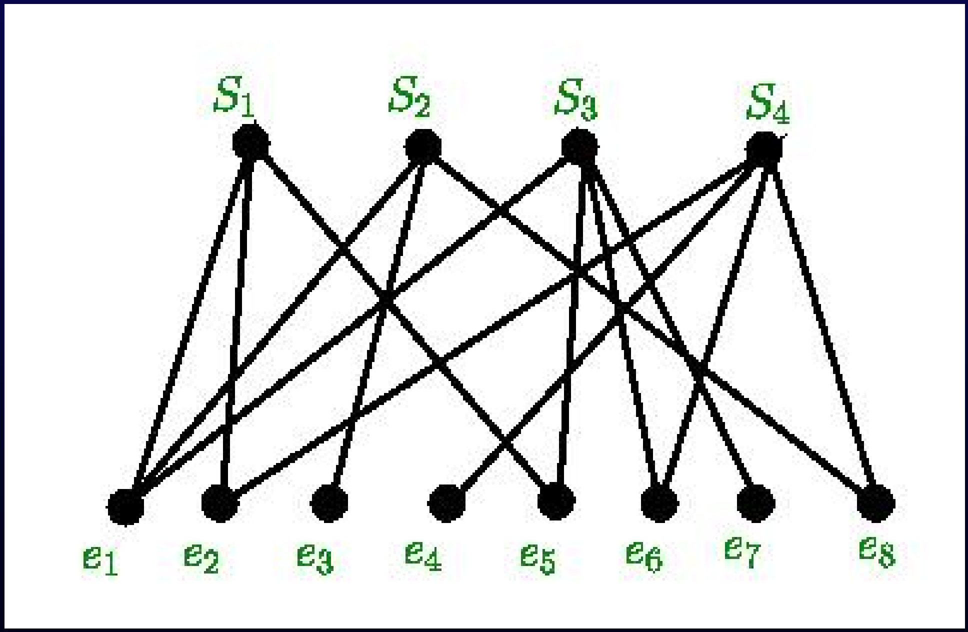

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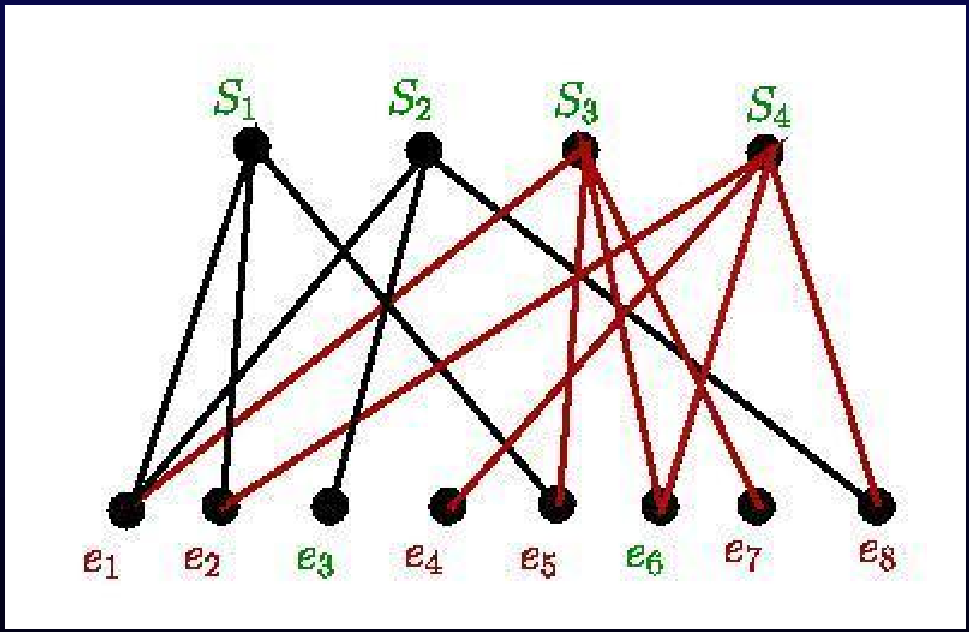

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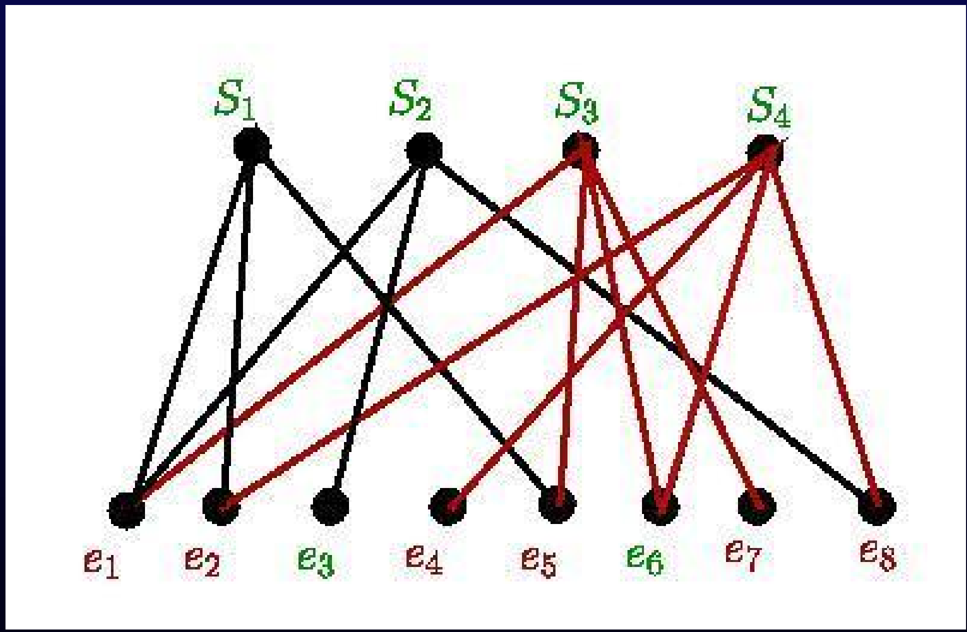
maximum unique coverage size = 6



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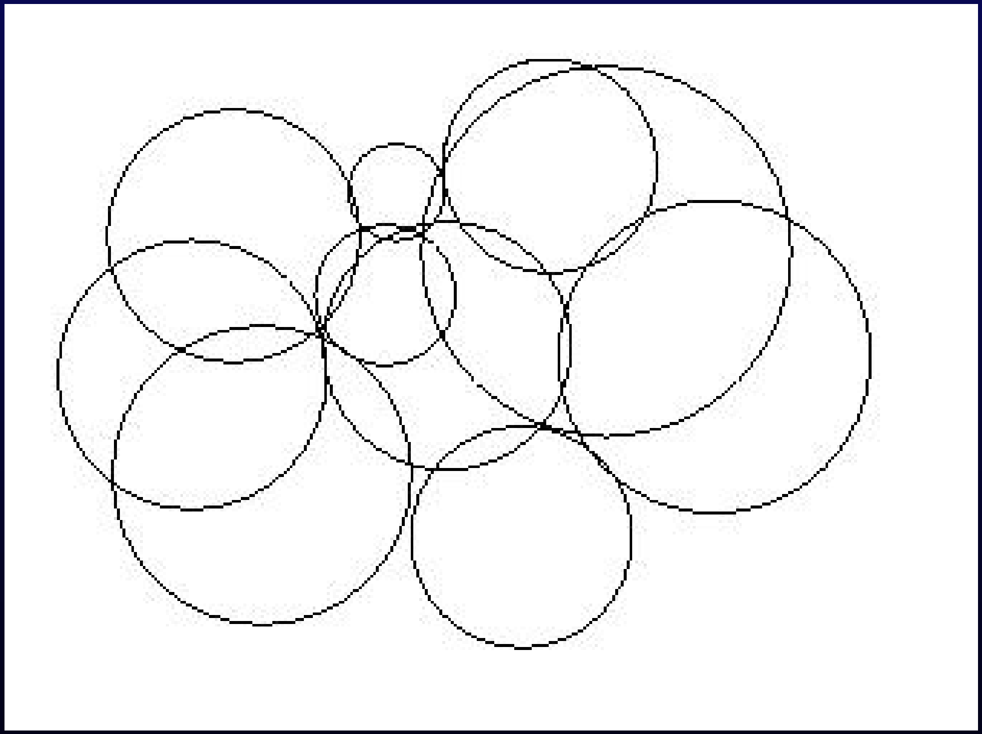

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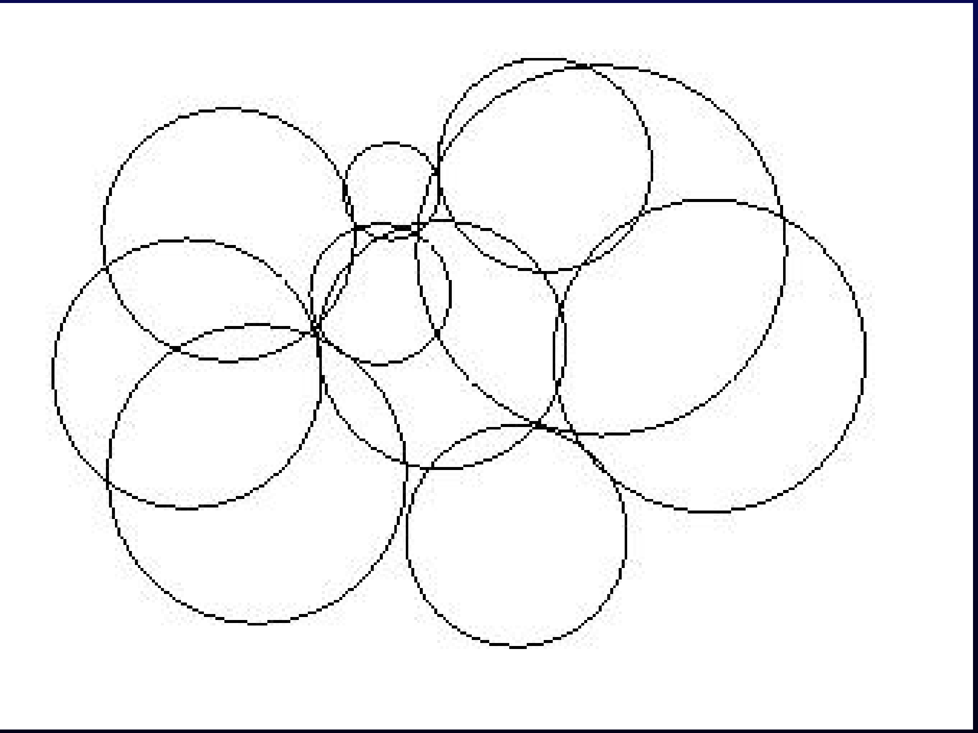

