Planar graphs without cycles of length from 4 to 7 are 3-colorable

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Abstract

Planar graphs without cycles of length from 4 to 7 are proved to be 3-colorable. Moreover, it is proved that each proper 3-coloring of a face of length from 8 to 11 in a connected plane graph without cycles of length from 4 to 7 can be extended to a proper 3-coloring of the whole graph. This improves on the previous results on a long standing conjecture of Steinberg.

1 Introduction

In 1976, Steinberg conjectured that every planar graph without 4 and 5-cycles is 3-colorable. This conjecture (open problem 2.9 in [4]) remains unsettled despite several attempts. Erdös (see [6]) suggested the following relaxation of this problem: does there exist a constant C such that the absence of cycles with size from 4 to C in a planar graph guarantees its 3-colorability? Abbott and Zhou [1] proved that such a C exists and $C \leq 11$. This result was later on improved to $C \leq 10$ by Borodin [2] and to $C \leq 9$ by Borodin [3] and, independently, Sanders and Zhao [5]. Here, we improve on all these results:

Theorem 1.1 *Every planar graph without cycles of length from 4 to 7 is 3-colorable.*

Let $\mathcal{G}_7$ denote the class of planar graphs without cycles of size from 4 to 7. To obtain Theorem 1.1, we prove the following stronger theorem:

Theorem 1.2 *Every proper 3-coloring of the vertices of any face of size from 8 to 11 in a connected graph in $\mathcal{G}_7$ can be extended to a proper 3-coloring of the whole graph.*

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Assuming Theorem 1.2, we can easily prove Theorem 1.1:

Proof of Theorem 1.1: Suppose that G is a counter-example to the theorem with the smallest number of vertices. Clearly, G is connected and by [3, 5] it has a cycle C of length 8 or 9. By the absence of cycles of length from 4 to 7 in G, the subgraph induced by C can have at most one chord, and therefore it has a proper 3-coloring φ. By Theorem 1.2, φ can be extended (after deleting the possible chords) both inside and outside of C to obtain a proper 3-coloring of G. ■

Our proof of Theorem 1.2 is constructive and easily yields a polynomial time algorithm for finding such a 3-coloring. In the remaining of this section, we define some notation used throughout the paper. In the next section we prove some properties for a possible minimum counter-example to Theorem 1.2. In the final section we complete the proof by showing that these properties are incompatible, using the Discharging Method.

Denote the degree of a vertex v by $d(v)$ and the size of a face f (bridges are counted twice) by $|f|$. A k-vertex is a vertex of degree k. By a $\geq k$-vertex (a $\leq k$-vertex) we mean a vertex of degree at least (at most) k. Similar notation is used for faces. For a cycle S of a plane graph G, the vertices lying inside and outside of S are denoted by $\mathrm{Int}(S)$ and $\mathrm{Out}(S)$, respectively. If $\mathrm{Int}(S) \neq \emptyset$ and $\mathrm{Out}(S) \neq \emptyset$, then S is a *separating* cycle. Two cycles that have an edge in common are called *adjacent.*

Let an embedded graph $G \in \mathcal{G}_7$, its face f_0, and a 3-coloring φ of the vertices of f_0 yield a minimal counter-example to Theorem 1.2. Without loss of generality assume that f_0, whose proper 3-coloring φ cannot be extended to a proper 3-coloring of G, is the outside face. We denote by D the sequence of vertices of f_0 obtained by a facial walk around f_0 staring at a vertex of it. Any face in G other than f_0 is called *internal.* The vertices in $G - D$ are also called *internal.* An internal 3-vertex which is incident with a 3-face is called *bad.* The notion of bad vertices is crucial to our proof.

2 Basic Properties of the Minimal Counter-example

From now on, we assume that G, f_0, and D are as defined in the last paragraph.

