Packing Steiner Trees

Mohammad R. Salavatipour

from two joint works with

K. Jain (Microsoft) and M. Mahdian (MIT)

and

J. Cheriyan (Waterloo)

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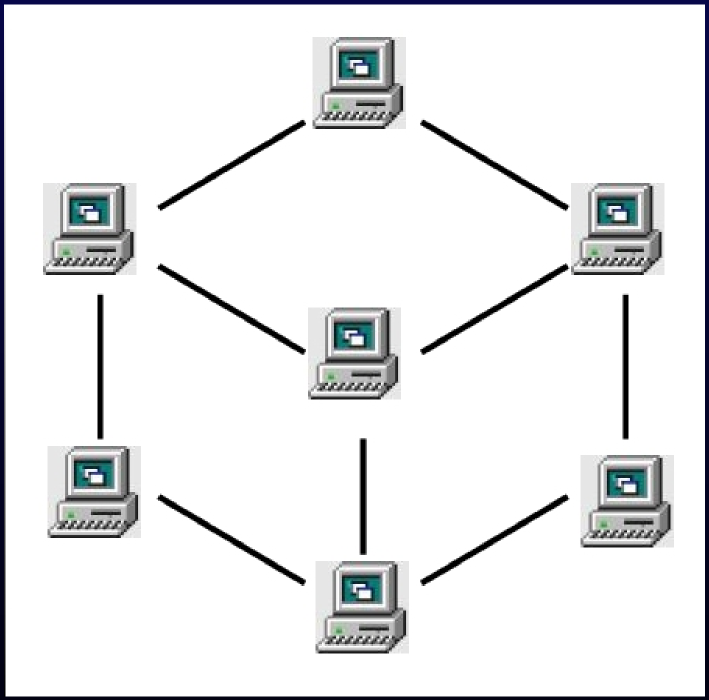

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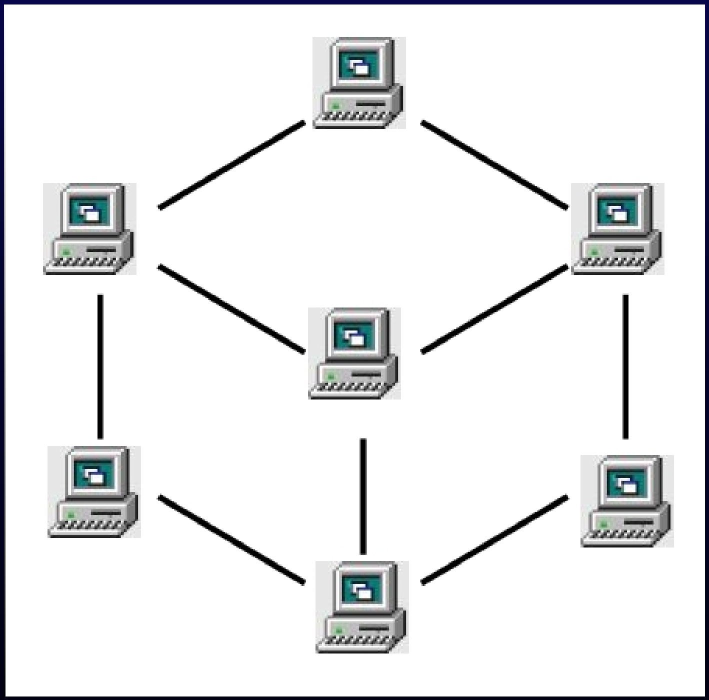

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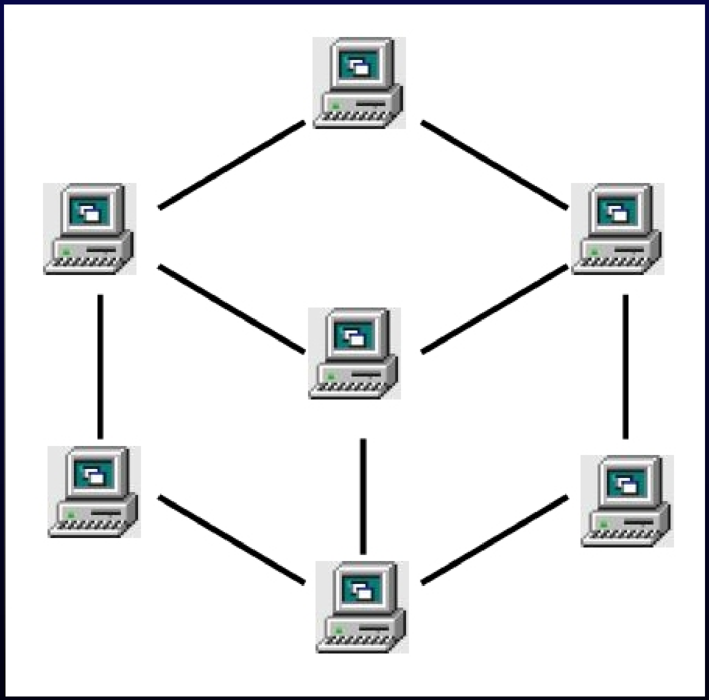

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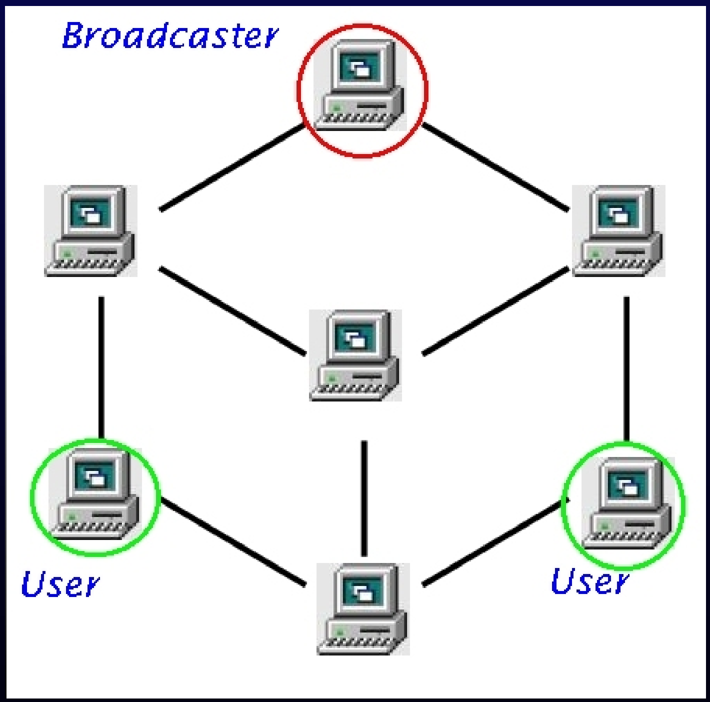

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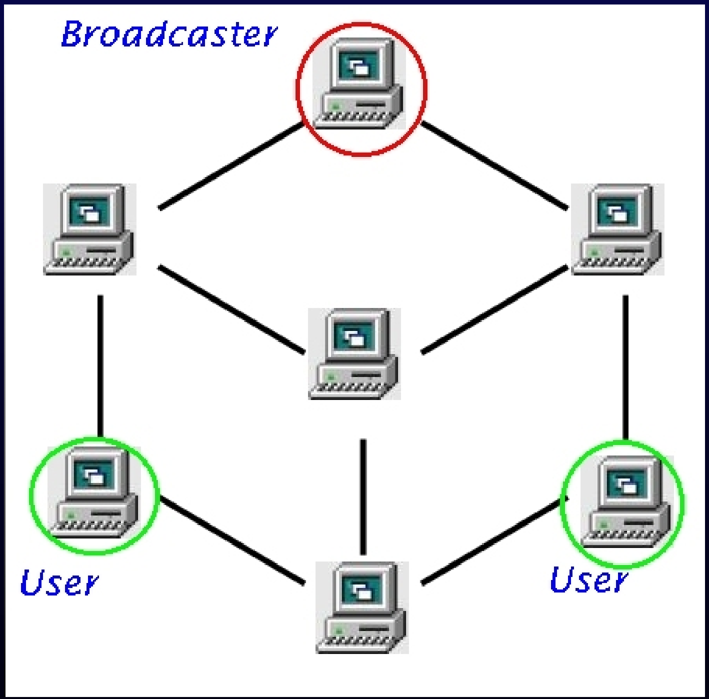

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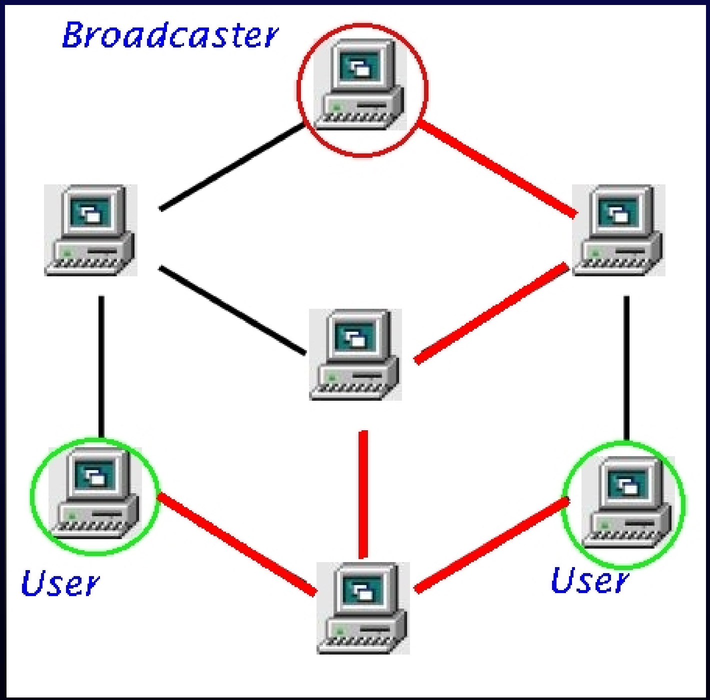

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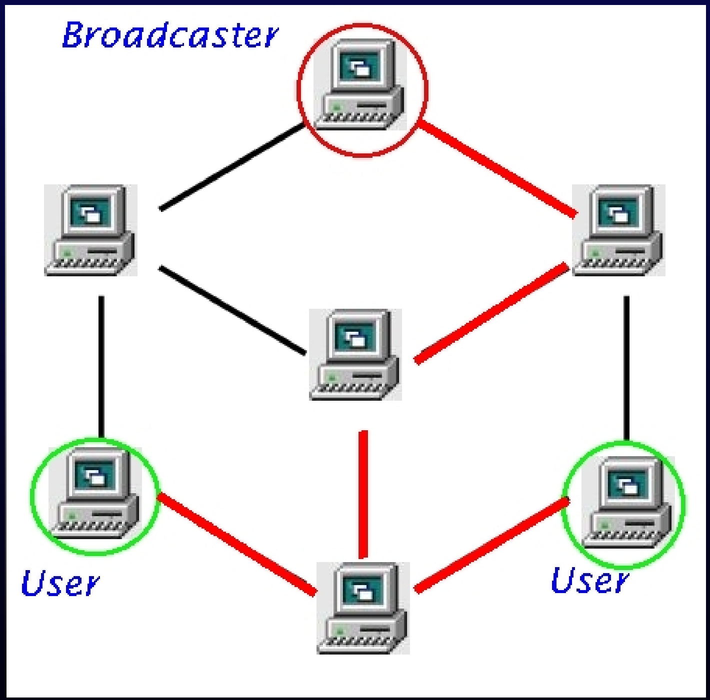
Goal: Find maximum number of edge-disjoint Steiner trees.



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If $S = V(G) \implies$ Steiner trees are spanning trees $\implies$
Theorem (Nash-Williams & Tutte 1960's): G has k edge-disjoint spanning trees iff for every partition $\mathcal{P} = \{V_1, \ldots, V_p\}$ of V:

$$E_G(\mathcal{P}) \geq k(p-1),$$

where $E_G(\mathcal{P})$ is the number of edges between classes of $\mathcal{P}$.

