Large Induced Forests in Triangle-free Planar Graphs

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Abstract. Given a planar graph G, what is the largest subset of vertices of G that induces a forest? Albertson and Berman [2] conjectured that every planar graph has an induced subgraph on at least half of the vertices that is a forest. For bipartite planar graphs, Akiyama and Wanatabe [1] conjectured that there is always an induced forest of size at least $\frac{5n}{8}$. Here we prove that every triangle-free (and therefore every bipartite) planar graph on n vertices has an induced forest of size at least $\frac{17n+24}{32}$.

1. Introduction

All graphs in this paper are simple and finite. Consider a graph G with vertex set V and edge set E and let R be a subset of vertices. The subgraph of G induced by R is denoted by G_R. If G_R is a forest then we call it an induced forest of G. The following conjecture was posed by Albertson and Berman [2]:

Conjecture 1. [2] Every planar graph has an induced forest with least half of the vertices.

Note that this conjecture, if true, would immediately imply that every planar graph has an independent set with at least one-quarter of the vertices, a fact whose only known proof relies on the Four Color Theorem.

Another formulation of this conjecture is in terms of *decycling number*. The decycling number of a graph G is the smallest number of vertices which can be removed from G so that the resultant graph has no cycles. So Conjecture 1 states that the decycling number of every planar graph G on n vertices is at most $\frac{n}{2}$. Finding the decycling number of a graph is NP-hard, even restricted to planar graphs, bipartite graphs, and perfect graphs. Therefore, finding the largest induced forest of a given planar graph is a hard problem. For this reason, it is natural to try to find lower bounds for the size of the largest induced forest of a planar graph.

A coloring of a graph G is acyclic if the union of every two color-classes is an acyclic graph. The best known lower bound on the size of the largest induced forest of a planar graph is due to Borodin [6], who proved that every planar graph has an acyclic 5-coloring. This implies that every planar graph on n vertices has an induced forest of size at least $\frac{2n}{5}$. For outerplanar graphs, Hosono [8] proved that there is always an induced forest

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with at least $\frac{2n}{3}$ vertices and this is best possible. Borodin and Glebov [7] have proved a stronger result for planar graphs of girth at least 5: the vertex set of a such graph can be partitioned into an independent set and another set that induces a forest. For bipartite planar graphs, Akiyama and Wanatabe [1] raised the following conjecture:

Conjecture 2. [1] Every bipartite planar graph with n vertices has an induced forest of size at least $\frac{5n}{8}$.

This conjecture, if true, is sharp as shown by Q_3 (the cube). Motivated by this conjecture, Alon [3] and Alon et al. [5] studied the size of the largest induced forests in bipartite and sparse graphs. Alon et al. [5] showed, among other things, that every (not necessarily planar) triangle-free graph with n vertices and m edges has an induced forest of size $n-\frac{m}{4}$. Since in every triangle-free planar graph $m \leq 2n-4$, their result implies a lower bound of $\frac{n}{2}+1$ for the size of the largest induced forest in triangle-free planar graphs. Here we prove a better lower bound in terms of the number of vertices. Our main result is:

Theorem 1. *Every triangle-free planar graph on n vertices and m edges has an induced forest of size at least $\lceil\frac{29n-6m}{32}\rceil$.*

Again, since every triangle-free planar graph has at most $2n-4$ edges, Theorem 1 implies:

Corollary 1. *Every triangle-free planar graph on n vertices has an induced forest of size at least $\lceil\frac{17n+24}{32}\rceil$.*

Let's define $f(n)$ as the minimum, over all n vertex triangle-free planar graphs, of the maximum size of an induced forest and $\gamma = \lim_{n\to\infty}\frac{f(n)}{n}$. It can be seen that γ exists and (by [5] and because of the cube): $\frac{1}{2} \leq \gamma \leq \frac{5}{8}$. Therefore, Corollary 1 improves the lower bound for γ to $\frac{17}{32} \leq \gamma$. Our proof uses the Discharging Method and is constructive. That is, it yields a quadratic time algorithm that, given a triangle-free planar graph G, finds an induced forest of G on at least $\frac{29n-6m}{32}$ vertices in time $O(n^2)$. In the rest of this section, we explain the notation and definitions. Then we prove some properties for a possible minimum counter-example to Theorem 1 in the next section and prove the theorem based on those properties. Sections 3 to 5 contain more details of the proof.