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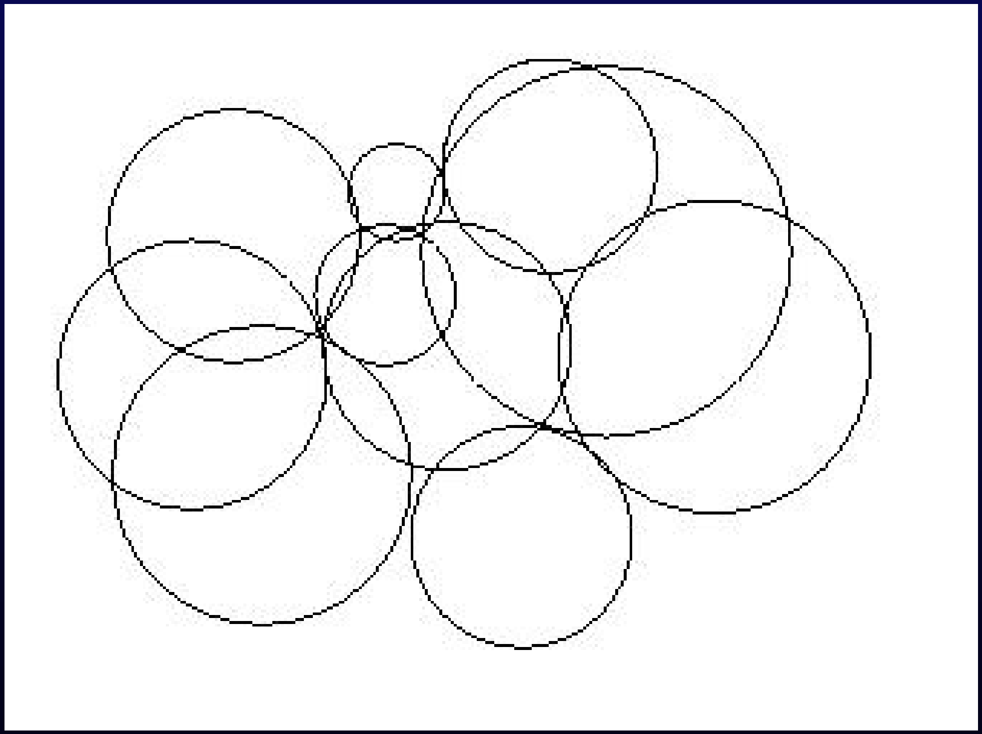

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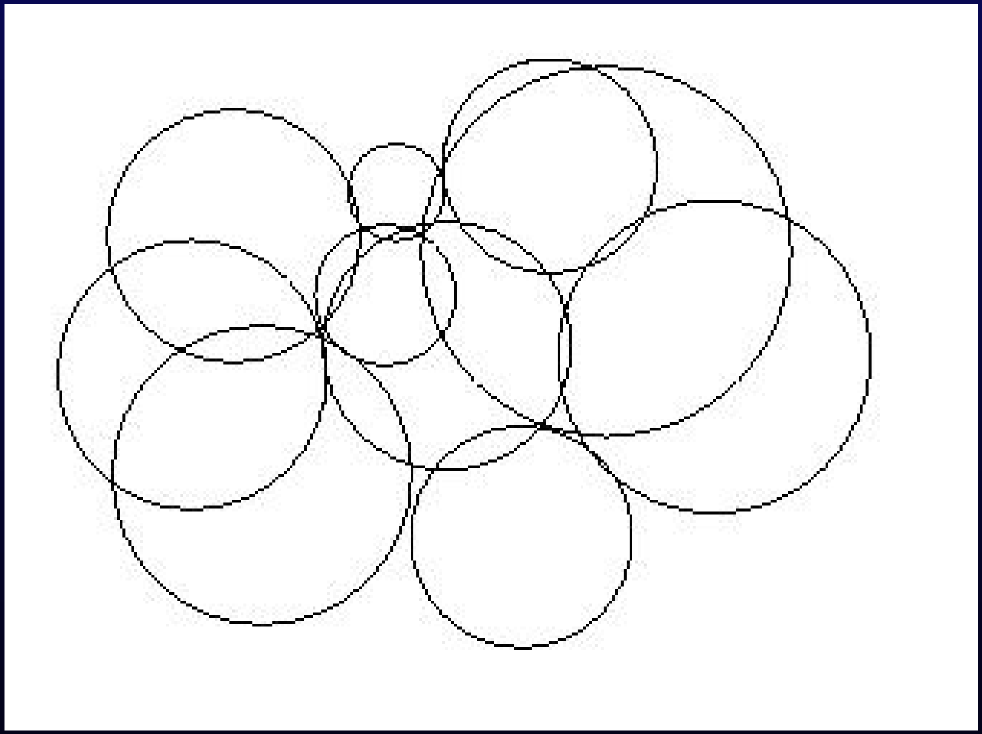

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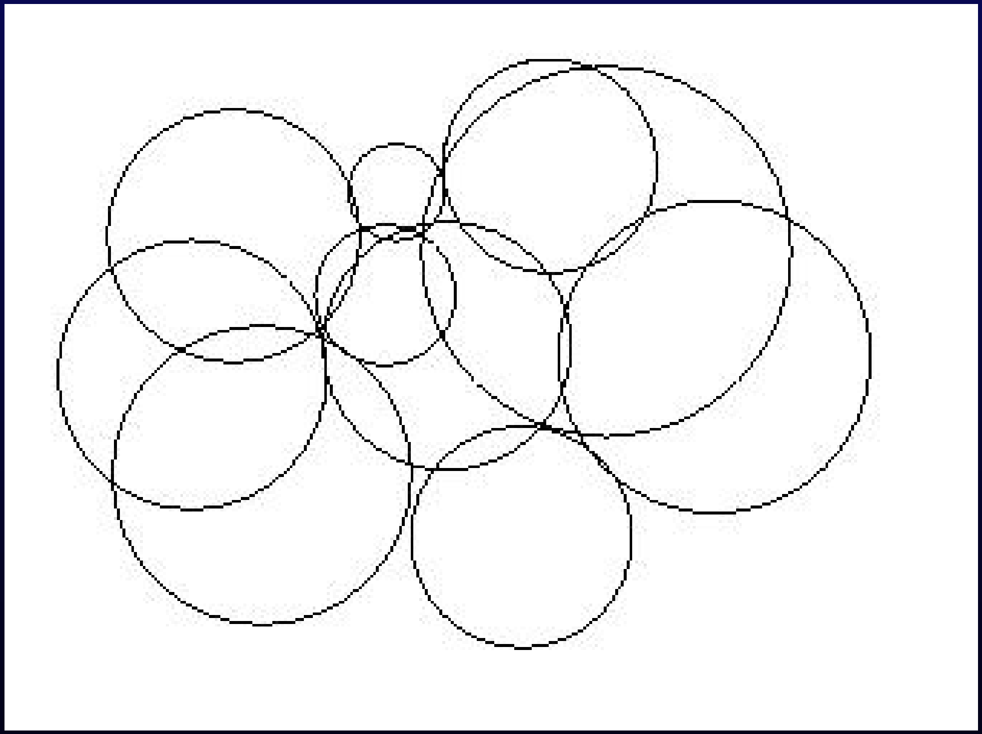

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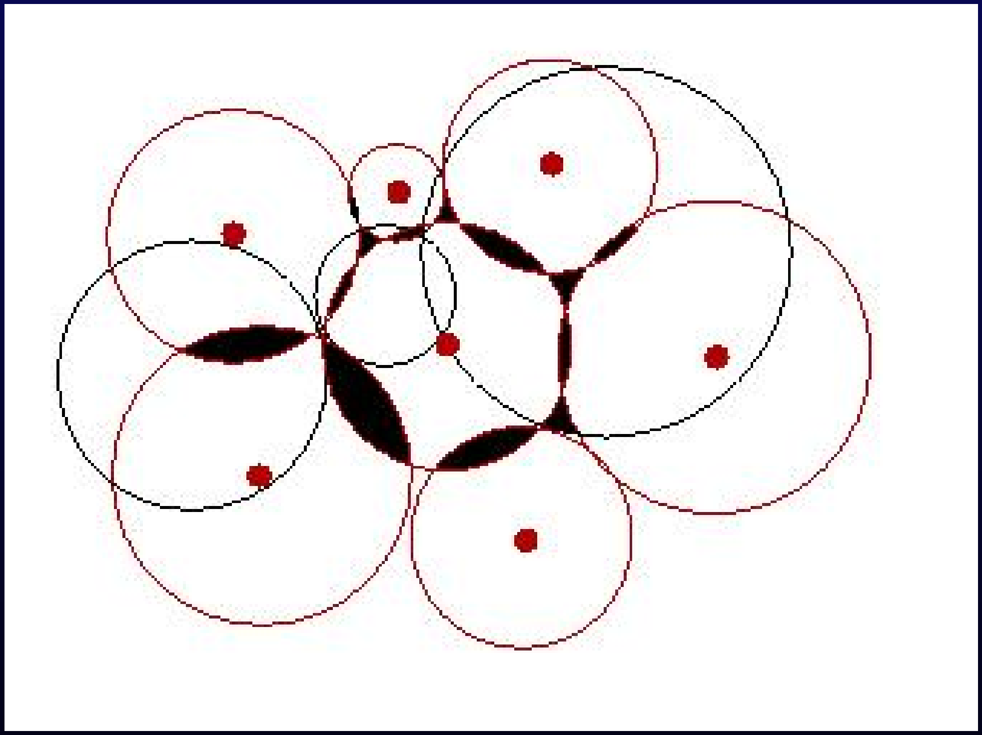

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- maximize total (average) service quality with budget B

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This case is APX-hard and has an $O(\log n + \log m)$-approx [GHKKKM'05]

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Corollary: our hardness results for U.C. imply the same hardness of approximation for envy-free pricing.

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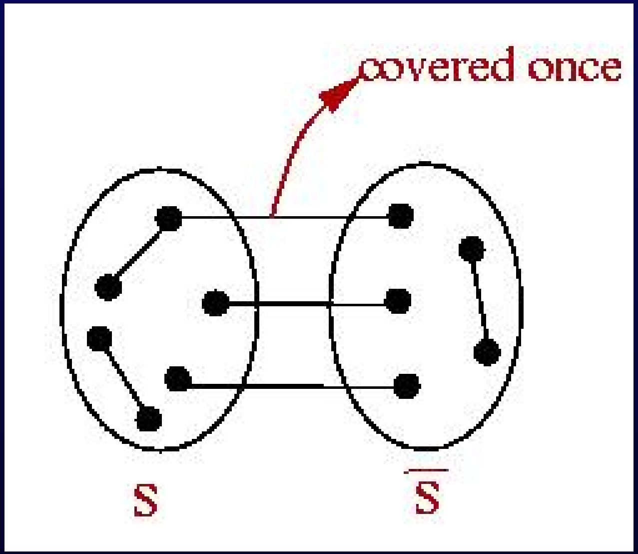

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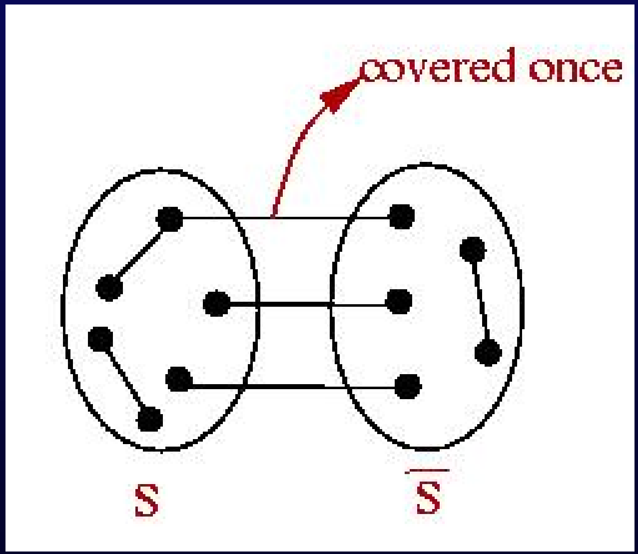
- So U.C. is at least as hard as Max-cut.