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Theorem 1 (Jain & Mahdian & S.'03): If $|T| = t$ and G has T-edge-connectivity at least $(\frac{t}{4} + o(t))k$, then we can find k edge-disjoint Steiner trees in poly. time.

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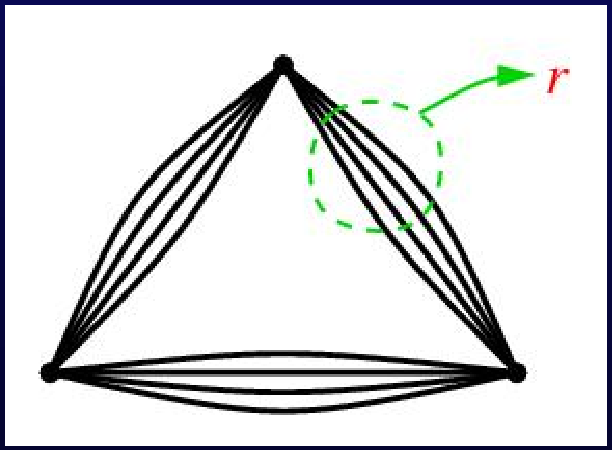

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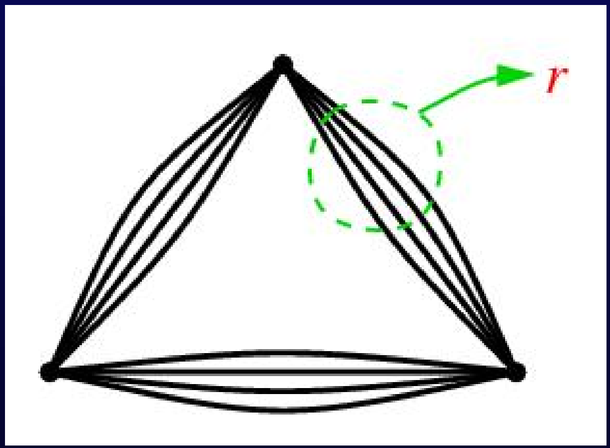

Theorem (Lau’04, unpublished): If G has T-edge-connectivity at least $26k$, then we can find k edge-disjoint Steiner trees in poly. time.

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- if max matching of G is $\leq (1-\epsilon)m$ then H has at most $(1-\frac{\epsilon}{100})m$ trees.

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Corollary: Fractional PEU is APX-hard and has an 1.59-approx algorithm.

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Packing Edge-disjoint Direct Steiner trees (PED): Given directed graph G and terminals $T \subseteq V$ containing a root r, find max number of edge-disjoint (rooted) Steiner trees.

Other variations

Packing Vertex-disjoint Undirected Steiner trees (PVU): Given undirected graph G and terminals $T \subseteq V$, find max number of Steiner trees that are *internally* vertex disjoint (i.e. on Steiner nodes).

The same results we proved for PEU also hold for PVU:

Theorem 6 (Cheriyan & S.): Given G and $T \subseteq V$, it is NP-hard to decide if G has two vertex-disjoint Steiner trees.

Theorem 7 (Cheriyan & S.): PVU is APX-hard even if $|T| = 4$.

Will come back to PVU at the end of the talk.

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Packing Edge-disjoint Direct Steiner trees (PED): Given directed graph G and terminals $T \subseteq V$ containing a root r, find max number of edge-disjoint (rooted) Steiner trees.

Packing Vertex-disjoint Direct Steiner trees (PVD): Similar to PED, except that trees have to be disjoint on Steiner nodes.

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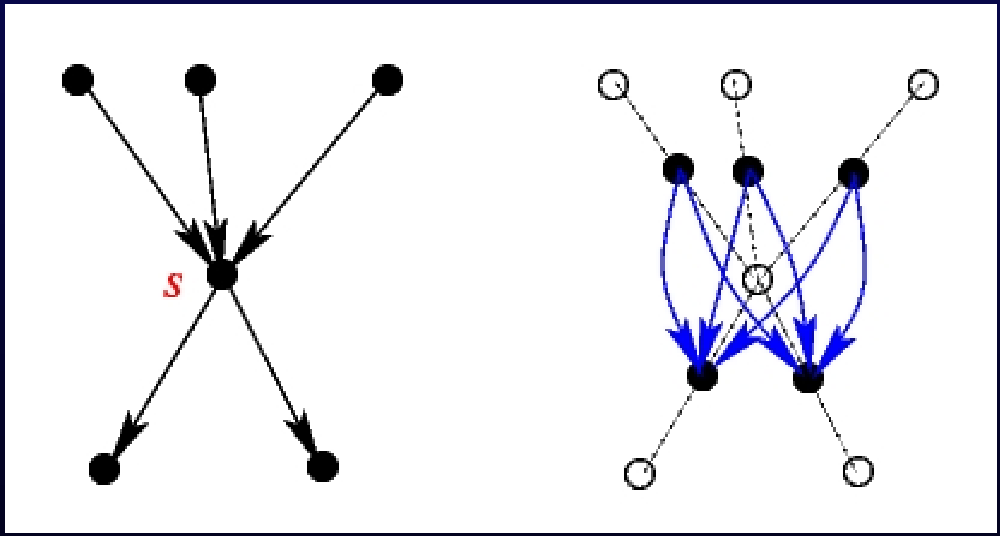

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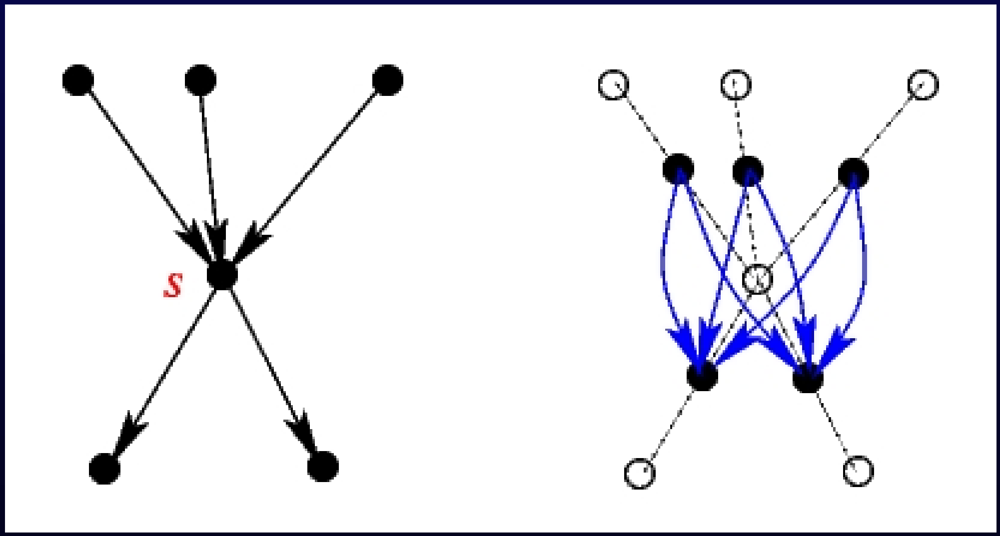

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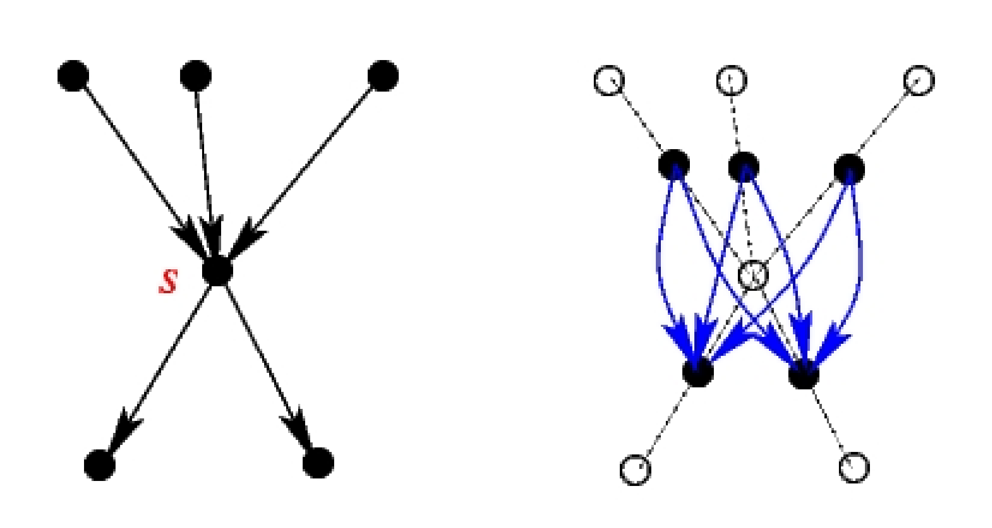

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Similarly, we can reduce PVD to PED.

Therefore, we only focus on finding algorithms and proving hardness for PED.

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This is NP-hard, even hard to approximate within $O(\log^2 n)$ factor.

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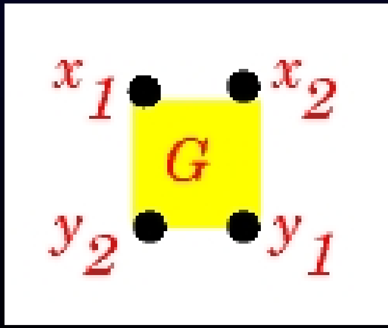

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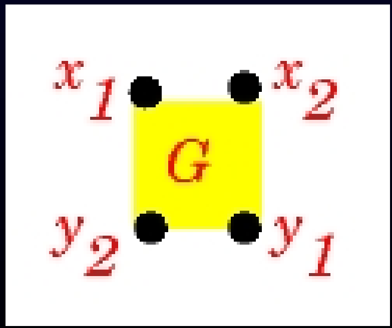
We construct a digraph H which has several copies of G.



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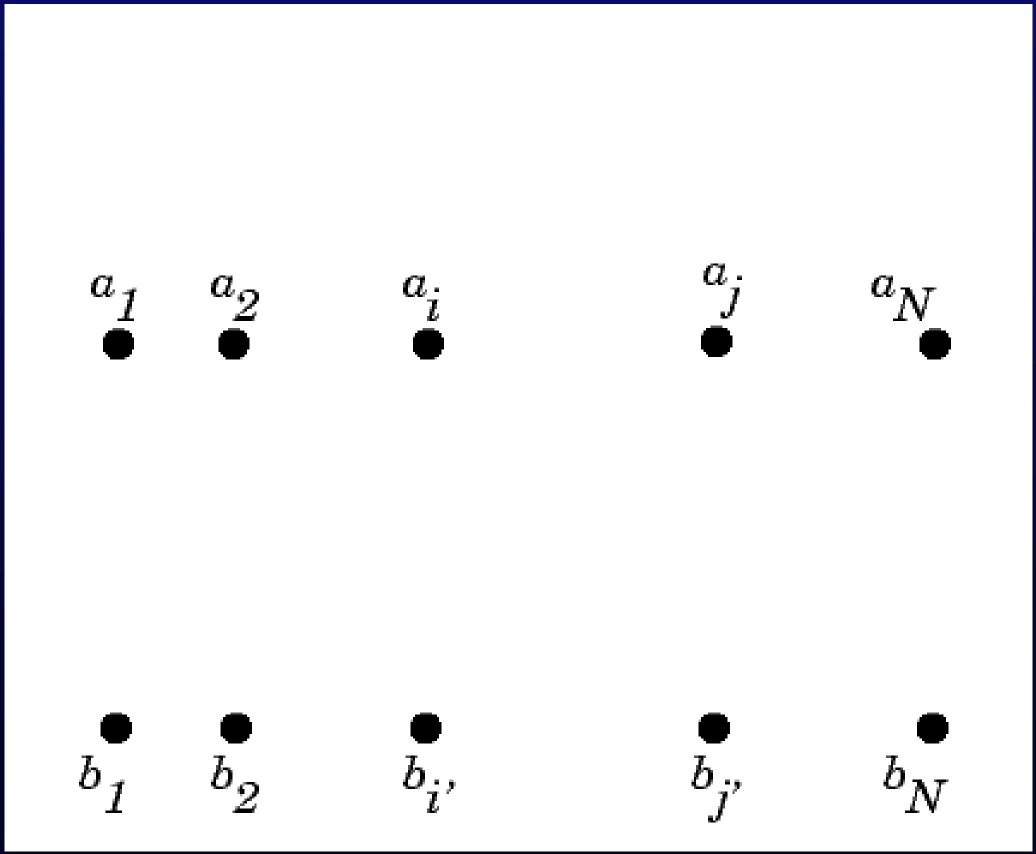