The vertex set, edge set, and face set of an embedded planar graph G are denoted by $V(G)$, $E(G)$, and $F(G)$ (or simply V, E, and F), respectively. Degree of a vertex $v \in V$, denoted by $d(v)$, is the number of edges incident with it. The minimum and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$ (or simply δ and Δ), respectively. A k-cycle is a cycle of size k. The size of a face $f \in F$, denoted by $|f|$, is the number of edges in the boundary of F, counting cut-edges (bridges) twice. We call a vertex of degree i, at least i, and at most i, an i-vertex, a $\geq i$-vertex, and a $\leq i$-vertex, respectively. We define an i-face, $\geq i$-face, and $\leq i$-face, similarly. Through a slight abuse of notation, we say "vertices of a face f" to refer to the vertices that are on the boundary of f. We also write $f = v_1v_2\dots v_k$, if $v_1,\dots,v_k$ are the vertices on the boundary of f, consecutively. The size of an induced forest G_R of a graph G (induced by $R \subseteq V$) is the number of vertices of R. We denote the class of triangle-free planar graphs by $\mathcal{G}$. For a subset $V' \subseteq V$, by $G - V'$ we mean the subgraph of G induced by the vertices in $V - V'$. By $G + E'$, where E' is a subset of edges on V, we mean the graph with vertex set V and edge set $E \cup E'$. If $E' \subseteq E$, by $G - E'$ we mean the graph obtained from G by deleting the edges in E'.

2. Proof of Theorem 1

By way of contradiction, assume that Theorem 1 is false and let $G \in \mathcal{G}$ together with a planar embedding of it be a counter-example to this theorem with the minimum number of vertices. Let $n = |V(G)|$, $m = |E(G)|$, and $\varphi = \frac{29n-6m}{32}$. Clearly G is connected and has at least five vertices.

Lemma 1. $\Delta(G) \leq 4$.

Proof. Let v be a ≥ 5-vertex of G. By definition of G, $G' = G - v$ has an induced forest of size $\frac{29(n-1)-6(m-5)}{32} > \varphi$. Clearly that is an induced forest of G as well. □

Lemma 2. *G is 2-edge-connected, i.e. bridge-less.*

Proof. Let e be a bridge in G and C_1 and C_2 be the two connected components of $G - e$, having n_1 and n_2 vertices and m_1 and m_2 edges, respectively. By definition of G, C_1 and C_2 have induced forests of size $\frac{29n_1-6m_1}{32}$ and $\frac{29n_2-6m_2}{32}$, respectively. The union of these two gives an induced forest in G of size $\frac{29n-6(m_1+m_2)}{32} > \varphi$. □

The following three lemmas are proved in the next three sections.

Lemma 3. $\delta(G) \geq 3$.

Lemma 4. *No 5-face in G has more than three 3-vertices.*

Lemma 5. *No 4-face in G has a 3-vertex.*

Now we prove that the properties listed above lead to a contradiction. To each vertex $v \in V$ we assign a charge of $ch(v) = d(v) - 6$ and to each face $f \in F$ we assign $ch(f) = 2|f| - 6$. By Euler's formula, $|V| - |E| + |F| = 2$, the total charge is $\sum_{x \in V \cup F} ch(x) = -12$. Recall that by Lemma 3, G has no $\leq$2-vertices. Therefore, by this set of initial charges and by Lemmas 3 and 1, the only elements in $V \cup F$ that have negative charges are 3-, and 4-vertices, having charges -2 and -1, respectively. Now we redistribute the charges by applying the following discharging rule:

Discharging rule: Every face sends 1 to each of its 3-vertices and $\frac{1}{2}$ to each of its 4-vertices.