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Set Cover: universe $U = \{e_1, \ldots, e_n\}$ and subsets $\mathcal{S} = \{S_1, \ldots, S_m\}$, $S_i \subseteq U$

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Unlike set cover and maximum coverage, the greedy doesn't seem to work for U.C. (all immediate algorithms have ratio $\Omega(n)$).

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Hardness: Elkin and Kortsarz prove (multiplicative) $\Omega(\log n)$ and additive $\Omega(\log^2 n)$ hardness, assuming $NP \not\subseteq DTIME(n^{O(\lg \lg n)})$

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Remark: the known algorithms for Radio broadcast implicitly imply an $O(\log n)$-approx for the (simple) U.C.

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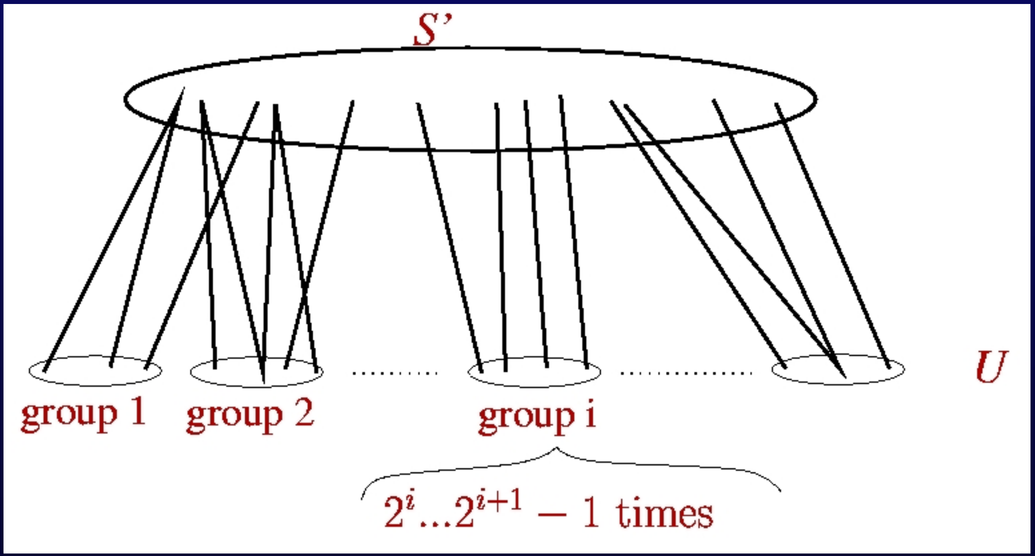

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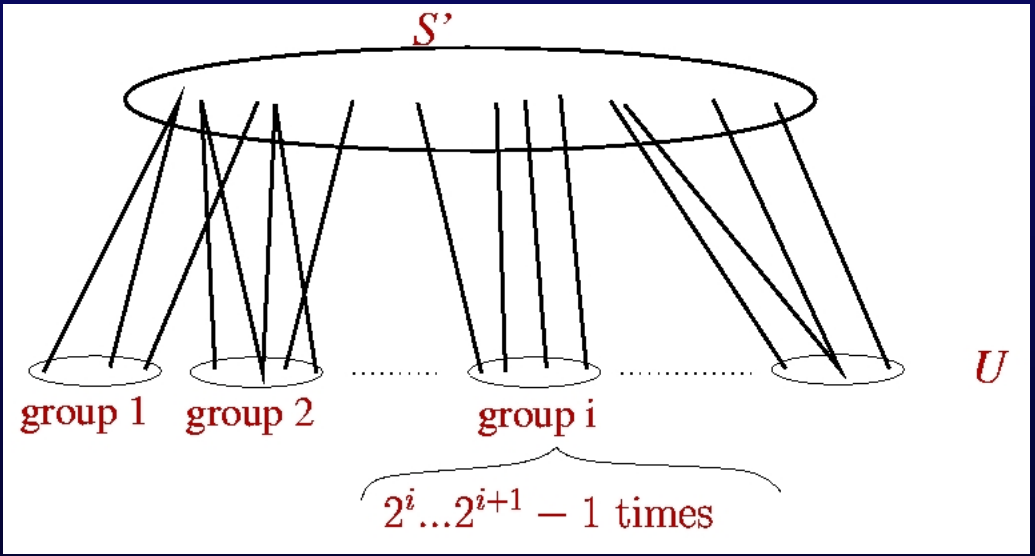

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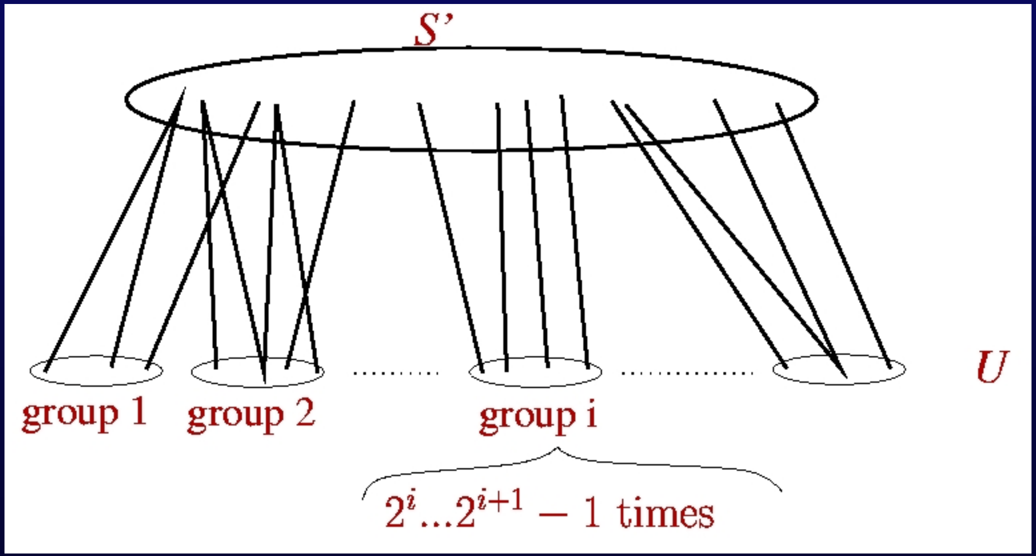

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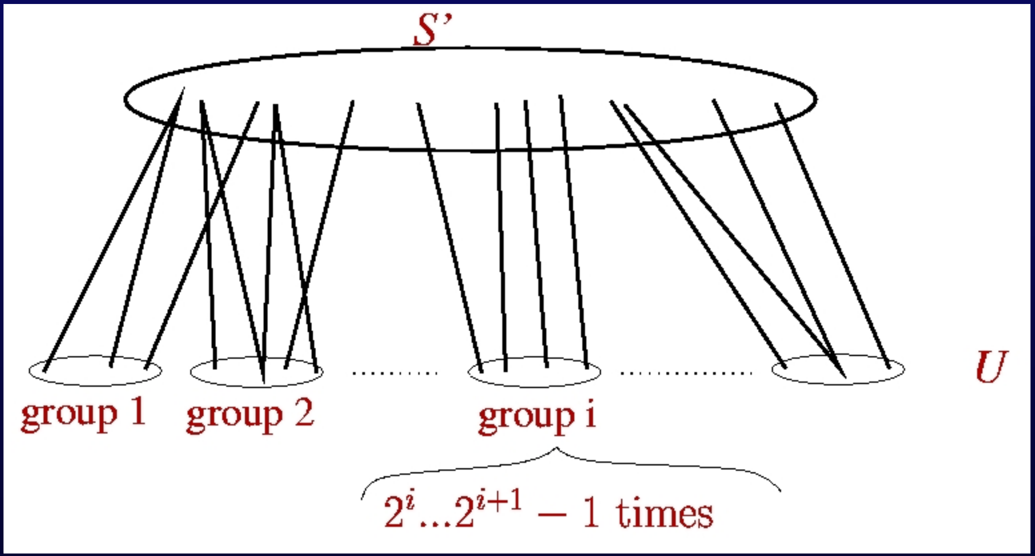

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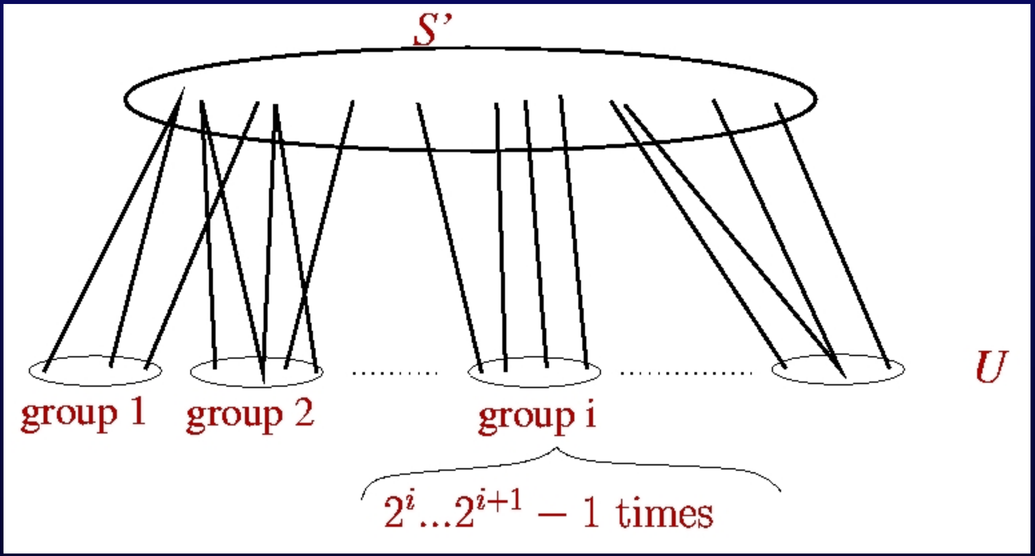