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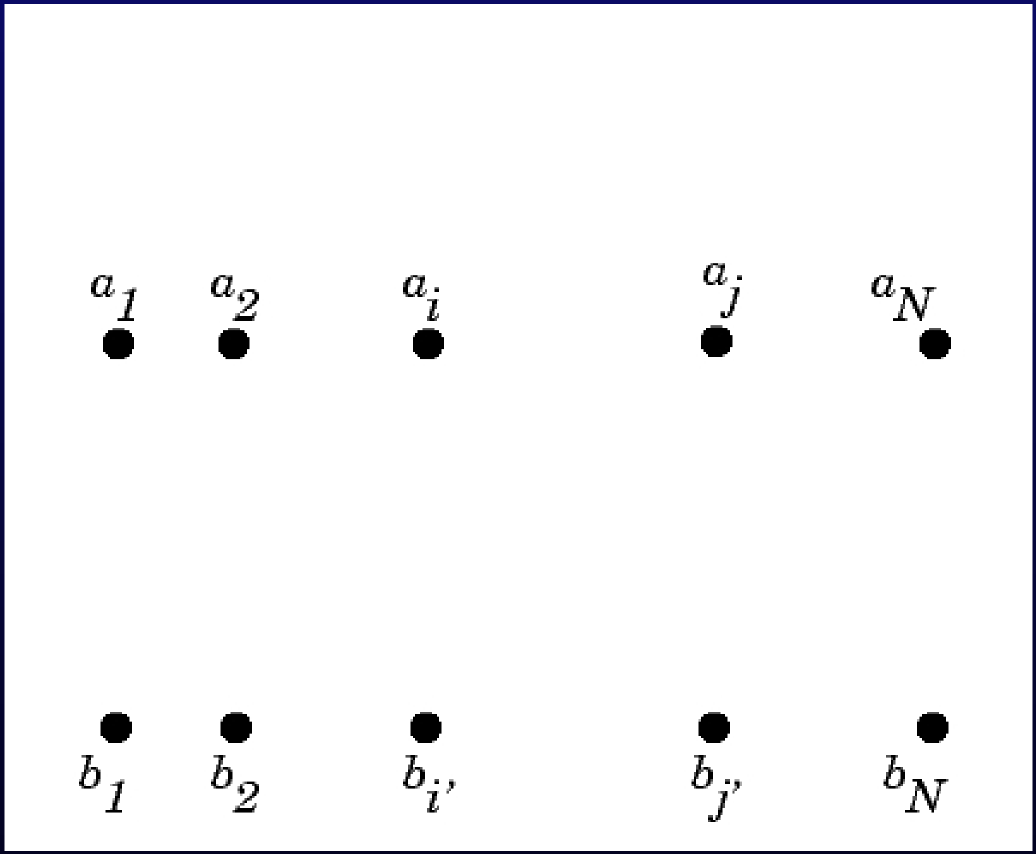

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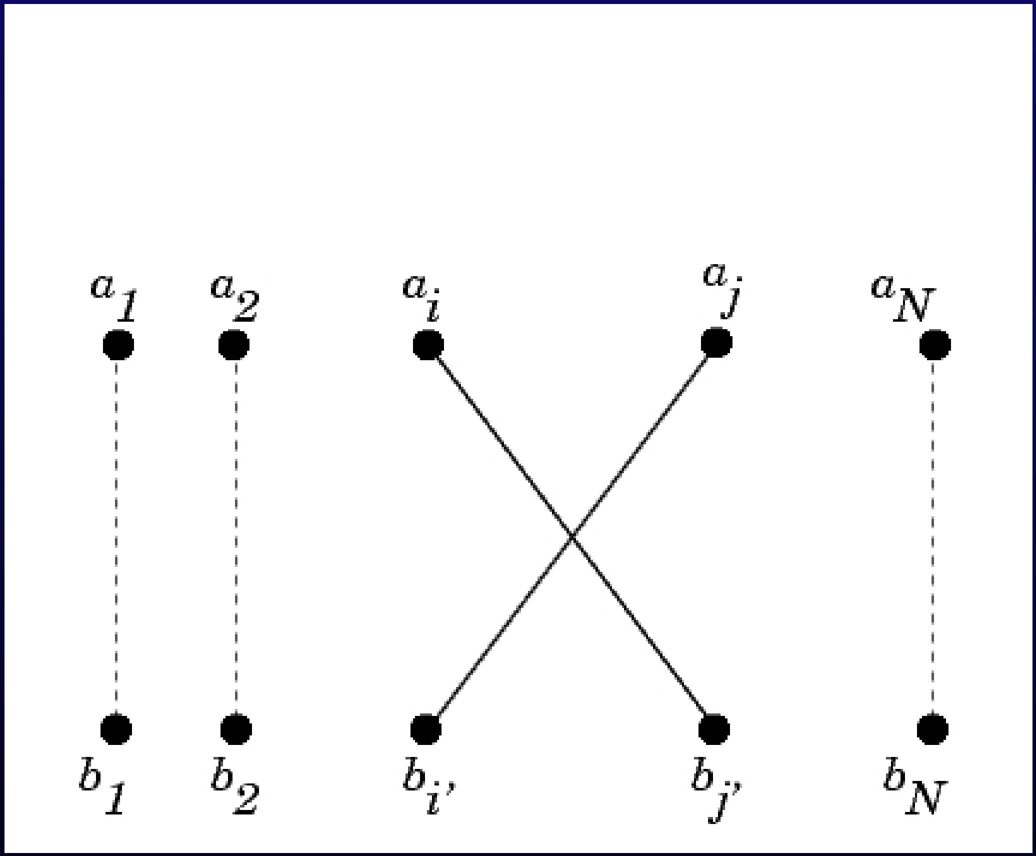

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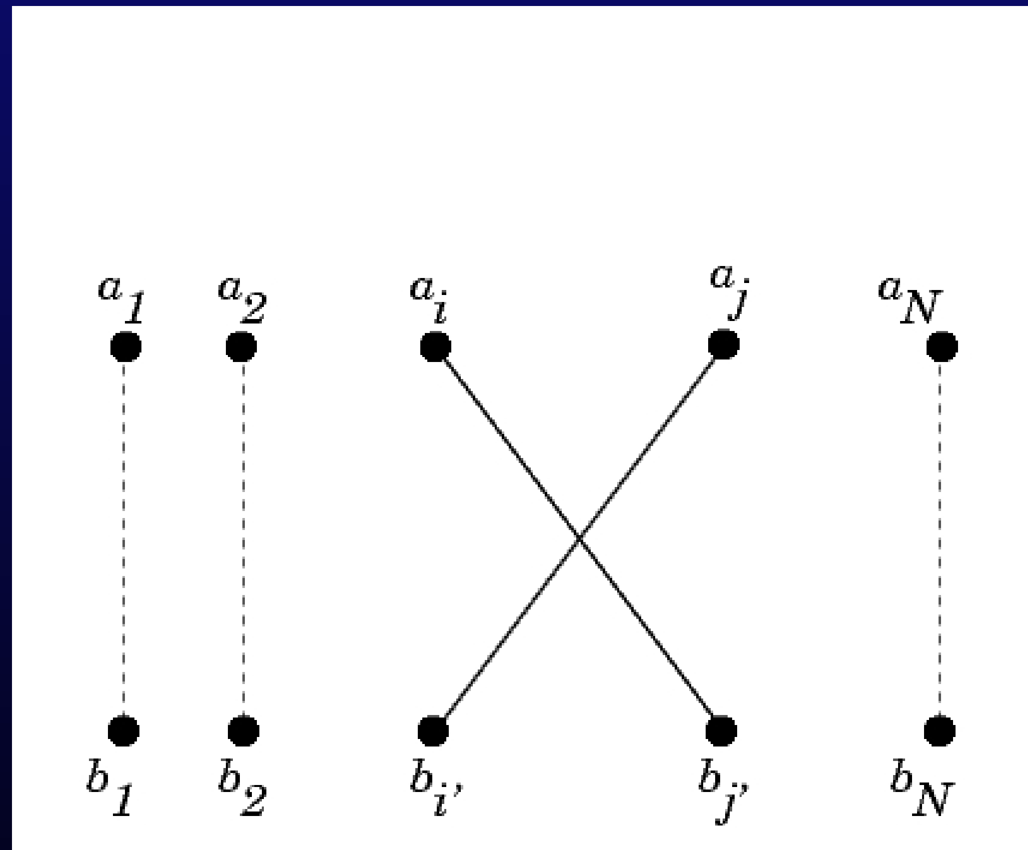

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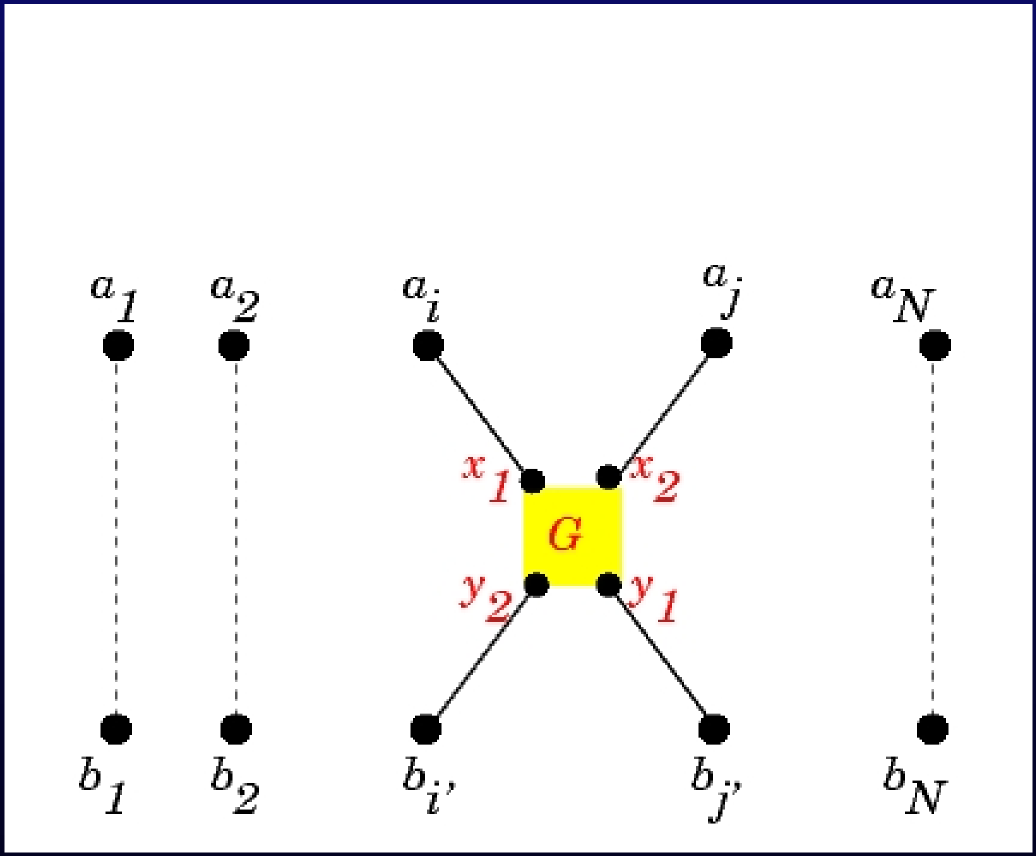