By this rule, vertices do not lose any charges in the discharging phase. Also, every 3-vertex and every 4-vertex receives a total of 3 and 2 units of charge, respectively, from the faces it is incident with (recall that G is bridge-less). Therefore:

Lemma 6. *For every vertex v has non-negative charge.*

Lemma 7. *Every face f has non-negative charge.*

Proof. Let $|f| = k$. If $k \geq 6$ then f sends at most k units of charge to its vertices, which is at most $2|f| - 6$ for $k \geq 6$. If $k = 5$ then, by Lemma 4, f has at most three 3-vertices and therefore, it sends at most $3 \times 1 + 2 \times \frac{1}{2} = 4 = 2|f| - 6$ to its vertices. If $k = 4$ then, by Lemma 5, it has only 4-vertices and sends $4 \times \frac{1}{2} = 2|f| - 6$ units of charge to them. □

By Lemmas 6 and 7, the total charge is non-negative, while the initial charge was -12. This contradiction completes the proof of Theorem 1. The algorithm for finding an induced forest of size at least φ has at most n iterations. In each iteration we apply the initial charges and the discharging rule as described above. Since in a planar graph the number of edges and faces is linear in the number of vertices, applying the initial charges and the discharging rule takes $O(n)$ time. Then we can find an element with negative charge in $O(n)$. This element must be in a configuration of vertices and faces as described in Lemmas 1 to 5. Once we find such a structure, using the proof of the corresponding lemma we construct graph G' which has fewer vertices. Then we solve the problem on G', recursively. Having an induced forest in G', again the proof of the corresponding lemma shows how to modify it into an induced forest of size at least φ in G. Therefore, the overall running time of the algorithm will be $O(n^2)$.

3. Proof of Lemma 3

Note that by Lemma 1 every vertex in G is a ≤ 4-vertex. If v is a 1-vertex in G then $G' = G - v$ has a forest induced by a set R' of size at least $\frac{29(n-1)-6(m-1)}{32} > \varphi - 1$. Clearly $R = R' \cup \{v\}$ induces a forest in G. So $\delta(G) \geq 2$. Our next goal is to show that every 2-vertex is adjacent to 3-vertices only.

Lemma 8. *G does not have a 2-vertex adjacent to a 4-vertex.*

Proof. Otherwise, if u is a 2-vertex adjacent to a 4-vertex v then we set $G' = G - \{u, v\}$. Clearly $G' \in \mathcal{G}$ and so there is a set $R' \subseteq V(G')$ of size $\frac{29(n-2)-6(m-5)}{32} \geq \varphi - 1$ that induces a forest in G'. It is easy to see that $R = R' \cup \{u\}$ induces a forest (of size $\geq \varphi$) in G, since u adjacent to at most one vertex in R', i.e. it's a leaf in G_R (subgraph of G induced by R). $\square$

Lemma 9. *G does not have a 2-vertex adjacent to a 2-vertex and a 3-vertex.*

Proof. Assume that u is a 2-vertex adjacent to a 2-vertex v and a 3-vertex w. Then $G' = G - \{u, v, w\}$ has a forest induced by a set R' of size $\frac{29(n-3)-6(m-5)}{32} > \varphi - 2$. It follows that $R = R' \cup \{u, v\}$ induces a forest (of size $\geq \varphi$) in G. $\square$

Lemma 10. *G does not have a 2-vertex adjacent to two 2-vertices.*

Proof. Assume that u is a 2-vertex adjacent to 2-vertices v and w.
Case 1: If v and w have another common neighbor x then, by previous lemma (applied to v) and by Lemma 1, $d(x) = 3$. But this implies that x is incident to a bridge, contradicting Lemma 2.
Case 2: If x is the other neighbor of w and is not adjacent to v then $G' = G - \{u, v, w, x\}$ has a forest induced by a set R' of size $\frac{29(n-4)-6(m-5)}{32} > \varphi - 3$. Then $R = R' \cup \{u, v, w\}$ induces a forest (of size $\geq \varphi$) in G. $\square$

Lemmas 8 to 10 imply that every 2-vertex is adjacent to 3-vertices only.

Lemma 11. *G does not have a 3-vertex adjacent to two 2-vertices.*

Proof. Let u be a 3-vertex adjacent to two 2-vertices v and w, and let $G' = G - \{u, v, w\}$. Then G' has a forest induced by a set R' of size $\frac{29(n-3)-6(m-5)}{32} > \varphi - 2$. It's easy to see that $R = R' \cup \{v, w\}$ induces a forest (of size $\geq \varphi$) in G (with both v and w being leaves in G_R). $\square$

Lemma 12. *G does not have a 4-face with a 2-vertex.*

Proof. Assume that $f = v_1v_2v_3v_4$ is a 4-face with $d(v_1) = 2$. Therefore, by Lemmas 8 to 10: $d(v_2) = d(v_4) = 3$, and by Lemma 11: $d(v_3) \geq 3$.