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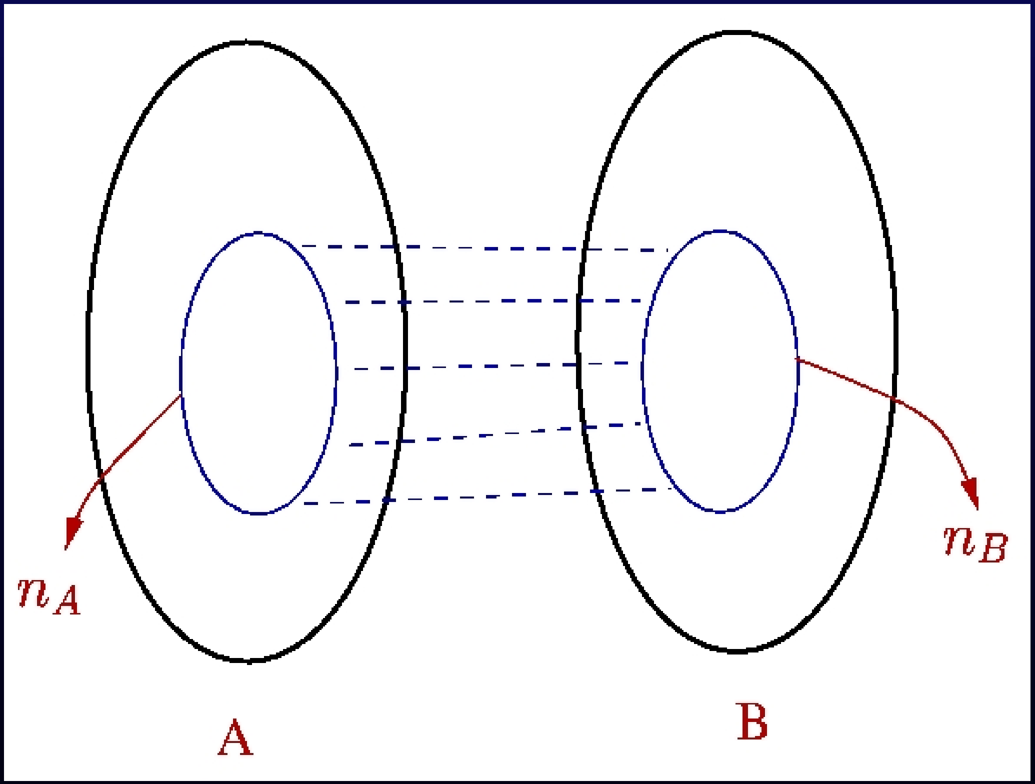

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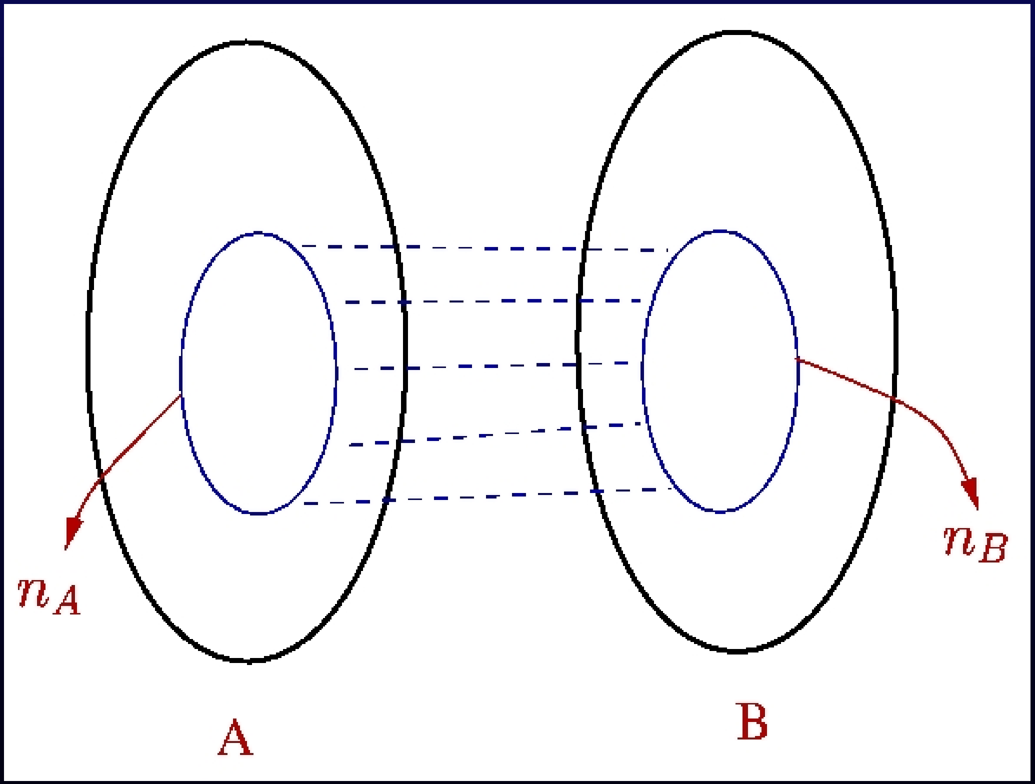

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This question, called *Balanced Bip. Ind. Set* turns out to be difficult.

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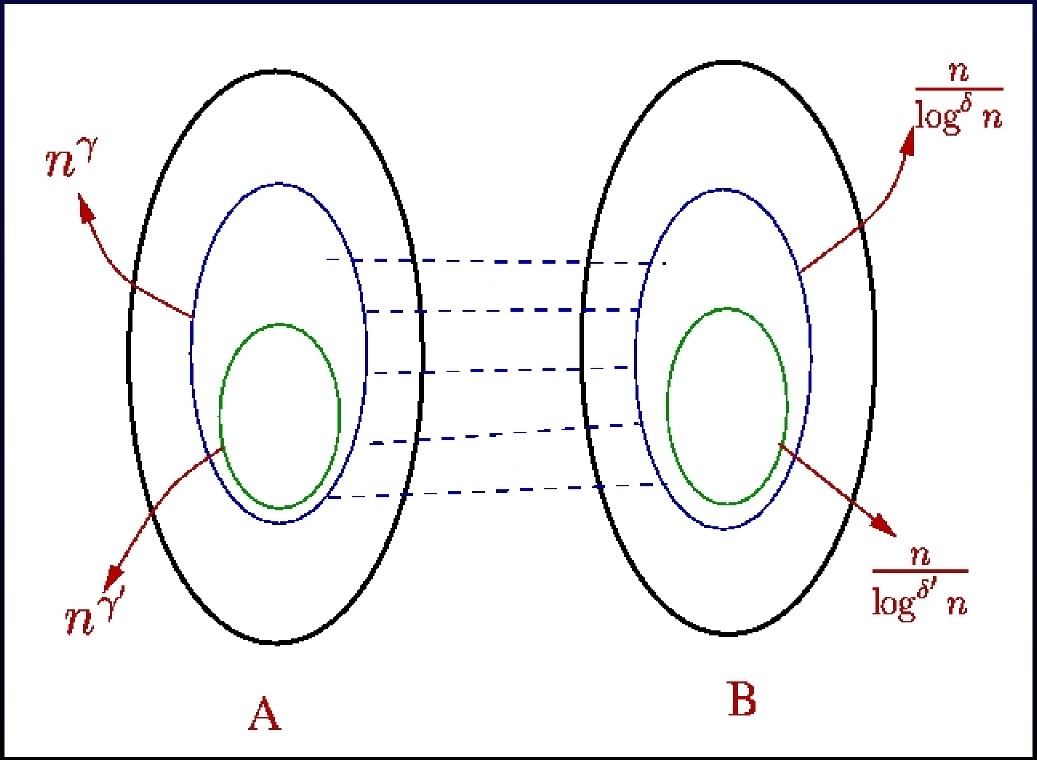

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Corollary: Assuming that $BBIS(\gamma, \gamma', \delta, \delta')$ is "hard" then U.C. has a hardness of factor $\Omega(\log^{\delta' - \delta} n)$.

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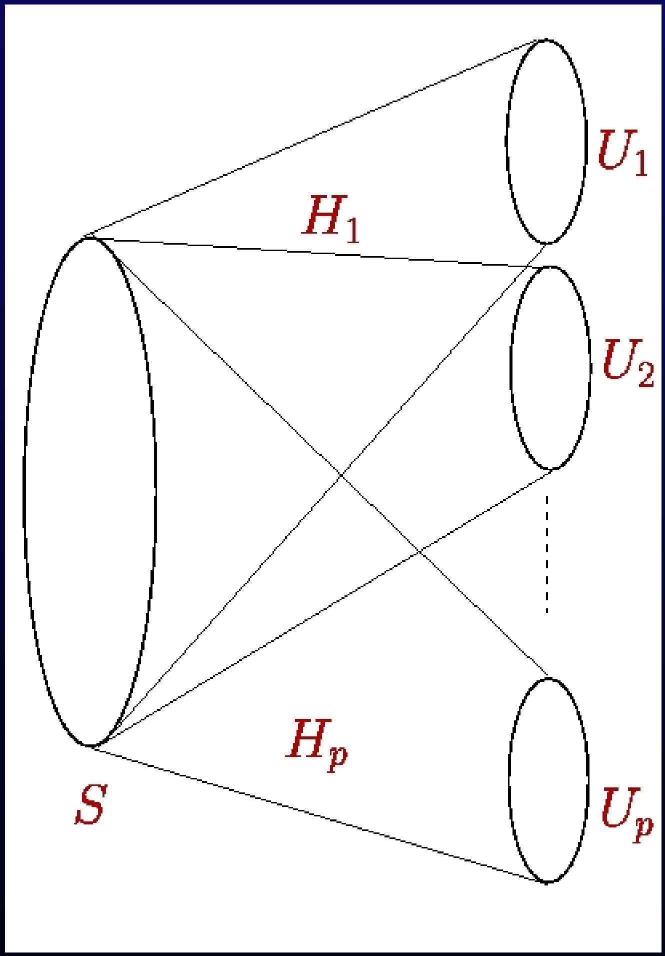

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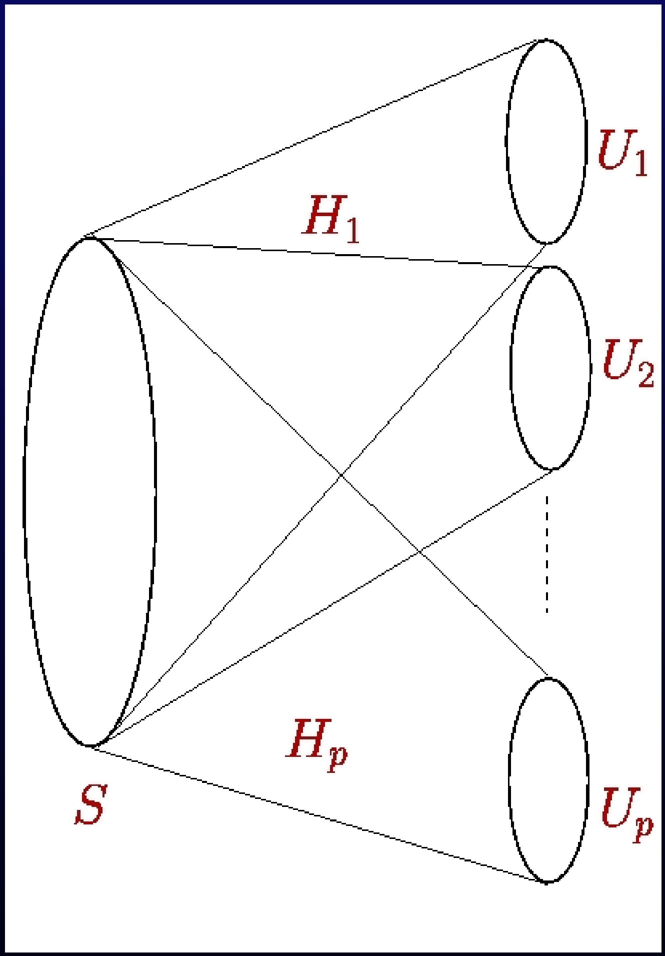