Lemma 2.1 *G has no separating cycle S of length at most 11.*

Proof: By minimality of G, we can extend φ to $G - \mathrm{Int}(S)$. Then we delete the (possible) chords from S and extend the 3-coloring of S induced by φ to $G - \mathrm{Out}(S)$, using the minimality of G if $|S| \neq 3$ or the minimality combined with [3, 5] (see the proof of Theorem 1.1) otherwise. ■

Lemma 2.2 *G is 2-connected; in particular G has no 1-vertices.*

Proof: Because of minimality of G, there cannot be a cut vertex in D. Now assume that B is a pendant block with the cut vertex $v \in G - D$. We first extend φ to $G - (B - v)$, then 3-color B (using again the minimality of G combined with [3, 5]), and finally get an extension of φ to G. ■

Corollary 2.3 *D is a cycle in G.*

Lemma 2.4 *Each 2-vertex in G belongs to D, and none of them is incident with a 3-face.*

Proof: Otherwise, we can first extend φ to $G - v$ and then color v if $v \notin D$. Therefore, every 2-vertex belongs to D. ■

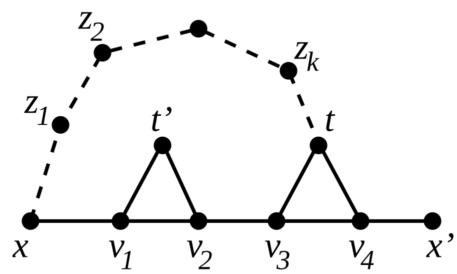

Figure 1: A tetrad

Lemma 2.5 *No cycle of length at most 13 in G has a non-triangular chord, neither D has a chord at all.*

Proof: The first statement follows from the fact that two adjacent cycles of length at least 8 must form a cycle of length at least 14. Now if a chord cuts a 3-cycle T from D, then T is a 3-face by Lemma 2.1, which contradicts Lemma 2.4. ■

A *tetrad* is a path $T = v_1v_2v_3v_4$ in $\mathrm{Int}(D)$ such that $d(v_1) = d(v_2) = d(v_3) = d(v_4) = 3$, where $\dots xv_1v_2v_3v_4x' \dots$ is on the boundary of a face, and there are triangles $t'v_1v_2$, tv_3v_4, where $t' \neq x$, $t \neq x'$.

Lemma 2.6 *G has no tetrad.*

Proof: Take a tetrad, delete v_1, v_2, v_3, and v_4 (along with the incident edges) and identify x with t. It is easy to see that the graph G^* obtained has no face of size from 4 to 7. To prove that G^* is in $\mathcal{G}_7$ we now prove that G^* cannot have a separating cycle of size from 4 to 7. By way of contradiction, suppose that $S^* = xz_1 \dots z_kt$ is such a cycle, where $3 \leq k \leq 6$ (see Figure 1). Then $S = xz_1 \dots z_ktv_3v_2v_1$ separates t' from v_4 in G. Indeed, t' cannot lie on S by Lemma 2.5. But this means that S is a separating cycle of size from 8 to 11 in G, which contradicts Lemma 2.1.

Also, G^* has neither loops nor multiple edges. Therefore, $G^* \in \mathcal{G}_7$. Next observe that any 3-coloring ψ of G^* can be extended to a 3-coloring of G: we first color v_4 and v_3 (in this order); then, since x and v_3 have different colors, it is easy to color v_1 and v_2.

So, if the coloring φ of D is not damaged by identifying x with t, then we have got a 3-coloring of G that extends φ, a contradiction. It follows that while identifying x with t we either (a) identify two vertices of D colored differently, or (b) insert an edge between two vertices of D colored the same. In other words, the total distance from x and t to D is at most 1.

Let $D = d_1 \dots d_{|D|}$, with the subscripts increasing in the clockwise order. Suppose $d_{|D|}$ is a vertex of D nearest to x, while d_j, closest to t. Since $|D| \leq 11$, it follows that D is split by $d_{|D|}$ and d_j into paths P_1, P_2 one of which, say $P_1 = d_{|D|}d_1 \dots d_j$, consists of at most 5 edges. This path, combined with the path $d_j\{t\}v_3v_2v_1\{x\}d_{|D|}$, yields a cycle C of length at most 10. By Lemma 2.5, since $t'v_2$ is an edge and $v_2 \in C$, it follows that t' cannot belong to C. Recall that $xv_1v_2v_3v_4x'$ is on the boundary of a face. Therefore, C separates t' from v_4. But this contradicts Lemma 2.1. ■

Let f be an 8-face with boundary $v_1, \dots, v_8$, where $v_1, v_2, v_3, v_5, v_6, v_7$ are bad, while v_4 and v_8 are good (i.e. non-bad) vertices. Note that by definition of bad vertices, f is internal. Assume that $v_2v_3t_{23}$, $v_5v_6t_{56}$, $v_1v_8t_{18}$, and $v_7v_8t_{78}$ are 3-faces adjacent to f (see Figure 2). Then f is called an *M-face.*