Case 1: Suppose there is a vertex $u \notin \{v_1, v_3\}$ adjacent to both v_2 and v_4. Note that $uv_3 \notin E$ as G is triangle-free. If $d(u) \geq 4$ or $d(v_3) \geq 4$ then $G' = G - \{v_1, \ldots, v_4, u\}$ has a forest induced by a set R' of size $\frac{29(n-5)-6(m-9)}{32} > \varphi - 3$. Therefore, $R = R' \cup \{v_1, v_2, v_4\}$ induces a forest (of size $\geq \varphi$) in G. So let's assume that $d(u) = d(v_3) = 3$.

If v_2 and v_4 are the only common neighbors of v_3 and u then $G' = G - \{v_1, v_2, v_4\} + \{uv_3\}$ (the graph obtained from G by removing vertices v_1, v_2, v_4 and adding edge uv_3) is triangle-free and planar. By definition of G, G' has a forest induced by a set R' of size $\frac{29(n-3)-6(m-5)}{32} > \varphi - 2$. Let $R = R' \cup \{v_1, v_4\}$ (note that R has at least φ vertices). Since $uv_3 \in G'$ and $uv_3 \notin G$, u and v_3 are not in the same connected component of $G_{R'}$. Therefore, the possible neighbors of v_4 in R', i.e. v_3 and u, are not in the same connected component of $G_{R'}$. So G_R is a forest.

If w is another common neighbor of v_3 and u then $d(w) \geq 3$, or else G is only the graph on $\{v_1, \ldots, v_4, u, w\}$ which trivially has an induced forest of size φ. If w is a 3-vertex adjacent to x (and to u and v_3) then x is incident to a bridge, contradicting Lemma 2. If $d(w) = 4$ then $G' = G - \{v_1, \ldots, v_4, u, w\}$ has a forest induced by a set R' of size $\frac{29(n-6)-6(m-10)}{32} > \varphi - 4$. So $R = R' \cup \{v_1, v_2, v_3, u\}$ induces a forest (of size $\geq \varphi$) in G.

Case 2: Suppose v_1 and v_3 are the only common neighbors of v_2 and v_4. Recall that $d(v_2) = d(v_4) = 3$. Let u be the other neighbor of v_2 and w be the other neighbor of v_4. First assume that at least one of u or w, say u, is a 4-vertex. Then $G' = G - \{v_1, \ldots, v_4, u\}$ has a forest induced by a set R' of size $\frac{29(n-5)-6(m-10)}{32} > \varphi - 3$. It's easy to see that $R = R' \cup \{v_1, v_2, v_4\}$ induces a forest (of size $\geq \varphi$) in G, with v_2 being a leaf.

So let's assume that $d(u) = d(w) = 3$. If $uw \in E$ then $G' = G - \{v_1, \ldots, v_4, u, w\}$ has a forest induced by a set R' of size $\frac{29(n-6)-6(m-10)}{32} > \varphi - 4$. Since only one neighbor of u remains in G', $R = R' \cup \{u, v_1, v_2, v_4\}$ induces a forest of size at least φ in G. If $uw \notin E$ then let $G' = G - \{v_1, v_2, v_3\} + \{uv_4\}$. It's easy to see that G' is planar. Also, the only case in which G' has a triangle is when uv_4 belongs to a triangle. This happens only if $uw \in G'$ (since the only neighbor of v_4 in G' is w), but $uw \notin G$ and therefore $uw \notin G'$. Thus $G' \in \mathcal{G}$ and has a forest induced by a set R' of size $\frac{29(n-3)-6(m-5)}{32} > \varphi - 2$. Then $R = R' \cup \{v_1, v_2\}$ induces a forest (of size $\geq \varphi$) in G, because $uv_4 \in G'$ but $uv_4 \notin G$, so u and v_4 are not in the same connected components of $G_{R'}$. This implies that G_R is a forest. □

