Discrete Mathematics 216 (2000) 287–292

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Note

On a conjecture of Keedwell and the cycle double cover conjecture

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Received 21 January 1998; revised 19 May 1999; accepted 4 October 1999

Abstract

At the 16th British Combinatorial Conference (1997), Cameron introduced a new concept called 2-simultaneous coloring. He used this concept to reformulate a conjecture of Keedwell (1994) on the existence of critical partial latin squares of a given type. Using computer programs, we have verified the truth of the above conjecture (the SE conjecture) for all graphs having less than 29 edges. In this paper we prove that SE conjecture is a consequence of the well-known oriented cycle double cover conjecture. This connection helps us to prove that the SE conjecture is true for semieulerian graphs. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

In this paper we consider finite loopless graphs. For notations not defined here we refer to [1]. By ‘edge-coloring’ here we mean a proper edge-coloring. A *cycle double cover* (CDC) $\mathscr{C}$ of a graph G is a collection of cycles in G such that every edge of G belongs to exactly two cycles of $\mathscr{C}$. Note that the cycles are not necessarily distinct. It can be easily seen that a necessary condition for a graph to have a CDC is that the graph be bridgeless. Seymour [5] in 1979 conjectured that this condition is also sufficient.

Conjecture 1 (CDC conjecture, Seymour [5]). Every bridgeless graph has a cycle double cover.

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PII: S0012-365X(99)00356-8

Actually Seymour stated: 'Let G be a bridgeless graph; then there is a list of circuits (closed trails) of G with each edge in precisely two of them'. Given the fact that every circuit is an edge disjoint union of cycles, it is obvious that this statement is equivalent to the conjecture stated above. Also, Szekeres [7] in 1973 had conjectured that every bridgeless cubic graph has a CDC, which is now known to be equivalent to the above conjecture.

The idea of the conjecture above comes from the fact that the set of faces of a planar graph (including the infinite face) is a set of circuits that cover every edge exactly twice. The CDC conjecture has many stronger forms, one of which is the following:

Conjecture 2 (Oriented CDC conjecture). Every bridgeless graph has a cycle double cover in which every cycle can be oriented in such a way that every edge of the graph is covered by two directed cycles in two different directions.

By replacing every edge of an arbitrary graph G by a path of length 2, we obtain a bipartite graph G'' such that every (oriented) CDC in G'' corresponds to an (oriented) CDC in G. Thus, we have established the following proposition.

Proposition 1. *The oriented CDC conjecture in the case of bipartite graphs is equivalent to the oriented CDC conjecture in the general case.*

Recently, Cameron [4], stated a conjecture called the *simultaneous edge-coloring* (SE) conjecture which is, in fact, a reformulation of a conjecture by Keedwell [2] on the existence of critical partial latin squares (CPLS) of a given type. Before stating the conjecture, we define the concept of a 2-simultaneous coloring.

Definition. Let G be a graph. A 2-*simultaneous coloring* of G is a pair of edge-colorings of G such that

- for each vertex, the sets of colors appearing on the edges incident to that vertex are the same in both colorings;
- no edge receives the same color in both colorings.

If G has a 2-simultaneous coloring, then G is called a 2-*simultaneous colorable* graph.

In fact, 2-simultaneous colorable graphs for graph colorings play a role such as the role of trades in block designs [6]. Therefore, this concept has applications in the study of the defining sets of graph colorings and uniquely colorable graphs [3].

Let G be a bipartite graph with bipartition (X,Y). The *bipartite degree sequence* of G is the sequence $(x_1,x_2,\dots,x_n; y_1,y_2,\dots,y_m)$ where $(x_1,x_2,\dots,x_n)$ are the vertex degrees in X and $(y_1,y_2,\dots,y_m)$ are the vertex degrees in Y. A sequence S of positive integers is called a *bipartite graphic sequence* if there exists a bipartite graph G whose bipartite degree sequence is S.

Conjecture 3 (*SE conjecture, Problems from the 16th British Combinatorial Conference* [4]). For each bipartite graphic sequence S with all its elements greater than 1, there exists a simple bipartite graph G whose bipartite degree sequence is S and it has a 2-simultaneous coloring.

We have verified the truth of the SE conjecture for all $M \leqslant 28$ using computer programs, where $M = x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_m$.

In this paper, we study the relationship between the oriented CDC conjecture and the SE conjecture. In the next section, this relationship is discussed and the SE conjecture is verified in some special cases. Finally, in Section 3 we mention some open problems related to these conjectures.

2. The strong SE conjecture

Cameron in [4] noted that it is not true that every bipartite graph G with $\delta(G) > 1$ is 2-simultaneous colorable. His example is a graph which consists of two 4-cycles with an extra edge joining them together. It is easy to verify that this graph is not 2-simultaneous colorable.

We will see in Theorem 2 that every bipartite graph which has a cut edge does not have a 2-simultaneous coloring. We conjecture that the converse of this proposition is true:

Conjecture 4 (Strong SE conjecture). Every bridgeless bipartite graph has a 2-simultaneous coloring.

We have checked that the above conjecture is true for every graph with less than or equal to 22 edges using computer programs. The following theorem shows that the above conjecture actually is stronger than the SE conjecture.

Theorem 1. *The strong SE conjecture implies the SE conjecture.*

Proof. It suffices to show that if we have a bipartite graphic sequence S, then there exists a 2-edge-connected simple bipartite graph G with this bipartite degree sequence.