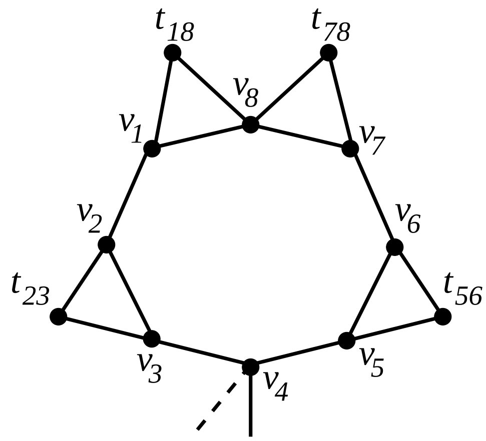

Figure 2: An M-face

Lemma 2.7 *G cannot have an M-face.*

Proof: Let f be an M-face as in Figure 2. We obtain G^* from G by deleting all the bad vertices of f and identifying v_4 with v_8. As in Lemma 2.6, it is easy to check that G^* does not have a 4 to 7-face and it cannot have a separating cycle of size from 4 to 7, or else G has a separating cycle of size from 8 to 11, which contradicts Lemma 2.1. Also, G^* has neither loops nor multiple edges. Therefore, $G^* \in \mathcal{G}_7$. The same argument as in the last paragraph of proving Lemma 2.6 shows that the coloring φ of D is not damaged by identifying v_4 with v_8.

Since G^* is smaller than G, it remains to prove that every 3-coloring ψ of G^* can be extended to a 3-coloring of G.

Let c be an arbitrary 3-coloring of G^*; w.l.o.g., assume that $c(v_4) = c(v_8) = 1$ and $c(t_{18}) = 2$. We transfer c to G. First color v_1 and v_7. Since $c(v_4) \neq c(v_1)$ and $c(v_4) \neq c(v_7)$, we can easily extend this coloring to v_2, v_3, v_5, and v_6. ■

Let f be an 8-face with boundary $v_1, \ldots, v_8$, where $v_1, \ldots, v_4$ and v_6, v_7 are bad vertices, while v_5 and v_8 are internal 4-vertices. Assume that $v_2v_3t_{23}$, $v_4v_5t_{45}$, $v_5v_6t_{56}$, $v_7v_8t_{78}$, and $v_8v_1t_{18}$ are 3-faces adjacent to f (see Figure 3). Then f is called an *MM-face.*

Lemma 2.8 *G cannot have an MM-face.*

Proof: We obtain G^* from G by deleting $v_1, \ldots, v_8$ and identifying t_{18} with t_{56}. As in the previous two lemmas, it is easy to check that $G^* \in \mathcal{G}_7$ and that the coloring φ of D is not damaged by this identification. We show that every 3-coloring ψ of G^* can be extended to a 3-coloring of G.

Let c be an arbitrary 3-coloring of G^*, where $c(t_{18}) = c(t_{56}) = 1$. We transfer c to G. If $c(t_{45}) \neq 1$, we first color v_5, v_4, and v_6, (in this order); then, using an argument as in proving Lemma 2.6, we can color v_8 and v_7, then v_1, and finally v_2 and v_3.

If $c(t_{45}) = 1$, we put $1 \neq c(v_8) = c(v_6) = c(v_4) \neq c(t_{78})$, then color v_1, v_5, v_7 (in this order), and finally v_2 and v_3. ■

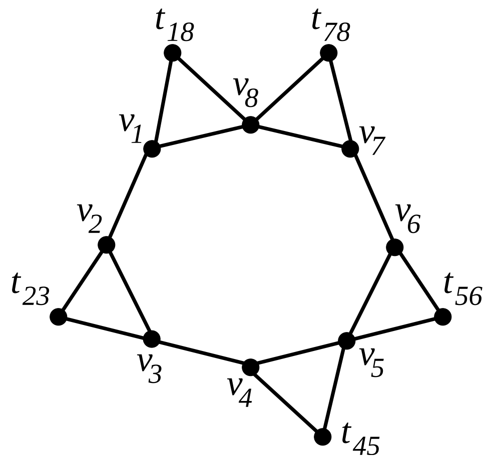

Figure 3: An MM-face

3 Incompatibility of the Basic Properties

The rest of our proof consists in showing that the structural properties of G proved in the previous section are incompatible. Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ for G may be rewritten as