Now we are ready to prove Lemma 3. Let u be a 2-vertex in G which belongs to two faces f_1 and f_2 and is adjacent to v and w. By Lemma 2: $f_1 \neq f_2$ and by Lemmas 8 to 10: $d(v) = d(w) = 3$. By Lemma 12 both of f_1 and f_2 are $\geq$ 5-faces. Therefore, v and w do not have any common neighbor other than u. Let $G' = G - \{u\} + \{vw\}$. Clearly G' is planar and because v and w do not have any other common neighbor, vw does not create any triangle in G', i.e. $G' \in \mathcal{G}$. So it has a forest induced by a set R' of size $\frac{29(n-1)-6(m-1)}{32} > \varphi - 1$. We claim that $R = R' \cup \{u\}$ induces a forest (of size $\geq \varphi$) in G: Since $vw \in G'$ and $vw \notin G$, v and w are not in the same connected components of $G_{R'}$. That is u is connecting two different connected components of $G_{R'}$, i.e. G_R is a forest. This completes the proof of Lemma 3.

4. Proof of Lemma 4

Note that by Lemmas 1 and 3, from now on, we can assume that every vertex in G has degree 3 or 4. Lemma 4 follows easily from Lemmas 13 and 14 below.

Lemma 13. *G does not have a 5-cycle with exactly four 3-vertices.*

Proof. Assume that $C = v_1v_2v_3v_4v_5$ is a 5-cycle, with $v_1, \dots, v_4$ all being 3-vertices and $d(v_5) = 4$. Let $u_1, \dots, u_4$ be the third neighbors of $v_1, \dots, v_4$, respectively. Note that u_i's may or may not be distinct, but since G is triangle-free: $u_i \neq u_{i+1}$ $(1 \leq i \leq 3)$. So if they are not distinct then $u_1 = u_3$, or $u_2 = u_4$, or $u_1 = u_4$. Below we consider these possibilities.

If u_1 is distinct from u_3 and u_4 then, with $G' = G - \{v_1, \dots, v_5, u_1\}$: $|E(G)| - |E(G')| \geq 13$. Clearly $G' \in \mathcal{G}$ and therefore, it has a forest induced by a set $R' \subseteq V(G')$ of size at least $\frac{29(n-6)-6(m-13)}{32} = \varphi - 3$. It's not hard to see that $R = R' \cup \{v_1, v_2, v_4\}$ induces a forest (of size $\geq \varphi$) in G, because each of v_2 and v_4 is adjacent to at most one vertex in R' and v_1 is adjacent to v_2 only.

If $u_1 = u_4$ or if $u_1 = u_3$ and u_2 is distinct from u_4, then let $G' = G - \{v_1, \dots, v_5, u_2\}$. Since $G' \in \mathcal{G}$ and $|E(G)| - |E(G')| \geq 13$, G' has a forest induced by a set R' of size $\frac{29(n-6)-6(m-13)}{32} = \varphi - 3$. It is easy to see that $R = R' \cup \{v_1, v_2, v_4\}$ induces a forest (of size $\geq \varphi$) in G.

Finally, consider the case that $u_1 = u_3$ and $u_2 = u_4$. This can happen if one of u_1 or u_2 is inside the 5-cycle C and the other is outside of C. In this case let $G' = G - \{v_1, \dots, v_5, u_1, u_2\}$. Since $G' \in \mathcal{G}$ and $|E(G)| - |E(G')| \geq 13$, G' has a forest induced by a set R' of size at least $\frac{29(n-7)-6(m-13)}{32} \geq \varphi - 4$. Now $R = R' \cup \{v_1, v_2, v_3, v_4\}$ induces a forest of size at least φ in G, a contradiction. $\square$

Lemma 14. *G does not have a 5-face with five 3-vertices.*

Proof. Suppose that $f = v_0v_1v_2v_3v_4$ and all v_i's have degree 3. Let u_i be the third neighbor of v_i $(0 \leq i \leq 4)$. The following statements are true:

-(i) u_i's are distinct: Because G is triangle-free u_i is distinct from u_{i+1}, $(0 \leq i \leq 4$, where all the additions for indices are in mod 5). If, for instance, $u_0 = u_2$, then no other u_i $(i \neq 2)$ can be equal to u_0. Also, by planarity, we cannot have $u_1 = u_3$ at the same time as $u_0 = u_2$. So, by symmetry, the only possibility is when $u_0 = u_2$. In this case let $G' = G - \{v_0, \dots, v_4, u_0\}$. Because $G' \in \mathcal{G}$ and $|E(G)| - |E(G')| \geq 11$, G' has a forest induced by a set R' of size at least $\frac{29(n-6)-6(m-11)}{32} > \varphi - 4$. It's not hard to see that $R = R' \cup \{v_0, v_1, v_2, v_3\}$ induces a forest (of size $\geq \varphi$) in G. This is because each of v_1 and v_3 is adjacent to at most one vertex in R' (namely u_1 and u_3), and u_1 and u_3 are in different connected components of $G'_{R'}$, because they are on different sides of the separating cycle $v_0v_1v_2u_0$.