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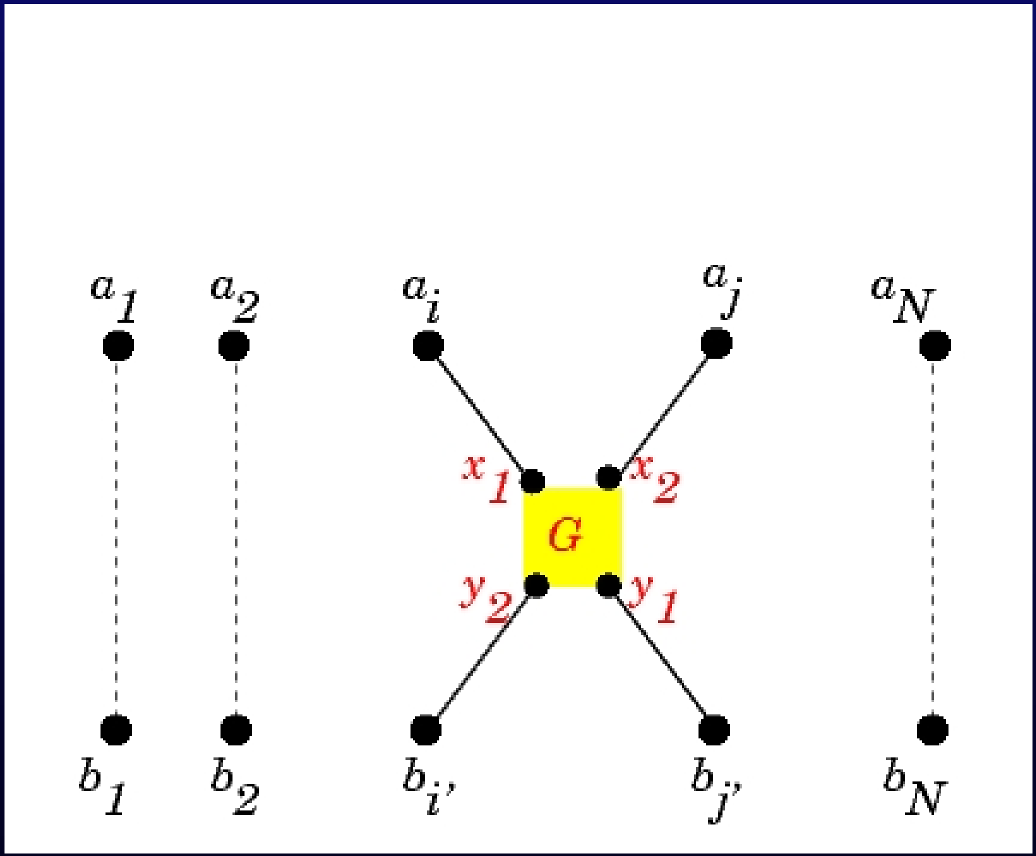

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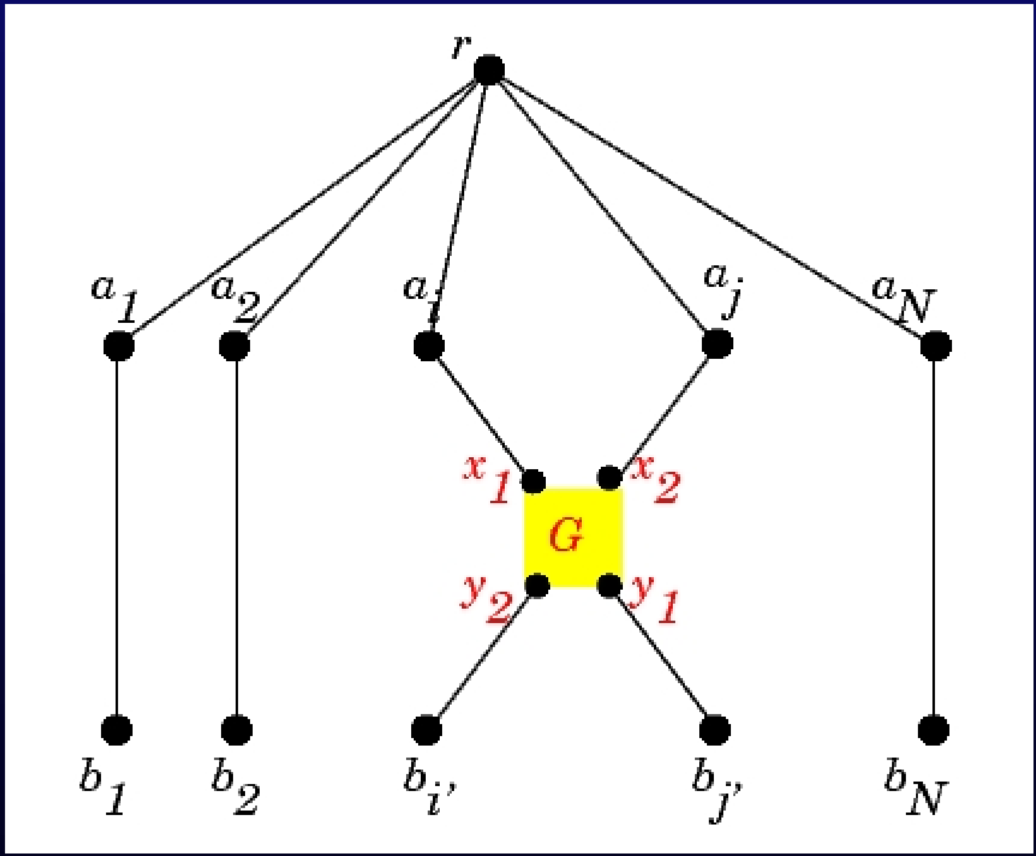

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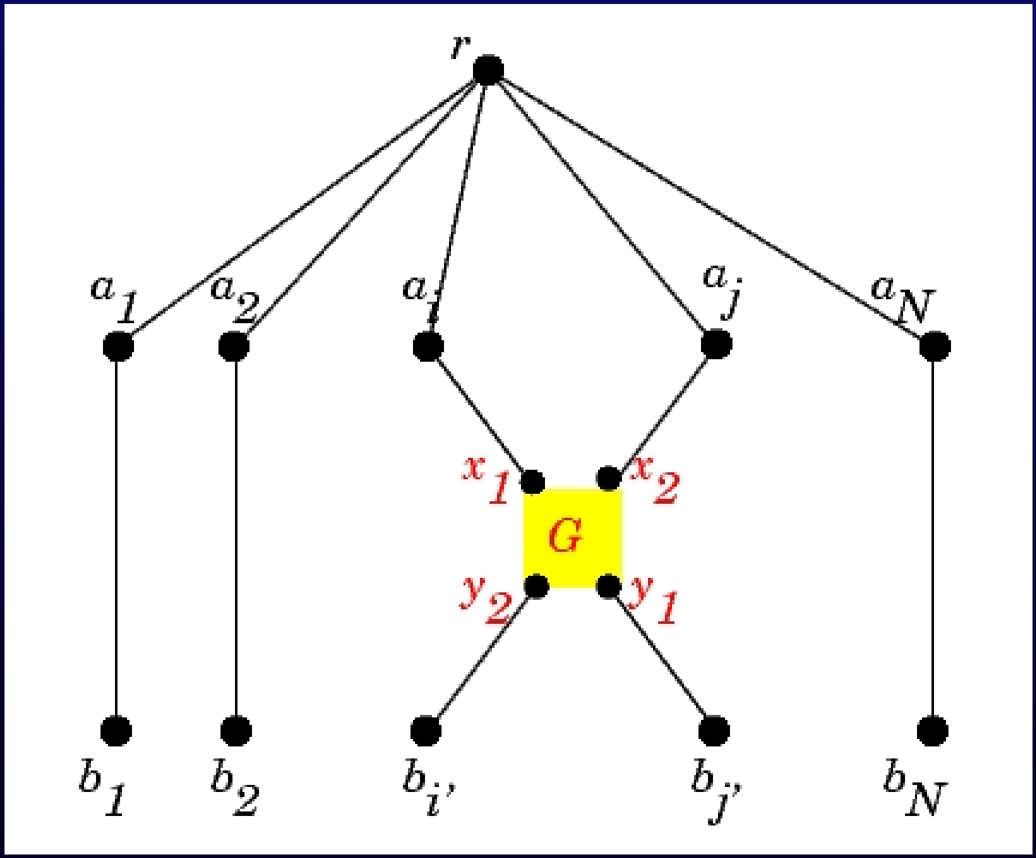

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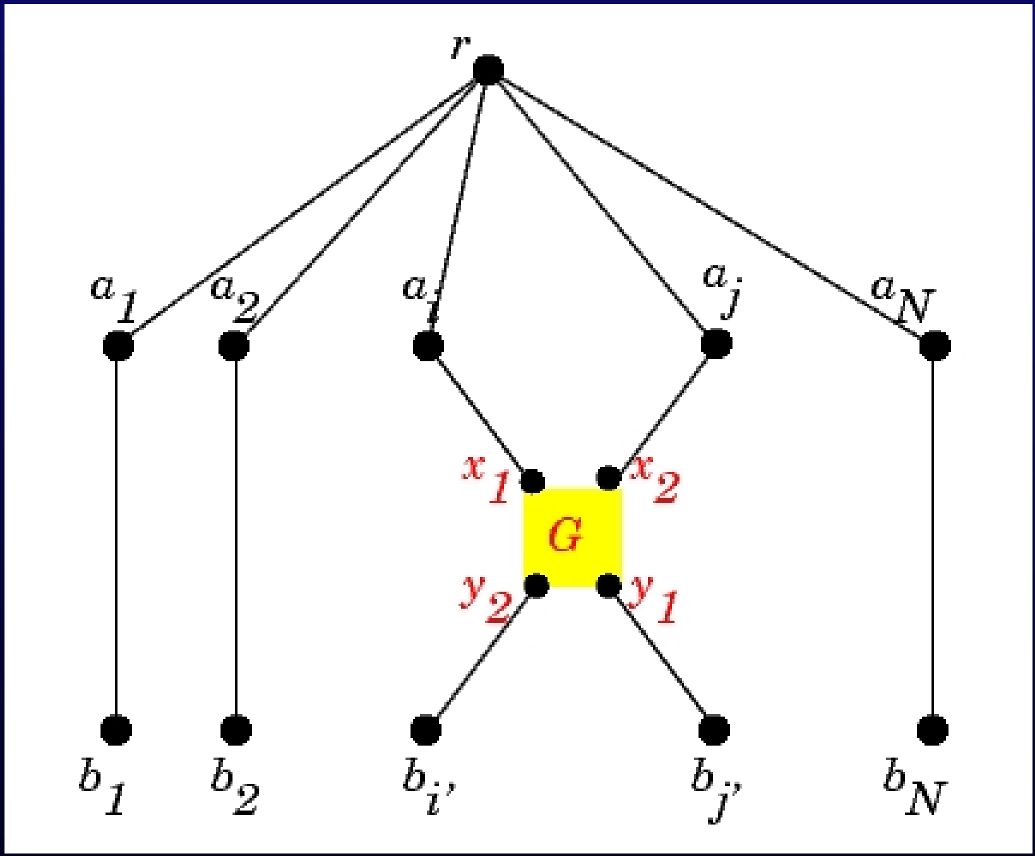