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (|f| - 4) = -8.$$

We set the *initial charge* of every vertex v of G to be $ch(v) = d(v) - 4$, the initial charge of every face $f \neq f_0$ to be $ch(f) = |f| - 4$, and put $ch(f_0) = |f_0| + 4$. Clearly,

$$\sum_{x \in V(G) \cup F(G)} ch(x) = 0.$$

Then we use the discharging procedure, leading to a *final charge* ch^*, defined by applying the following rules:

R1. Each 3-face receives $\frac{1}{3}$ from each incident vertex.

R2. Each internal non-triangular face f sends to each incident vertex v:

(a) $\frac{2}{3}$ if either $deg(v) = 2$ or v is a bad vertex.

(b) $\frac{1}{3}$ if v is internal and either $deg(v) = 3$ and v is not bad, or $deg(v) = 4$ and v is either incident with a 3-face not adjacent to f, or else is incident with two 3-faces both adjacent to f.

R3. Each internal non-triangular face f receives $\frac{1}{3}$ from its incident vertex v if:

(a) $deg(v) \geq 5$ and v is internal and incident with two 3-faces adjacent to f, or

(b) $deg(v) \geq 4$ and $v \in D$.

R4. The outside face f_0 gives $\frac{4}{3}$ to each vertex of D.

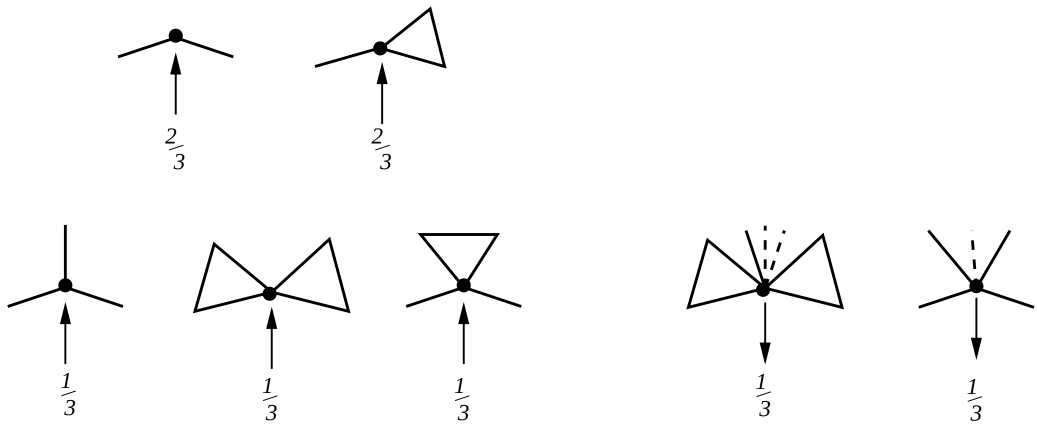

Figure 4: (a) Rule R2 (b) Rule R3

Rules 2 and 3 are illustrated in Figure 4. Since the above procedure preserves the total charge, we have:

$$\sum_{x\in V(G)\cup F(G)} ch^*(x) = 0.$$

The rest of the proof consists in checking that $ch^*(x) \geq 0$ whenever $x \in V(G) \cup F(G)$ and that $ch^*(f_0) > 0$, with the obvious final contradiction.

Lemma 3.1 *If $v \in V(G)$ then $ch^*(v) \geq 0$.*

Proof: If $d(v) = 2$ then by Lemma 2.4 it belongs to D and is not incident with a 3-face. Therefore, only rules R2 and R4 are applied to v and $ch^*(v) = 2 - 4 + \frac{2}{3} + \frac{4}{3} = 0$. Suppose that $d(v) = 3$. If $v \in D$, then v receives $\frac{4}{3}$ from D by R4 and possibly sends away $\frac{1}{3}$ by R1. So, assume $v \notin D$. If v is not incident with a 3-face, then $ch^*(v) = 3-4+3\times\frac{1}{3} = 0$ by R2. Otherwise, $ch^*(v) = 3-4+2\times\frac{2}{3}-\frac{1}{3} = 0$ by R1 and R2.