-(ii) $d(u_i) = 3$ $(0 \leq i \leq 4)$: If one of u_i's, say u_0, is a 4-vertex then let $G' = G - \{v_0, \dots, v_4, u_0\}$. Since $G' \in \mathcal{G}$ and $|E(G)| - |E(G')| = 13$, G' has a forest induced by a set R' of size $\frac{29(n-6)-6(m-13)}{32} = \varphi - 3$. Then $R = R' \cup \{v_0, v_1, v_3\}$ induces a forest (of size $\geq \varphi$) in G.

-(iii) $u_iu_j \notin E$ $(0 \leq i, j \leq 4)$: Without loss of generality, we fix $i = 0$ and consider the cases when $u_0u_1 \in E$ or $u_0u_2 \in E$ (the other situations reduce to one of these two by symmetry). If $u_0u_1 \in E$ then we set $G' = G - \{v_0, \dots, v_4, u_0, u_1\}$. Because $G' \in \mathcal{G}$ and $|E(G)| - |E(G')| = 13$, G' has a forest induced by a set R' of size $\frac{29(n-7)-6(m-13)}{32} > \varphi - 4$. Then, as each of u_0 and v_3 is adjacent to at most one vertex in R', $R = R' \cup \{u_0, v_0, v_1, v_3\}$

induces a forest (of size $\geq \varphi$) in G. If $u_0u_2 \in E$ then we set $G' = G - \{v_0, \ldots, v_4, u_0, u_2\}$. Again $G' \in \mathcal{G}$ and $|E(G)| - |E(G')| = 13$, so G' has a forest induced by a set R' of size at least $\varphi - 4$. It follows that $R = R' \cup \{u_0, v_1, v_2, v_4\}$ induces a forest (of size $\geq \varphi$) in G. This is because each of u_0, v_1, and v_4 is adjacent to at most one vertex in R' and v_2 is only adjacent to v_1 in R.

-(iv) There is a vertex w adjacent to both u_0 and u_1: If not, $G' = G - \{v_0, \ldots, v_5\} + \{u_0u_1\}$ is in $\mathcal{G}$ and has a forest induced by a set R' of size $\frac{29(n-5)-6(m-9)}{32} > \varphi - 3$. Then, $R = R' \cup \{v_0, v_1, v_3\}$ induces a forest (of size $\geq \varphi$) in G, because u_0 and u_1 are in different connected components of $G_{R'}$ (as u_0u_1 is in G' and not in G). So adding v_0 and v_1 does not create a cycle, i.e. G_R is a forest.

So $C' = v_0u_0wu_1v_1$ is a 5-cycle with all u_0, v_0, v_1, u_1 having degree 3. Therefore, by Lemma 13: $d(w) = 3$, and by (i) above applied to C': $wv_2 \notin E$ and $wv_4 \notin E$. Also, by (iii) applied to C': $wu_2 \notin E$ and $wv_3 \notin E$. So with $G' = G - \{v_0, \ldots, v_4, u_0, u_1, u_2, w\}$: $|E(G)| - |E(G')| \geq 17$. Since $G' \in \mathcal{G}$, G' has a forest induced by a set R' of size $\frac{29(n-9)-6(m-17)}{32} > \varphi - 5$. Because each of u_0, u_1, v_4 is adjacent to at most one vertex in R', $R = R' \cup \{u_0, u_1, v_1, v_2, v_4\}$ induces a forest (of size $\geq \varphi$) in G. This completes the proof of lemma 14. □

5. Proof of Lemma 5

Lemma 5 follows easily from Lemmas 15 to 18 below.