Let G be a graph with the bipartite degree sequence S which has the minimum number of cut edges. Assume that e is one of the cut edges of G. Let G_1 and G_2 be two components of $G - e$. Both G_1 and G_2 contain at most one vertex of degree 1. So, there exists a cycle C_1 in G_1 and a cycle C_2 in G_2. Now, we choose an edge $u_1v_1 \in C_1$ and an edge $u_2v_2 \in C_2$, neither being adjacent to e, and replace them with two edges u_1v_2 and u_2v_1. This replacement does not change the bipartite degree sequence of G, and also the resulting graph is still bipartite assuming that u_1 and u_2 belong to the same vertex part of G. After the replacement, e is not a cut edge anymore, and also it can be easily seen that no new cut edge is added by this process. This contradicts the minimality of the number of cut edges of G. □

Now, we are ready to prove the following theorem.

Theorem 2. *The strong SE conjecture is equivalent to the oriented CDC conjecture.*

Proof. First, assume that the oriented CDC conjecture is true. Let G be a bridgeless bipartite graph with bipartition (X, Y), let $\{C_1, C_2, \ldots, C_m\}$ be an oriented CDC for G and let $x \in X$, $y \in Y$. We color those edges xy of the oriented cycle C_i which are oriented in the direction x to y with colour i in the first coloring and those which are oriented in the direction y to x with color i in the second coloring. Since, by assumption, each edge xy is covered by two cycles, one in the direction x to y and the other in the direction y to x, each edge receives two distinct colors and so the two colorings define a 2-simultaneous coloring for G.

Conversely, assume that the Strong SE conjecture is true. By Proposition 1, we know that it is sufficient to prove the oriented CDC conjecture for bipartite graphs. Let G be a bridgeless bipartite graph. By the Strong SE conjecture there exists a 2-simultaneous coloring for G. Let A_i (B_i, respectively) be the set of all edges which are colored with the color i in the first (second, respectively) coloring. Let $C_i = A_i \cup B_i$. Obviously, $G[C_i]$ is a collection of disjoint cycles. For every i these cycles form a CDC for G. We will give an orientation for these cycles as follows. Orient each edge xy ($x \in X$ and $y \in Y$) in a cycle of $G[C_i]$ from x to y if i is its first color and from y to x if i is its second color. □

The oriented CDC conjecture is true for Eulerian graphs, because every Eulerian graph has a cycle decomposition. Now we prove that it is also true for semi-Eulerian graphs. This shows that the SE conjecture is true for all sequences which have all elements even except possibly two elements.

Theorem 3. *The Strong SE conjecture is true for semi-Eulerian bipartite graphs.*

Proof. Let G be a bridgeless bipartite graph whose vertices have even degrees, except for two vertices u and v. By Menger's theorem, there exist two edge-disjoint paths P_0 and P_1 between u and v. We claim that there are actually three edge-disjoint paths between them. We remove the edges of P_1, to obtain a new graph G'. In G', every vertex has even degree. Also we know that u and v are connected in G' by the path P_0. Thus u and v fall in the same connected component of G'. Since every Eulerian graph in bridgeless, again we use the Menger theorem in the connected component of G' which contains u and v. This shows that there are two edge-disjoint paths P_2 and P_3 between u and v.

Now, we remove the edges of the three paths P_1, P_2, and P_3 from G to obtain a new graph G''. Every connected component of G'' is an Eulerian graph and thus has an oriented CDC. It is sufficient to find an oriented CDC for $P_1 \cup P_2 \cup P_3$. Consider the directed graph $P_i \cup P_j$ where the edges of P_i are oriented from u to v and the edges of P_j are oriented from v to u. This digraph is an Eulerian digraph and therefore has a directed cycle decomposition $\mathscr{C}_{ij}$. Now $\mathscr{C}_{12} \cup \mathscr{C}_{23} \cup \mathscr{C}_{31}$ is an oriented CDC for $P_1 \cup P_2 \cup P_3$. □

3. Some open problems

In the previous sections, we discussed 2-simultaneous coloring. It is natural to generalize this concept to k-simultaneous coloring.

Definition. Let G be a graph. A k-simultaneous coloring of G is a set of k edge-colorings of G such that

- for each vertex, the sets of colors appearing on the edges incident to that vertex are the same in each coloring;
- no edge receives the same color in any two colorings.

If G has a k-simultaneous coloring, then G is called a *k-simultaneous colorable* graph.

Now, we ask the following question.

Problem. Characterize all k-simultaneous colorable graphs.

In fact the strong SE conjecture is a characterization of all 2-simultaneous colorable bipartite graphs. In the general case, it is easy to see that every 2-simultaneous colorable graph should have a (not necessarily orientable) CDC. Thus, having no cut-edge is still a necessary condition for 2-simultaneous colorability. But this condition is not sufficient. For example, every odd cycle is not 2-simultaneous colorable. Also, the graph $K_4 - e$ is another example which is not 2-simultaneous colorable.

The following theorem shows that an infinite family of k-simultaneous colorable graphs exists.

Theorem 4. *Every r-regular r-edge colorable graph is k-simultaneous colorable for every $k \leqslant r$.*

Proof. Let G be an r-regular r-edge colorable graph. Let $c : E(G) \mapsto \{0, 1, \ldots, r-1\}$ be an r-edge coloring of G. We define the coloring c_i for $0 \leqslant i \leqslant r-1$ by $c_i(e) \equiv c(e) + i \pmod r$. It is easy to verify that $c_0, c_1, \ldots, c_{r-1}$ are r different colorings which satisfy the conditions of an r-simultaneous coloring. □

Acknowledgements

Research supported by The National Research Council of I.R. Iran (NRCI) as a National Research Project under the grant number 2559.

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