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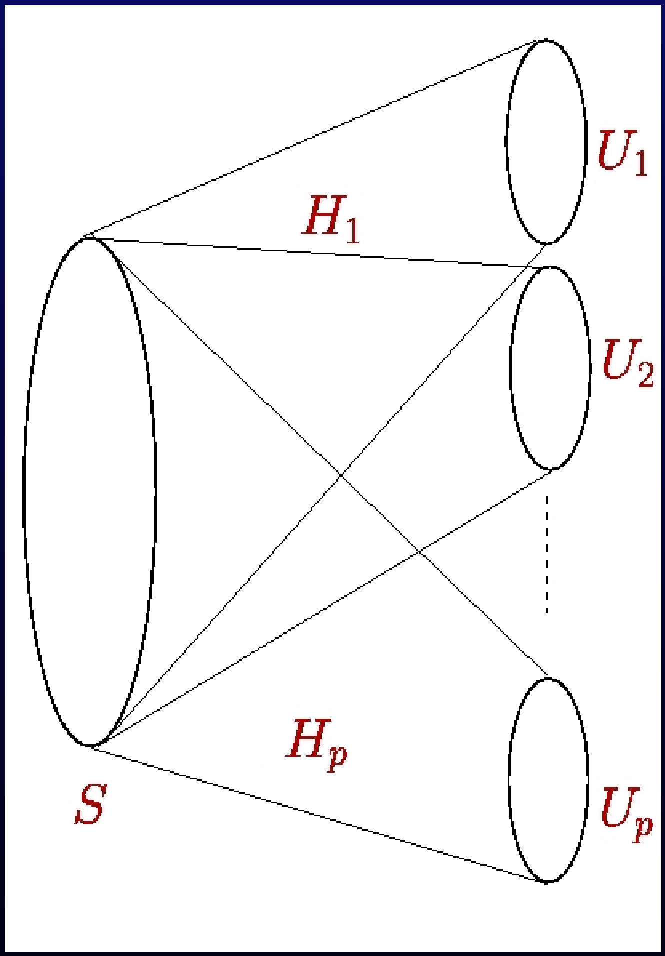

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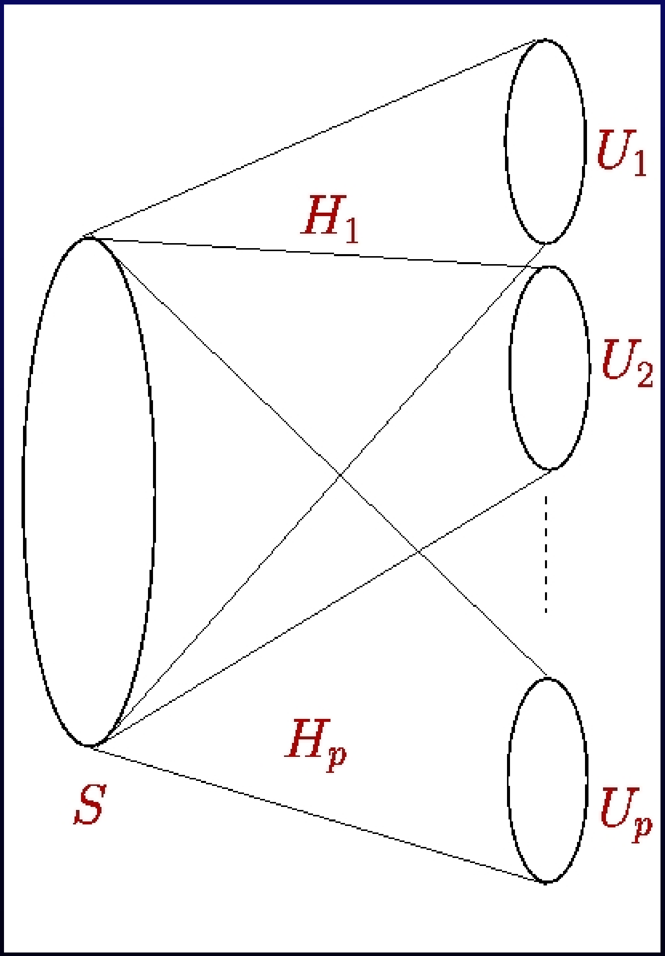

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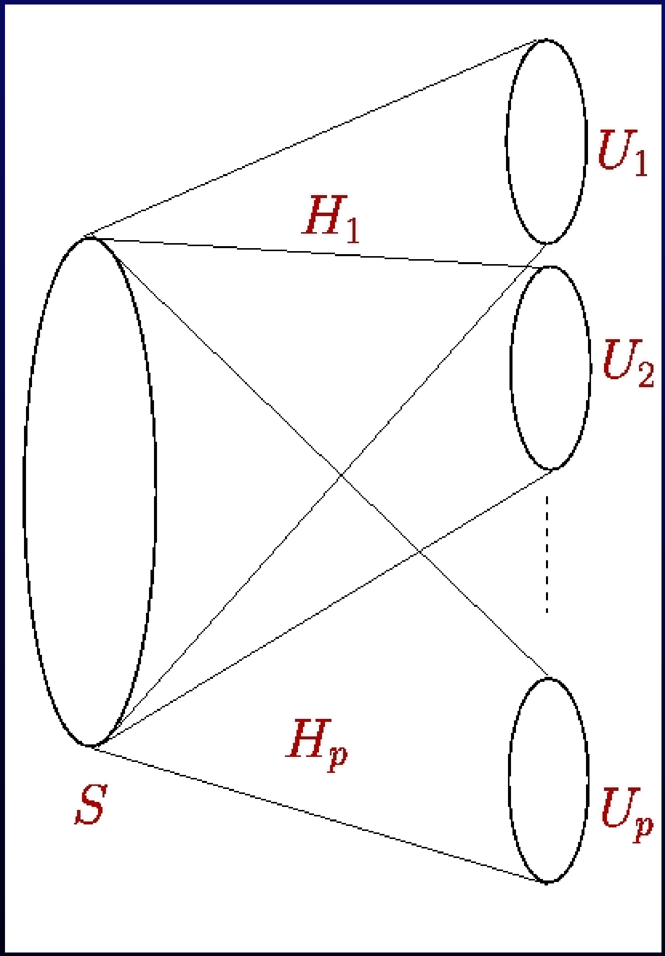

- We will add another set of random edges to each H_i

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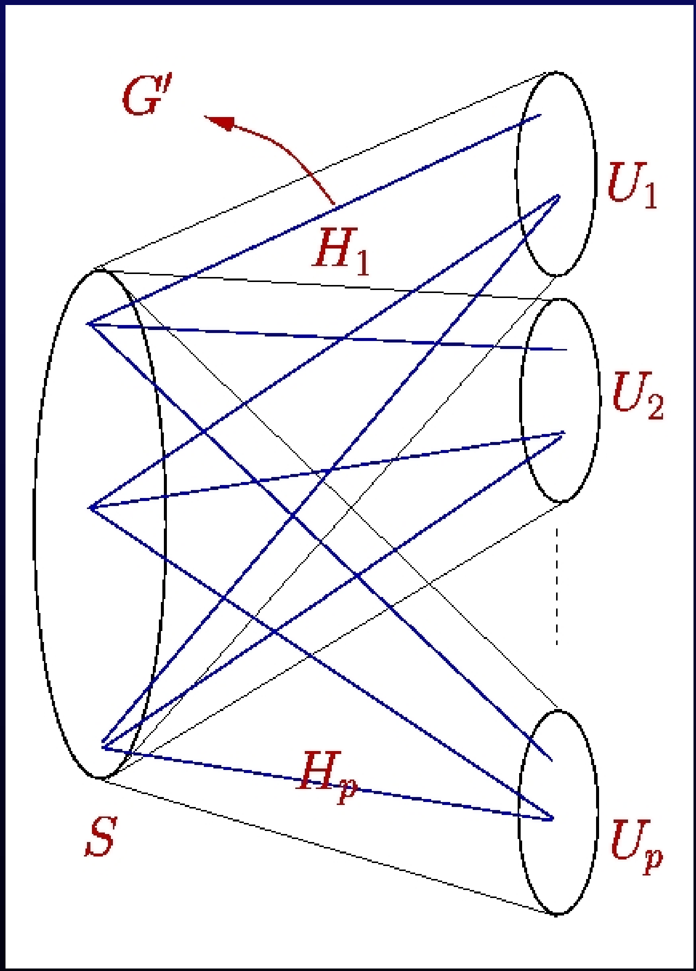

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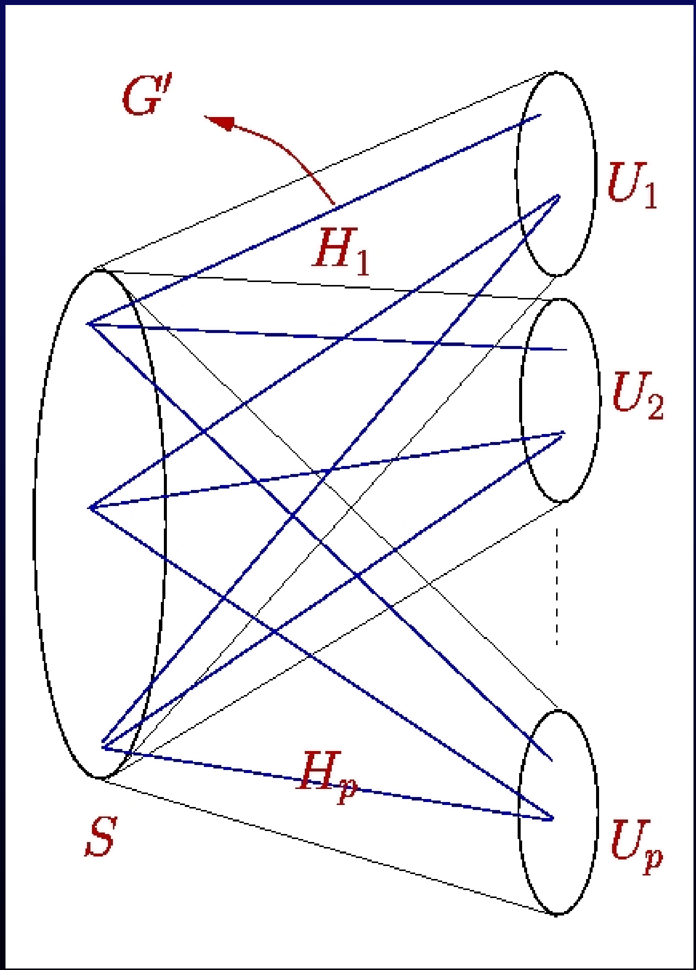