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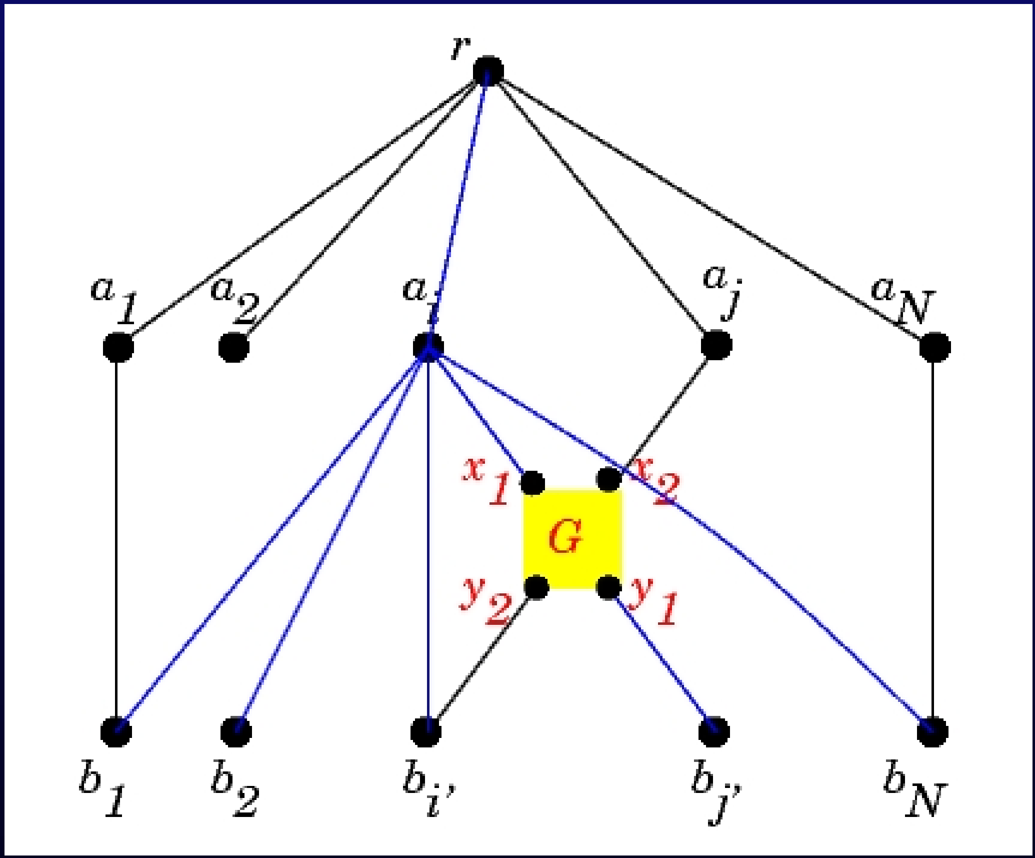

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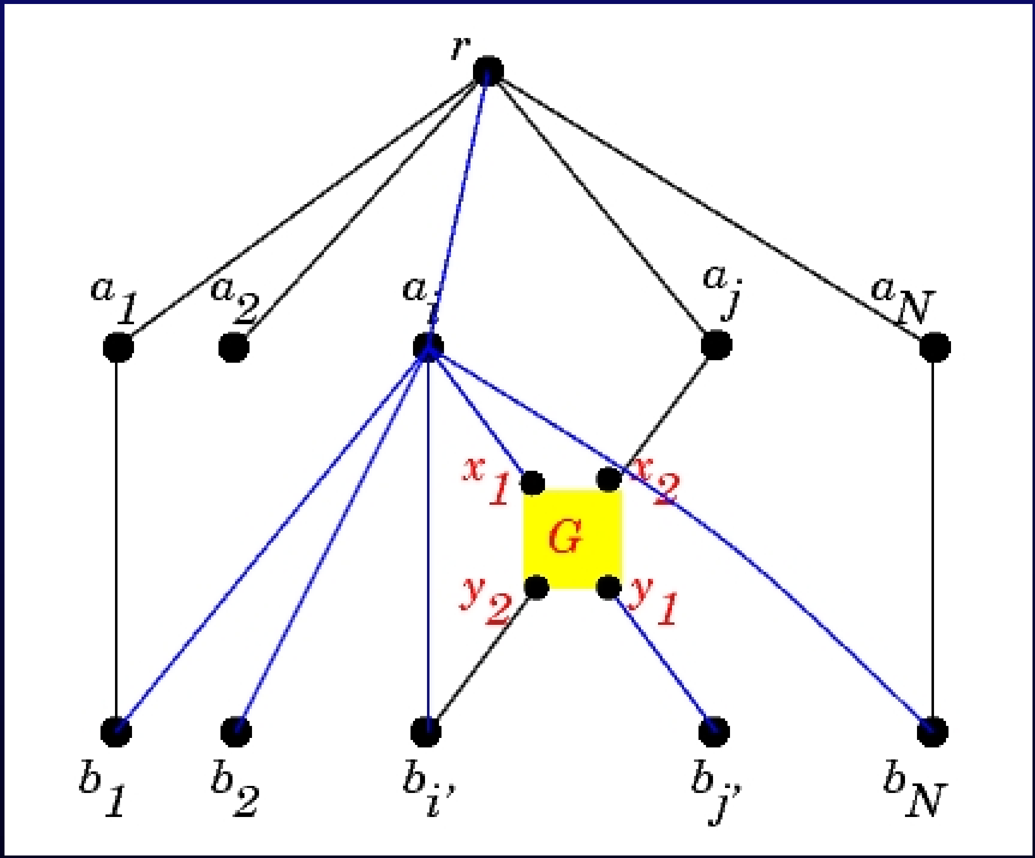

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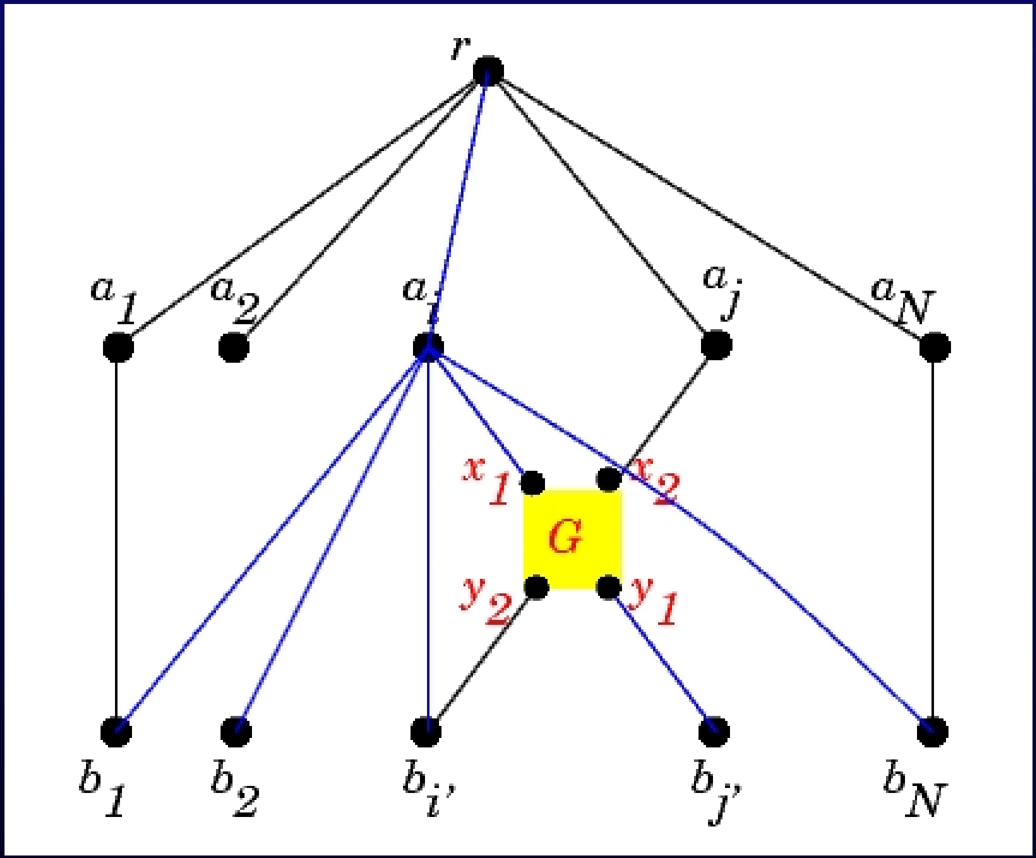

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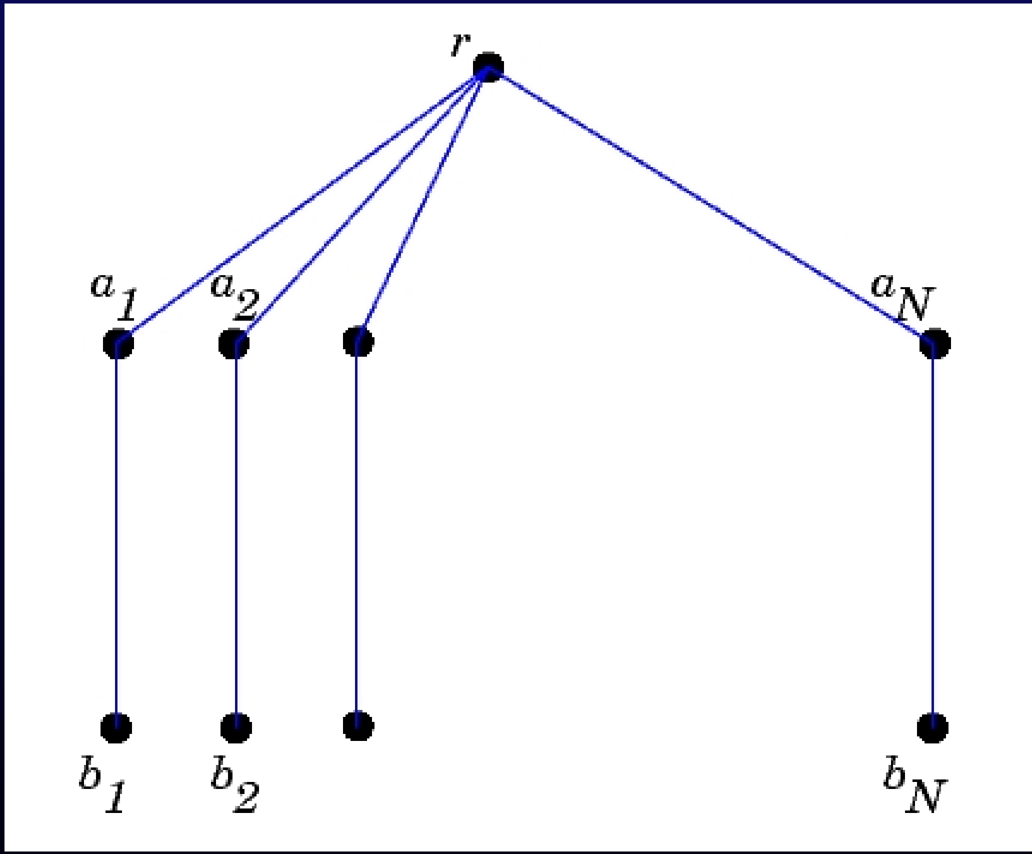

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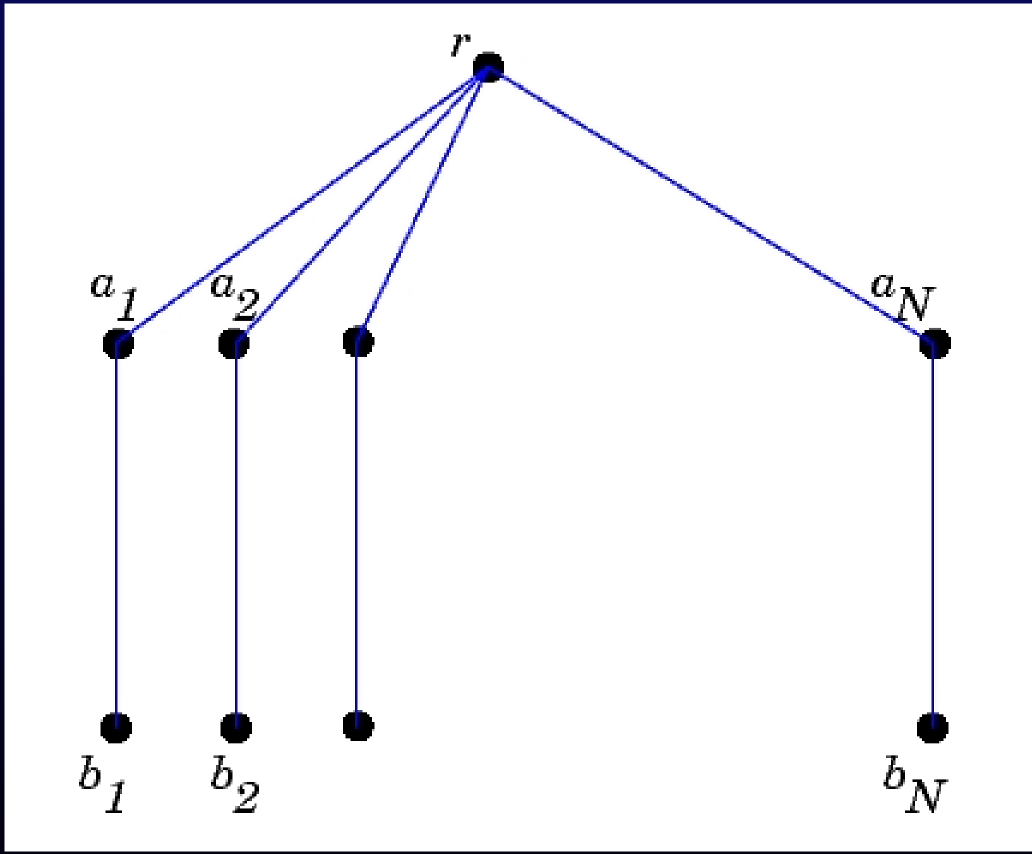