Suppose $d(v) = 4$. If $v \in D$, then v receives $\frac{4}{3}$ from f_0 by R4 and sends away $\frac{1}{3}$ to each internal incident face by R3 or R1, and therefore $ch^*(v) \geq \frac{4}{3} - 3 \times \frac{1}{3} > 0$. So, we assume $v \notin D$. If v is not incident with a 3-face, then $ch^*(v) = ch(v) = 0$. If v is incident with only one 3-face, then v receives $\frac{1}{3}$ by R2 and sends away $\frac{1}{3}$ due to R1. If v is incident with two (mutually nonadjacent) 3-faces, then v receives $2 \times \frac{1}{3}$ due to R2 and sends away $2 \times \frac{1}{3}$ due to R1. In any case $ch^*(v) \geq 0$.

Now suppose $d(v) = 5$. If $v \notin D$ then v sends $\frac{1}{3}$ to at most two 3-faces by R1 and to at most one non-triangular face by R3, so that $ch^*(v) \geq 0$. Otherwise, $ch^*(v) \geq 1 + \frac{4}{3} - 4 \times \frac{1}{3} > 0$

If $d(v) \geq 6$ then v sends away at most $d(v) \times \frac{1}{3}$ according to R1 and R3, so that $ch^*(v) \geq d(v) - 4 - \frac{d(v)}{3} = \frac{2(d(v)-6)}{3} \geq 0$. ∎

Lemma 3.2 *$ch^*(f_0) > 0$.*

Proof: Recall that $ch(f_0) = |f_0| + 4$. By R4, $ch^*(f) \geq |f_0| + 4 - |f_0| \times \frac{4}{3} = \frac{12-|f_0|}{3} > 0$. ∎

Lemma 3.3 *If $f \in F(G)$ and $f \neq f_0$, then $ch^*(f) \geq 0$.*

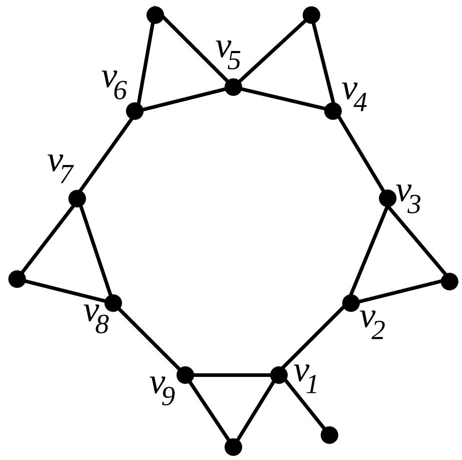

Figure 5: A 9-face as in the proof of Lemma 3.3

Proof: If $|f| = 3$ then $ch(f) = |f| - 4 + 3 \times \frac{1}{3} = 0$ by R1.

Suppose $|f| \geq 12$. As f sends each incident vertex at most $\frac{2}{3}$ due to R2, we have $ch^*(f) = |f| - 4 - |f| \times \frac{2}{3} = \frac{|f|-12}{3} \geq 0$.

Observe that if $|f| \geq 8$ and f is incident with a 2-vertex, which belongs to D by Lemma 2.4 and takes $\frac{2}{3}$ from f by R2, then f is incident with two $\geq$3-vertices of D, namely the ends of a maximal path of 2-vertices on the boundary of f. These vertices get nothing from f, and therefore $ch^*(f) \geq |f| - 4 - (|f| - 2) \times \frac{2}{3} \geq \frac{|f|-8}{3} \geq 0$. Thus, from now on, we may assume that f is not incident with any 2-vertices.

Suppose $|f| = 11$. If f sends to at least one incident vertex less than $\frac{2}{3}$, i.e., at most $\frac{1}{3}$ (see R2), then we have $ch^*(f) \geq 11 - 4 - 10 \times \frac{2}{3} - \frac{1}{3} = 0$. However, f cannot be incident with 11 bad vertices because of parity.

Now suppose $|f| = 10$. If f sends to at least two incident vertices at most $\frac{1}{3}$ each, we are done: $ch^*(f) \geq 10 - 4 - 8 \times \frac{2}{3} - 2 \times \frac{1}{3} = 0$. The only danger comes from f being incident with at least 9 bad vertices. But clearly every 5 consecutive bad vertices on the boundary of f include a tetrad, which is impossible by Lemma 2.6.