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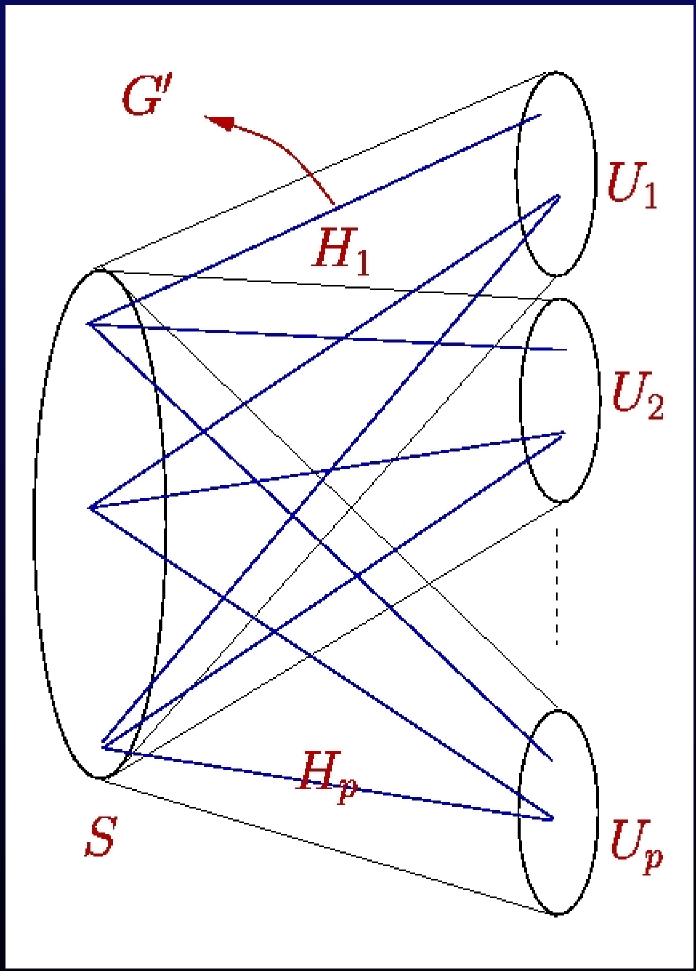

This completes the construction of H from graph G.

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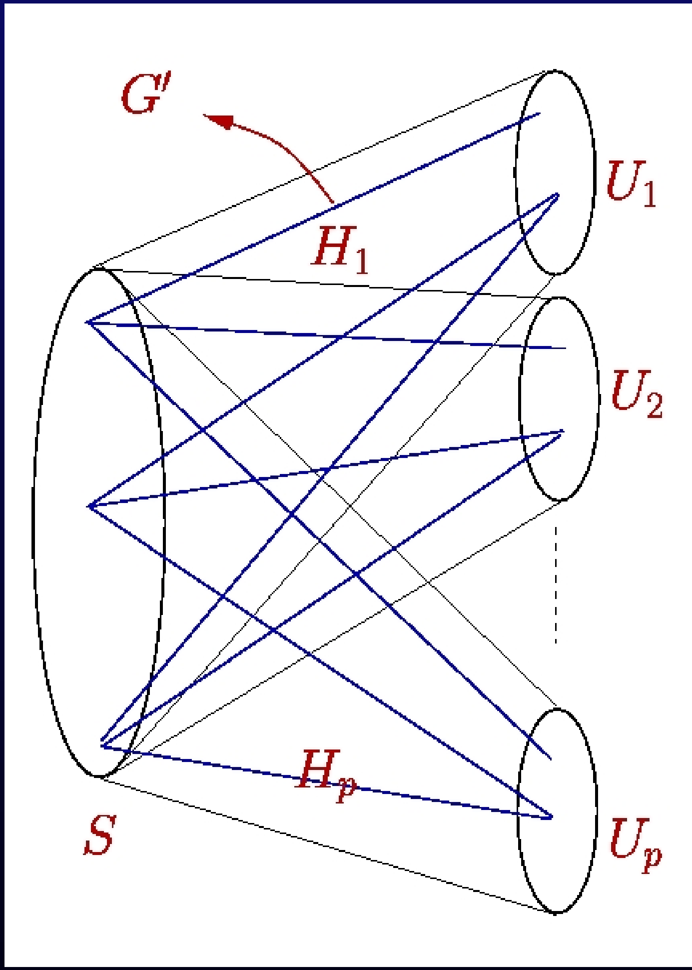
We will show (details to follow):



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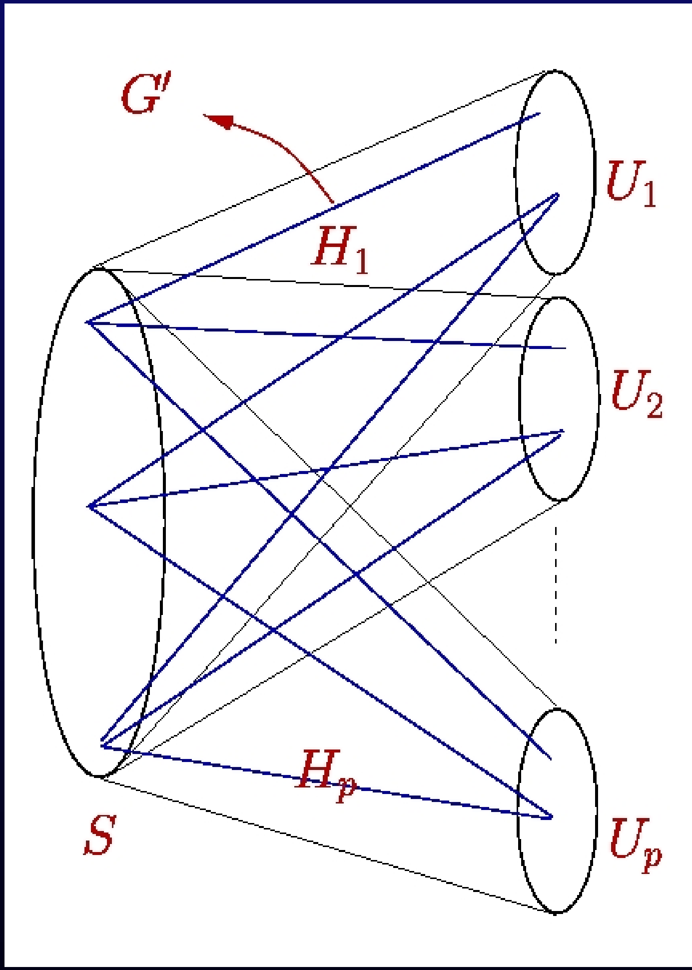

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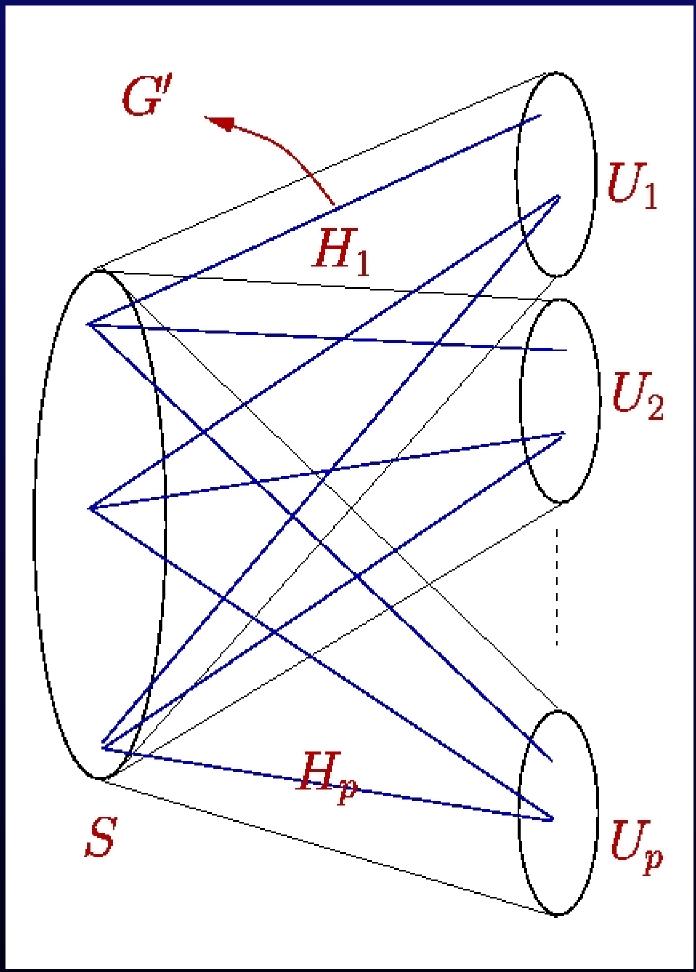

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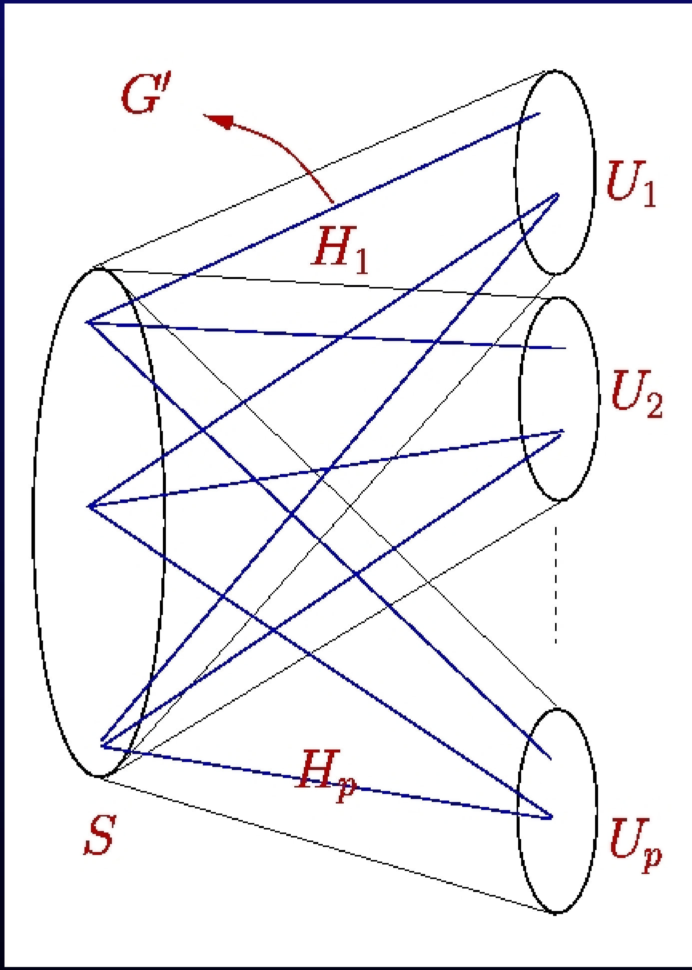

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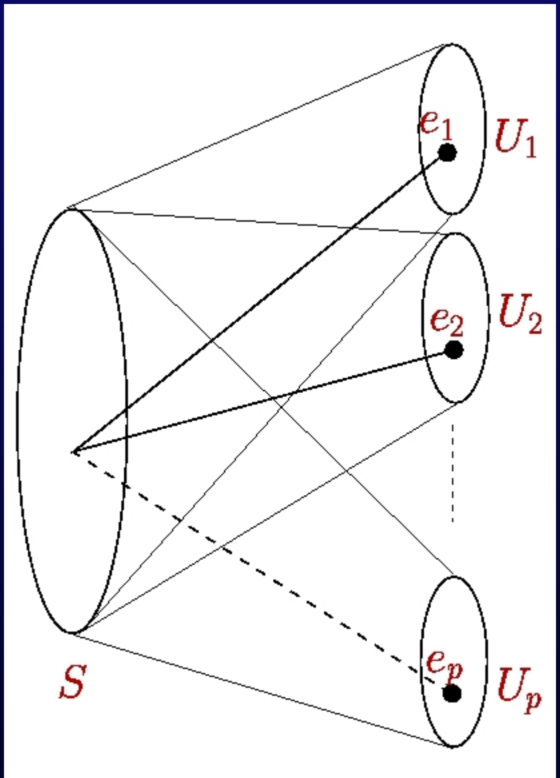