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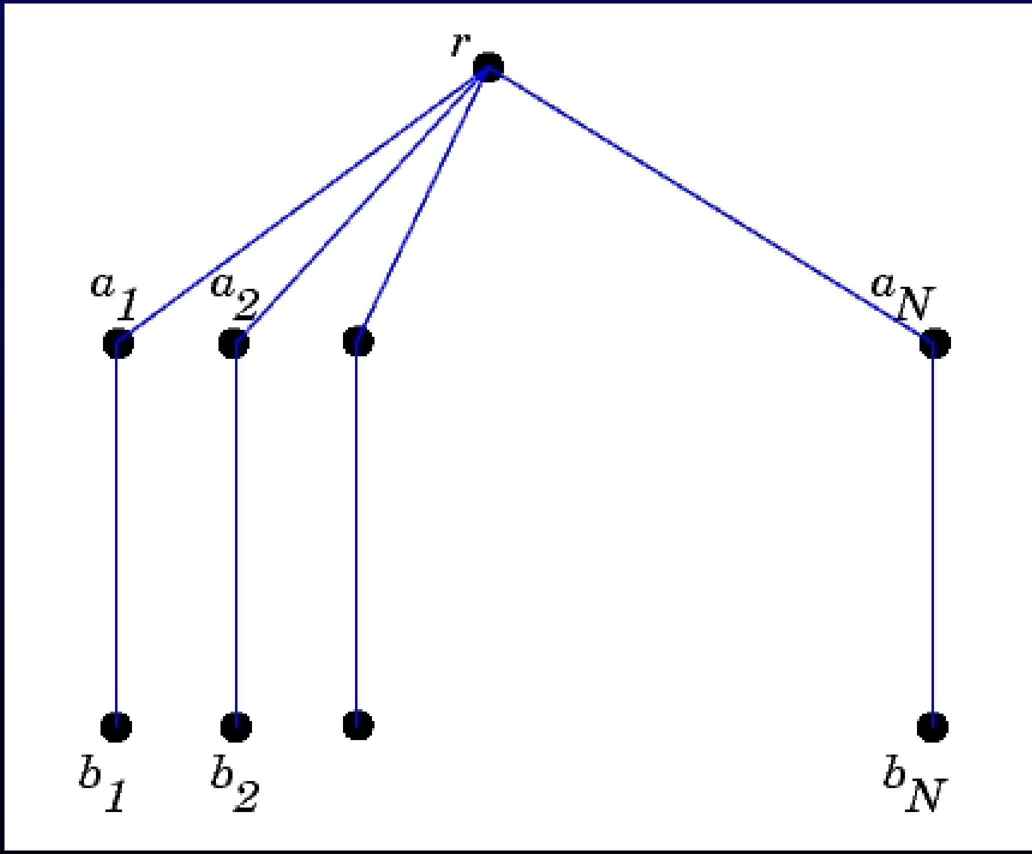

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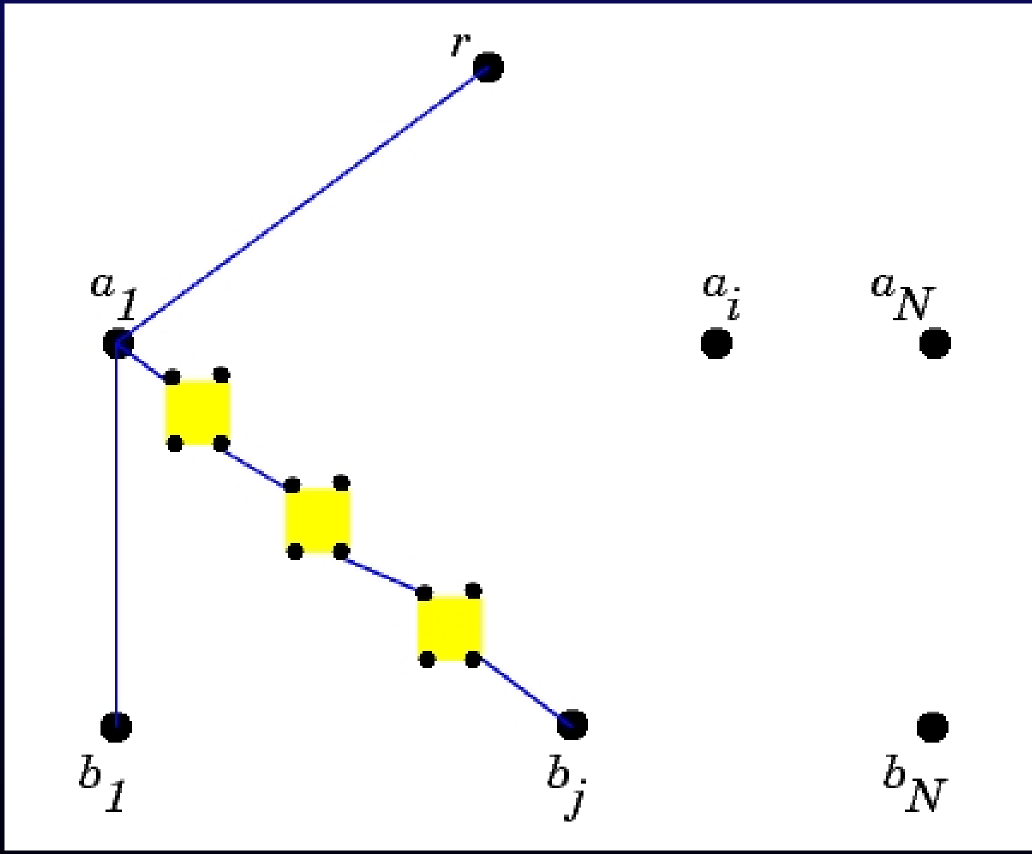

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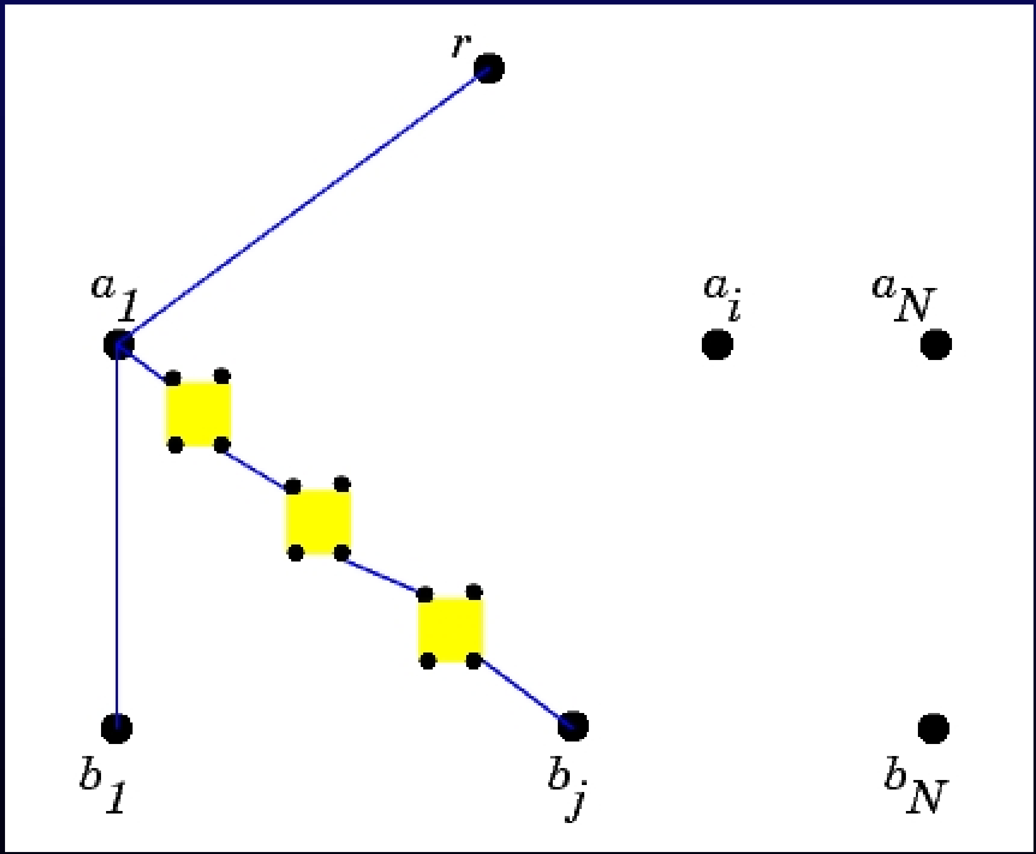

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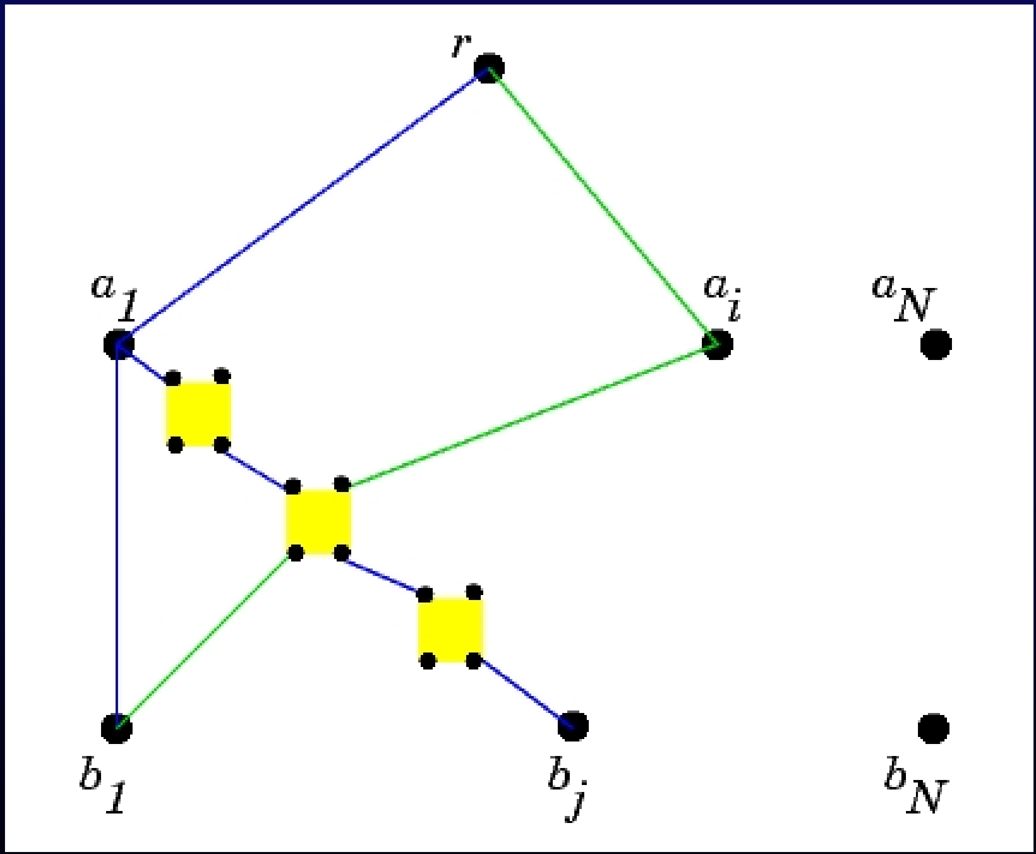