Next suppose $|f| = 9$. If f sends to at least three incident vertices at most $\frac{1}{3}$ each, or if it sends nothing to at least one vertex and at most $\frac{1}{3}$ to another one, then we are done: $ch^*(f) \geq 9-4-6\times\frac{2}{3}-3\times\frac{1}{3} = 0$ or $ch^*(f) \geq 9 - 4 - 7 \times \frac{2}{3} - \frac{1}{3} = 0$, respectively. If f has eight bad vertices it will certainly form a tetrad, which contradicts Lemma 2.6. So, there are at most seven bad vertices and the other two must be internal vertices and taking $\frac{1}{3}$ each. Clearly those seven must be split by the two good vertices as 4+3, otherwise they form a tetrad. Furthermore, the quadruple should fail to be a tetrad. W.l.o.g., we have a situation as in Figure 5. But in this case, one of the good vertices (v_1 in the figure) takes nothing from f and therefore $ch^*(f) \geq 0$.

Finally, suppose $|f| = 8$. If f sends to at least four incident vertices at most $\frac{1}{3}$ each, or if it sends nothing to at least two vertices, then we are done: $ch^*(f) \geq 8 - 4 - 4 \times \frac{2}{3} - 4 \times \frac{1}{3} = 0$ or $ch^*(f) \geq 8-4-6\times\frac{2}{3} = 0$, respectively. So we may again assume that f is totally surrounded by internal vertices. (If exactly one vertex v at f belongs to D, then clearly $deg(v) \geq 4$, so that v gives $\frac{1}{3}$ to f. Since the other seven vertices cannot all be bad by Lemma 2.6, it follows that $ch^*(f) \geq 8-4+\frac{1}{3}-6\times\frac{2}{3}-\frac{1}{3} = 0$.) If f is incident with at most one good vertex, we have a tetrad. It remains to assume that $f = v_1 \dots v_8$

is incident with exactly 6 or exactly 5 bad vertices.

Case 1. There are precisely 5 bad vertices around f.

If at least one good vertex of f fails to take $\frac{1}{3}$ from f, then we are done: $ch^*(f) \geq 8-4-5\times\frac{2}{3}-2\times\frac{1}{3} = 0$. So suppose that each of these three vertices takes $\frac{1}{3}$. It follows by R2 and R3 that all of them must be internal ≤ 4-vertices, each having either two or no incident triangular edges in common with f. However, this is impossible by parity: each bad vertex starts a unique path of triangular edges along the boundary of f, with another bad vertex at the end and all good 4-vertices in between.

Case 2. There are precisely 6 bad vertices around f.

These six must be split by the two good vertices as 4+2 or 3+3, since each path of 5 bad vertices contains a tetrad.

Subcase 2.1: 4+2

Not to form a tetrad, those 4 bad vertices, $v_1, \ldots v_4$ should form triangles with the good vertices v_5 and v_8. If the edge v_6v_7 is triangular, then both v_5 and v_8 get nothing from f by R2, and we are home. So suppose that both v_5v_6 and v_7v_8 are triangular. Observe that $d(v_5) \geq 4$ and $d(v_8) \geq 4$. If $d(v_5) \geq 5$, or $d(v_8) \geq 5$, or if one of these two vertices belongs to D, then we are done due to R3: $ch^*(f) \geq 8 - 4 - 6 \times \frac{2}{3} - \frac{1}{3} + \frac{1}{3} = 0$. Therefore, it remains to assume that both v_5 and v_8 are internal 4-vertices and, furthermore, we have 3-faces $v_1v_8t_{18}$, $v_2v_3t_{23}$, $v_4v_5t_{45}$, $v_5v_6t_{56}$, and $v_7v_8t_{78}$ as in Figure 3. But this is an MM-face, contrary to Lemma 2.8.

Subcase 2.2: 3+3

As in Subcase 2.1, one of the two good vertices at f, say v_8, must have two triangular edges in common with f, for otherwise each good vertex takes 0 from f. Due to the absence of tetrads, the other good vertex at f must be v_4. But then f is an M-face (as in Figure 2), which contradicts Lemma 2.7.

This completes the proof of Theorem 1.2.

■

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