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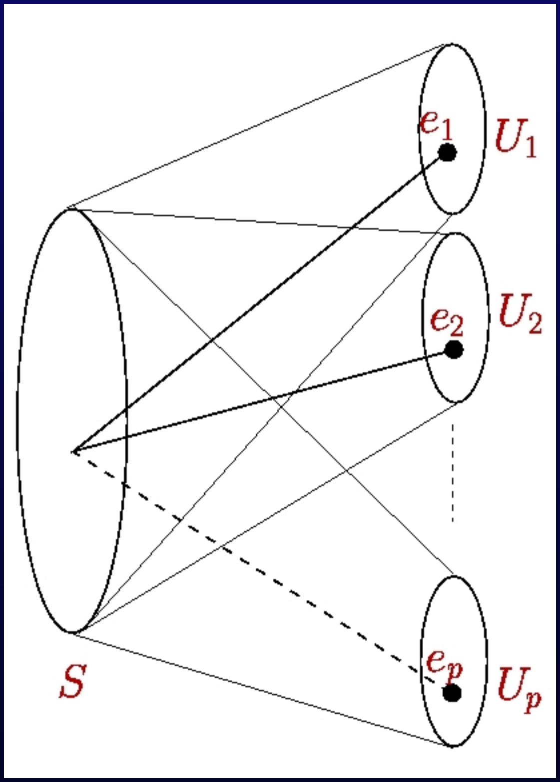

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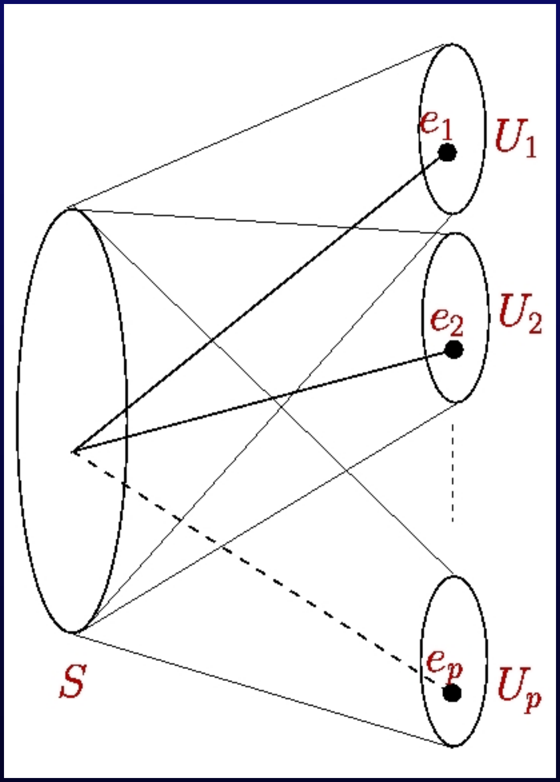

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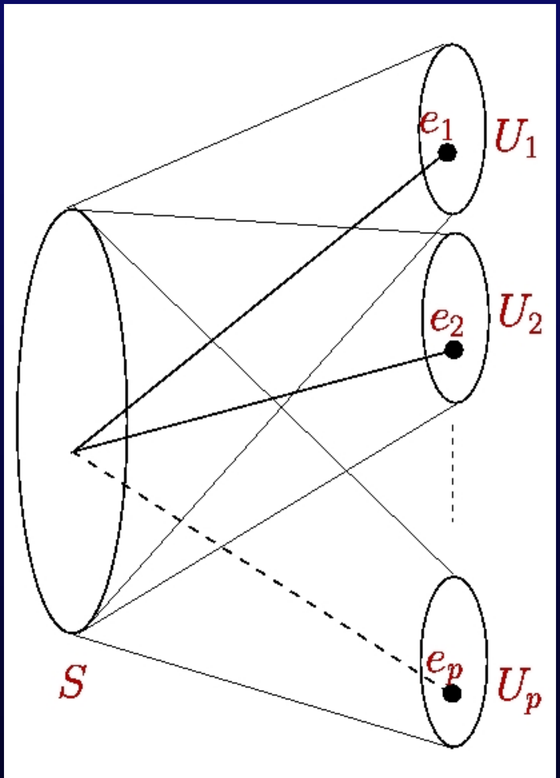

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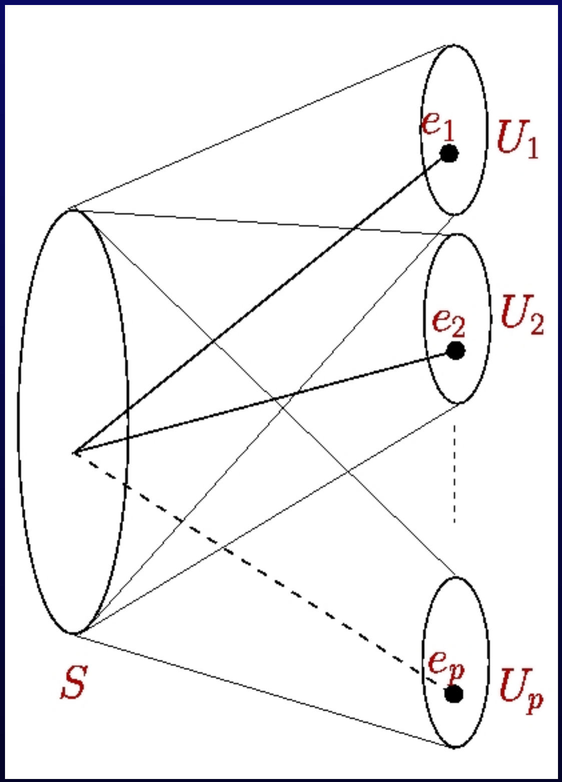

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  $X = \sum_{b \in B} X_b$ number of vertices u.c. by type 2 edges

$$\mathrm{E}[X] = n \sum_{i=1}^{p} \frac{i}{2^i} \in O(n)$$

Details of the proof (cont'd)

Lemma 2: If G has a $(n^\gamma, \frac{n}{\log^\delta n})$-BIS (Yes case) then type 1 edges (those coming from G') uniquely cover $\Omega(\frac{n}{\log^\delta n})$ vertices in each U_i (a total of $\Omega(n \log^{1-\delta} n)$).

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Proof:

- Assume $A^* \cup B^*$ ($A^* \subseteq A$, $B^* \subseteq B$) is a $(n^\gamma, \frac{n}{\log^\delta n})$-BIS; consider corresponding sets A', B' in G' and S^*, U_i^* in H_i

Details of the proof (cont'd)

Lemma 2: If G has a $(n^\gamma, \frac{n}{\log^\delta n})$-BIS (Yes case) then type 1 edges (those coming from G') uniquely cover $\Omega(\frac{n}{\log^\delta n})$ vertices in each U_i (a total of $\Omega(n \log^{1-\delta} n)$).

Proof:

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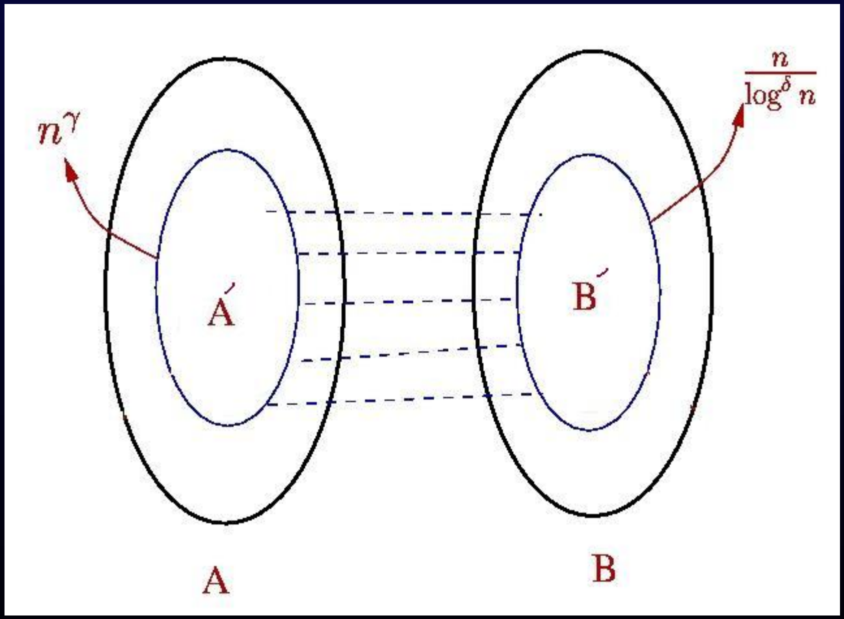

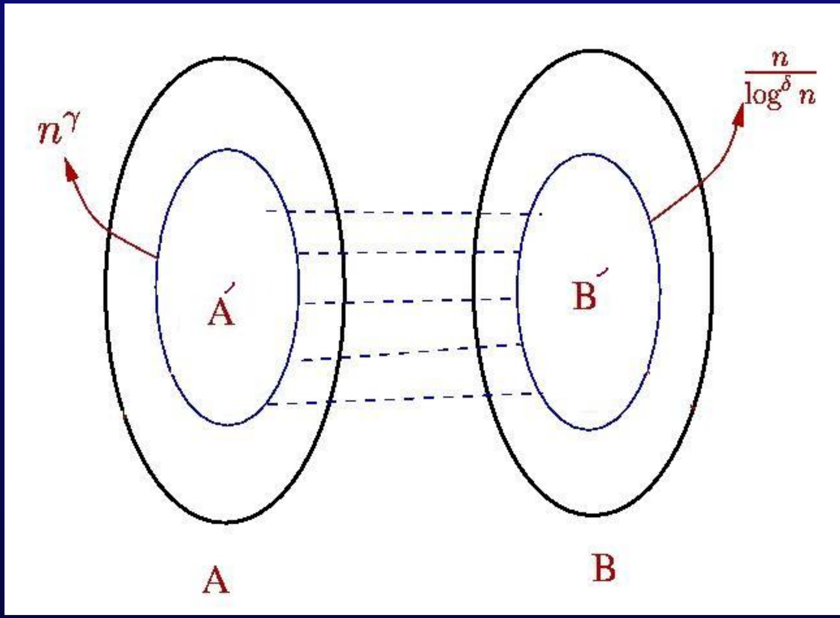

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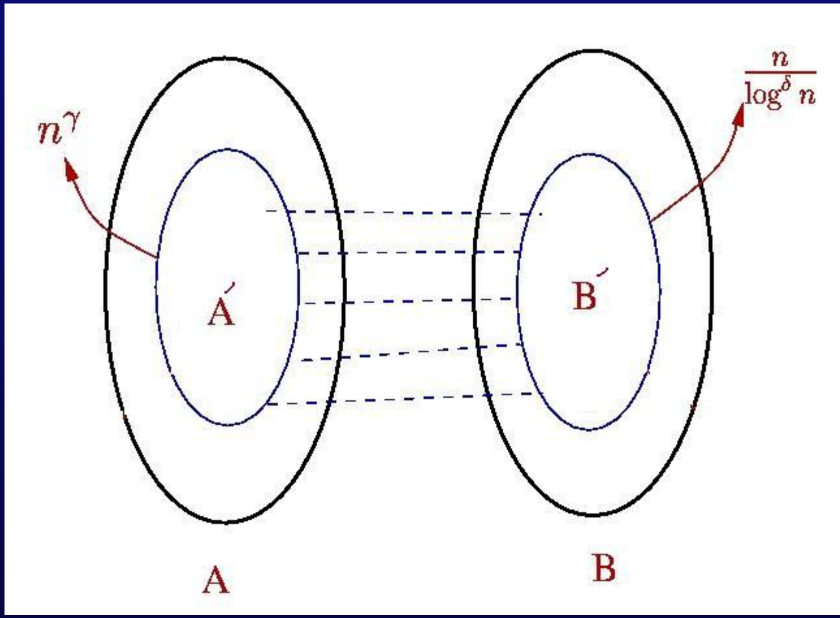