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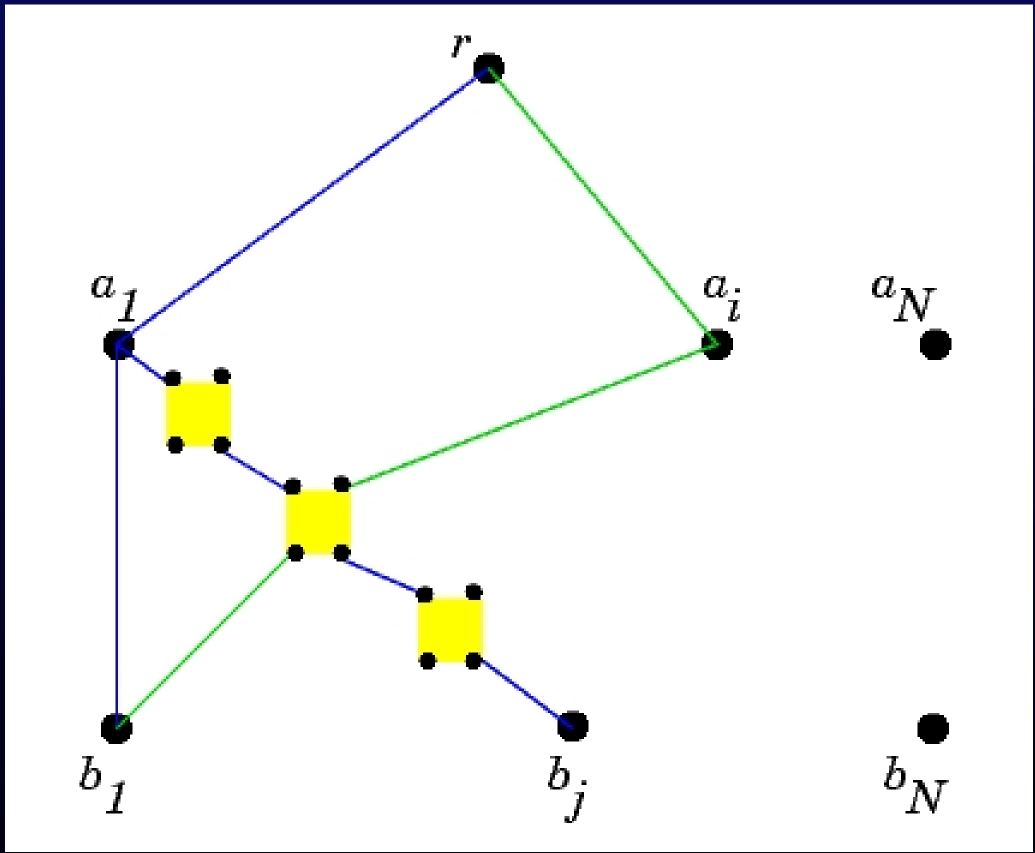

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proof: Reduction from Set-Cover Packing.

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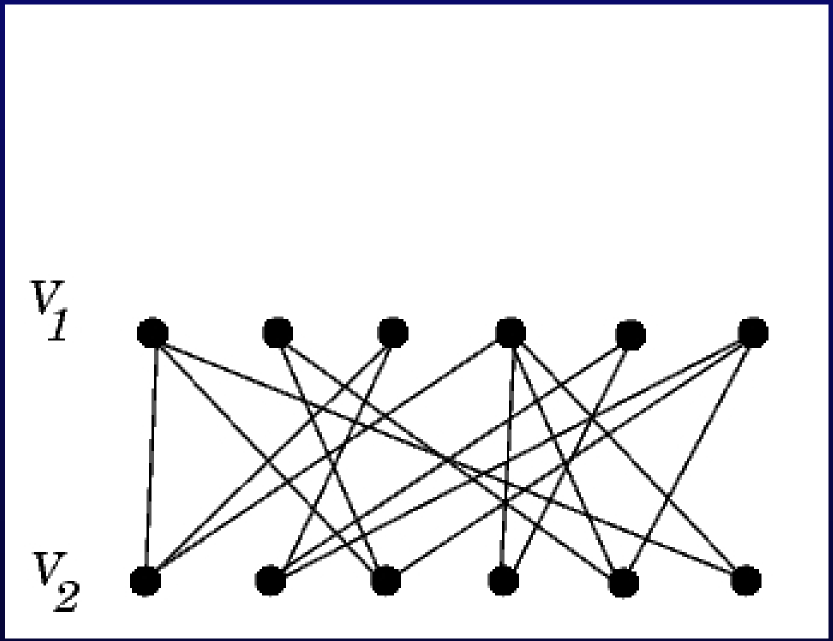

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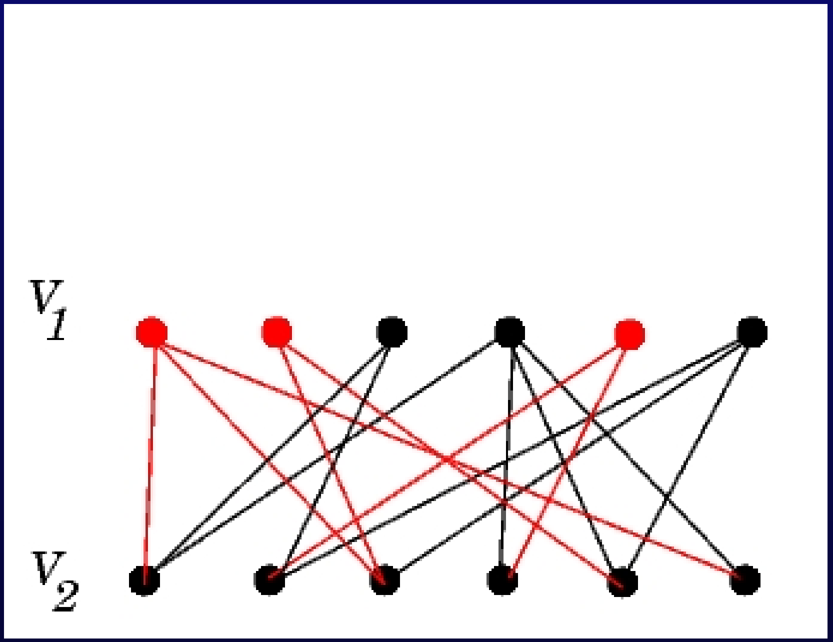

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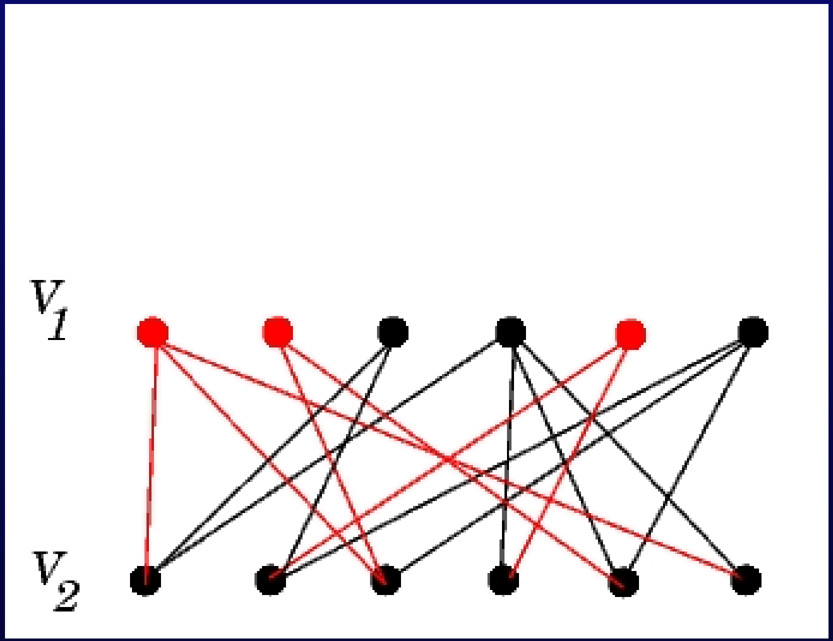

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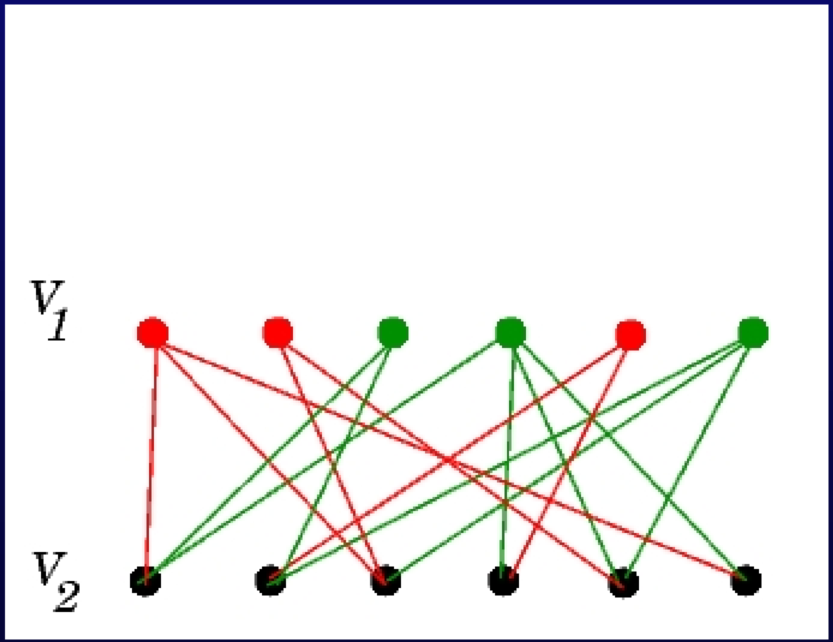

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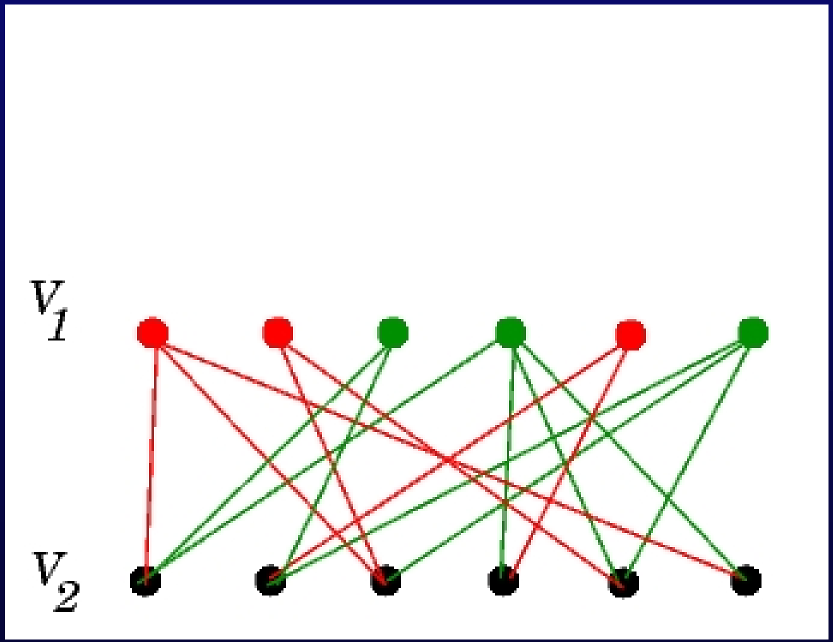