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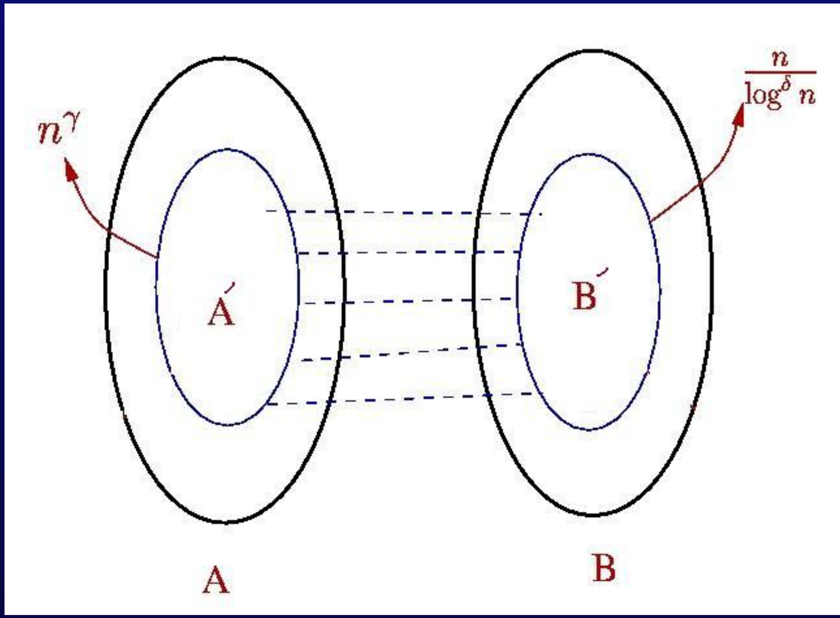

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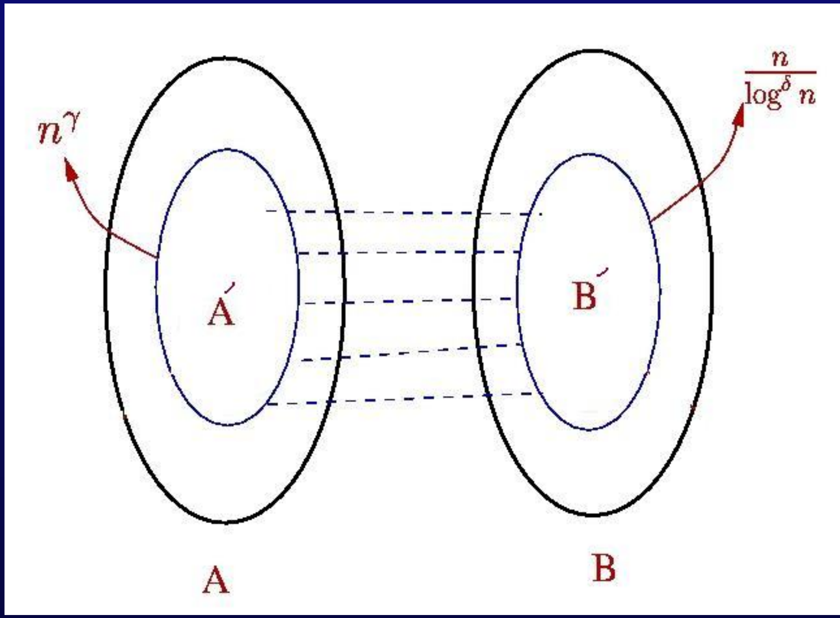

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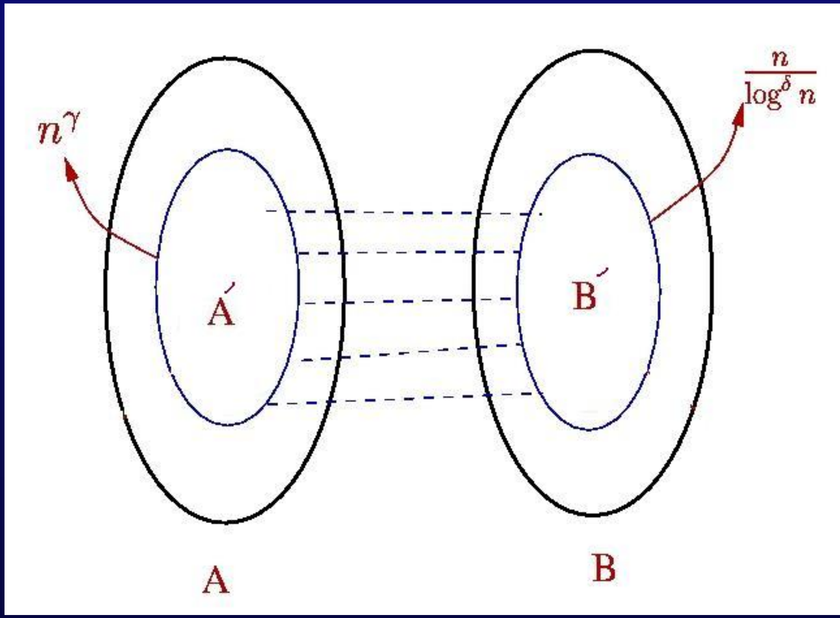

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$$\mathrm{E}[X] = \sum_{b \in B'} \Pr[X_b = 1] = |B'| \binom{|A'|}{1} \frac{1}{n^\gamma} \left(1 - \frac{1}{n^\gamma}\right)^{|A'|-1} \in \Omega\left(\frac{n}{\log^\delta n}\right)$$

Details of the proof (cont'd)

Lemma 3: If G has no $(n^{\gamma'}, \frac{n}{\log^{\delta'} n})$-BIS then w.h.p. every u.c. solution of H has size $O(n \log^{1-\delta'} n)$.

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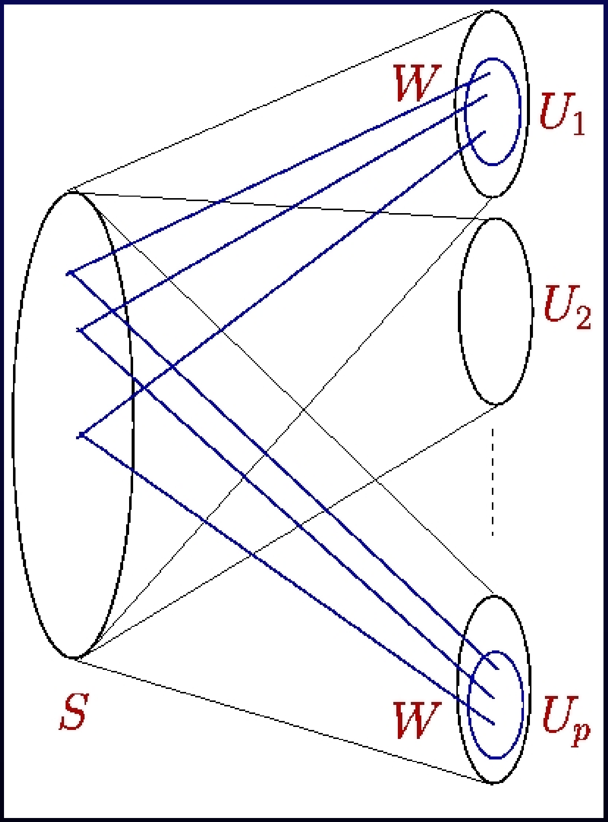

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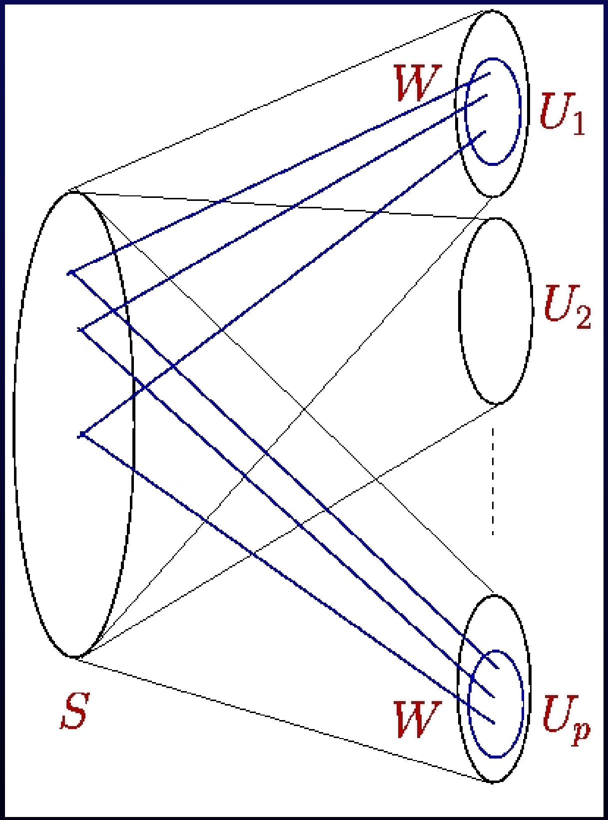

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But we need much larger gap (here it is only a constant!)

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- For suitable k_A, k_B we can show:

Theorem: Unless $NP \subseteq BPTIME(2^{n^\epsilon})$ it is hard to distinguish between:

- ⋆ G^{k_A,k_B} has a $(n^\gamma, \frac{n}{\log^\delta n})$-BIS
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What next (Open problems)

The hardness result is not matching the approximation algorithm ratio ($O(\log n)$ v.s $\Omega(\log^{\delta} n)$) and it requires relatively strong assumption (i.e. $NP \not\subseteq 2^{n^{\epsilon}}$).

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This would imply an $\Omega(\log n)$-hardness for U.C.

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- Question: Can we prove an $\Omega(\log^2 n)$-hardness?
- Proposition: An $\Omega(\log^{1+\delta} n)$-hardness for Radio Broadcast implies an $\Omega(\log^{\delta} n)$-hardness for U.C. (easy!).