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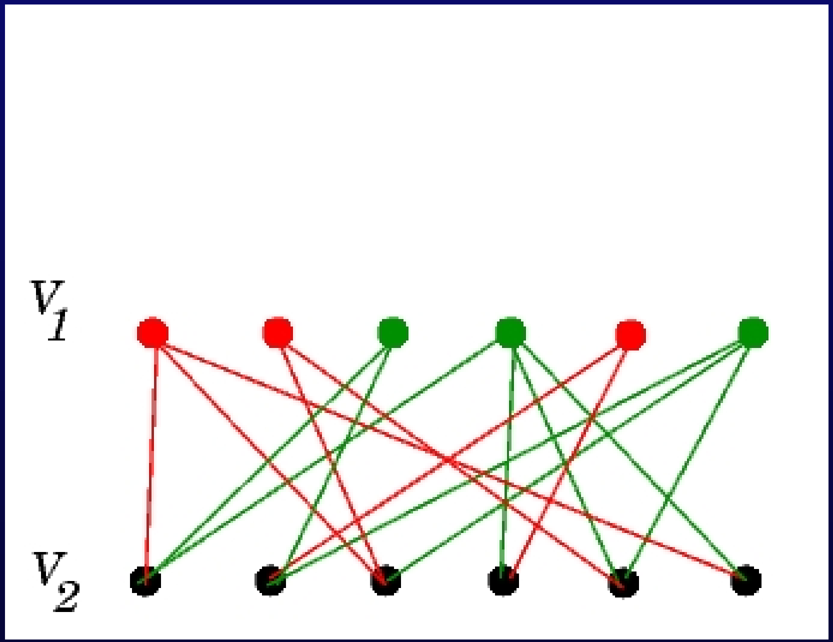

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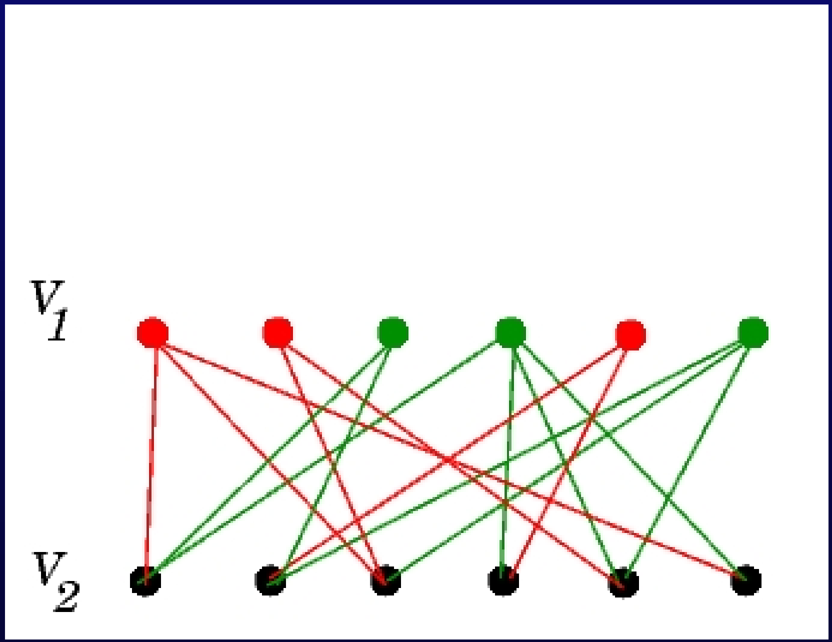

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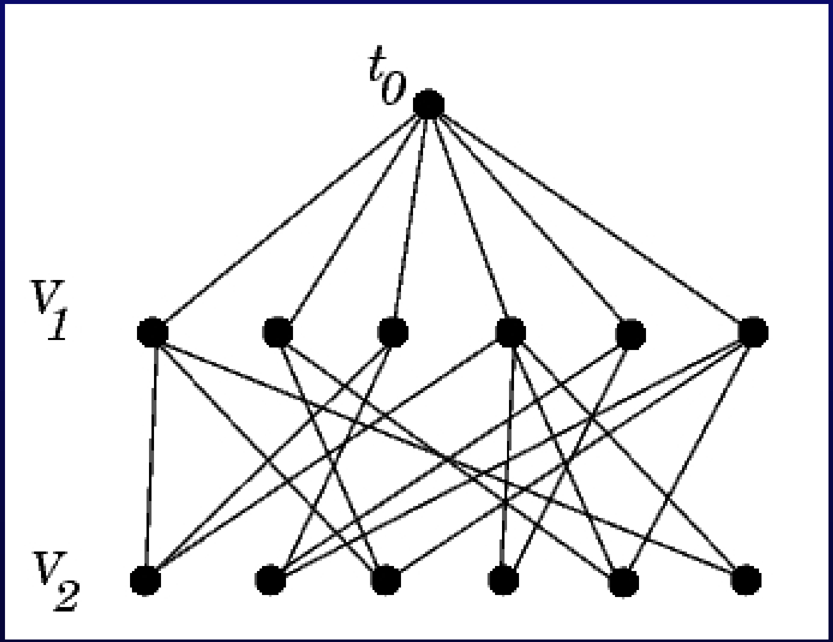

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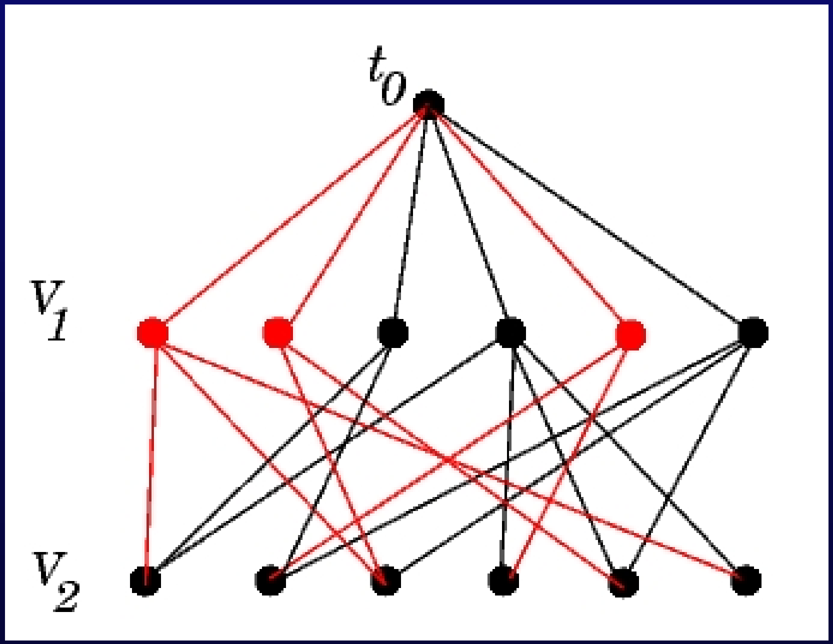

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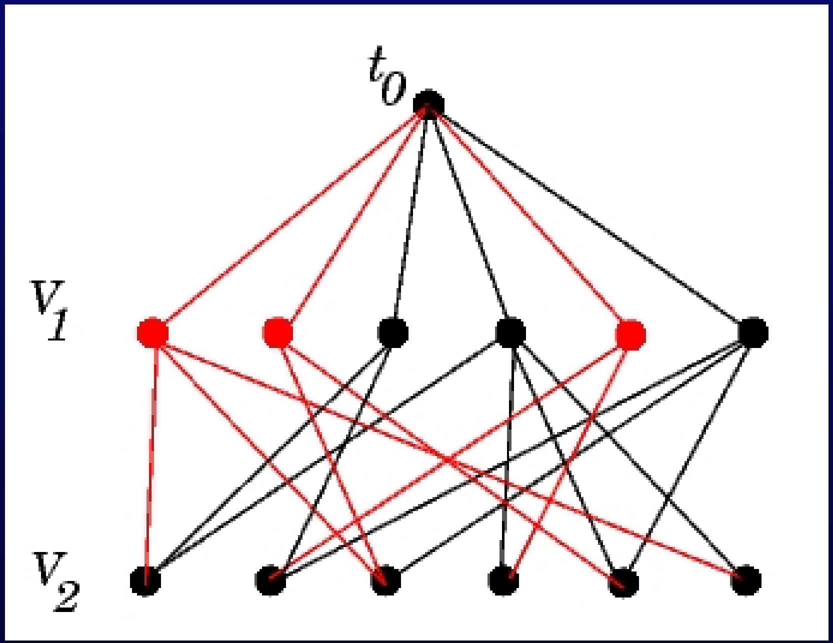

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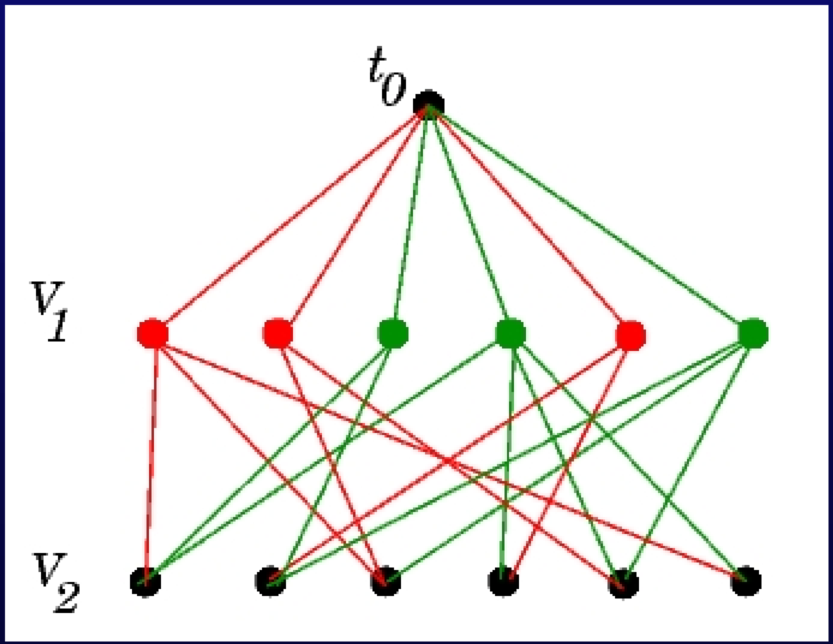

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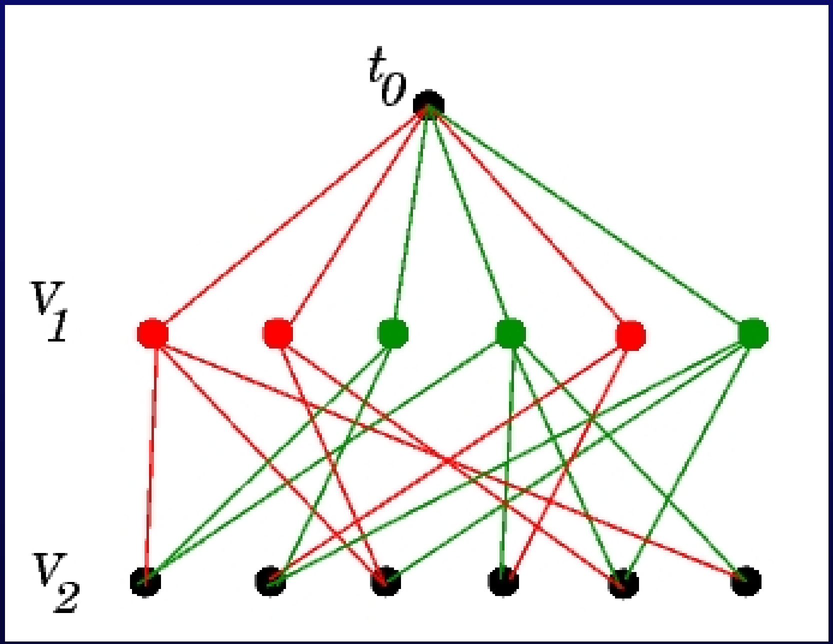

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Conversely, if $T_1, \ldots, T_p$ are V.D. Steiner trees, because V_2 is independent set, there is a set $S_i \subset V(T_i)$ s.t. covers V_2.

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- Use randomized rounding to get an $O(\log n\sqrt{n})$ approximation.

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