Lemma 15. *G does not have a 4-cycle with exactly one 4-vertex.*

Proof. let $C = v_0v_1v_2v_3$ be a 4-cycle with $d(v_0) = 4$ and $d(v_1) = d(v_2) = d(v_3) = 3$. Define $G' = G - \{v_0, \ldots, v_3\}$. Note that because G is triangle-free, $v_0v_2 \notin E$ and $v_1v_3 \notin E$. Therefore, $|E(G)| - |E(G')| = 9$ and as G' has a forest induced by a set R' of size $\frac{29(n-4)-6(m-9)}{32} > \varphi - 2$. Clearly $R = R' \cup \{v_1, v_3\}$ induces a forest (of size $\geq \varphi$) in G. □

Lemma 16. *G does not have a 4-cycle with four 3-vertices.*

Proof. Let $C = v_0v_1v_2v_3$ be a 4-cycle with $d(v_i) = 3$ and u_i be the third neighbor of v_i $(0 \leq i \leq 3)$.

-(i) u_i's are distinct: As G is triangle-free, u_i is distinct from u_{i+1}. So, without loss of generality, assume that $u_0 = u_2$ and let $G' = G - \{v_0, \ldots, v_3, u_0\}$. Because $G' \in \mathcal{G}$ and $|E(G)| - |E(G')| \geq 9$, G' has a forest induced by a set R' of size $\frac{29(n-5)-6(m-9)}{32} > \varphi - 3$. It follows easily that $R = R' \cup \{v_0, v_1, v_2\}$ induces a forest (of size $\geq \varphi$) in G, with v_0 and v_2 being leaves in G_R.

-(ii) C is a 4-face, i.e. all u_i's are either inside or outside of cycle C: By way of contradiction, assume that C is a separating cycle. At least two of $u_0, \ldots, u_3$ are inside or at least two of them are outside of C. Without loss of generality, suppose that u_0 and u_i (for some $1 \leq i \leq 3$) are outside of C while u_j (for some $1 \leq j \leq 3$ with $j \neq i$) is inside of C. Let $G' = G - \{v_0, \ldots, v_3, u_0\}$. Clearly G' has a set R' of size at least $\frac{29(n-5)-6(m-10)}{32} > \varphi - 3$ of vertices that induces a forest. It is easy to see that $R = R' \cup \{v_0, v_i, v_j\}$ induces a forest in G, because the only possible neighbors of $\{v_0, v_i, v_j\}$ in R' are u_i and u_j which are on different sides of cycle C, and therefore, u_i and u_j are in different connected components of $G'_{R'}$.

-(iii) All u_i's have degree 3: Without loss of generality, assume that $d(u_0) = 4$ and let $G' = G - \{v_0, \ldots, v_3, u_0, u_1\}$. If $u_0u_1 \notin G$ then $|E(G)| - |E(G')| \geq 13$ (note that u_i's are all distinct). If $u_0u_1 \in G$ then by applying Lemma 15 to the 4-cycle $u_0u_1v_1v_0$: $d(u_1) = 4$, and again $|E(G)| - |E(G')| \geq 13$. Therefore, G' has a forest induced by a set R' of size $\frac{29(n-6)-6(m-13)}{32} = \varphi - 3$. Clearly $R = R' \cup \{v_0, v_1, v_2\}$ induces a forest (of size $\geq \varphi$) in G, because the only possible neighbors of $\{v_0, v_1, v_2\}$ in R' is u_2.

-(iv) $u_0u_2 \notin E$ **and** $u_1u_3 \notin E$**:** Without loss of generality, assume that $u_0u_2 \in E$ and let x be the third neighbor of u_0 (other than v_0 and u_2). Define $G' = G - \{v_0, \ldots, v_3, u_0\}$. Since $G' \in \mathcal{G}$, it has a forest induced by a set R' of size at least $\frac{29(n-5)-6(m-10)}{32} > \varphi - 3$. Note that u_1 and u_3 are on different sides of separating cycle $C' = u_0v_0v_1v_2u_2$. If x and u_1 are on the same side (i.e. x and u_3 are on different sides) of C' let $R = R' \cup \{u_0, v_0, v_3\}$, and if x and u_3 are on the same side (i.e. x and u_1 are on different sides) of C' then let $R = R' \cup \{u_0, v_0, v_1\}$. It follows that in each case G_R is a forest (of size $\geq \varphi$).

-(v) $u_iu_{i+1} \notin E$ $(0 \leq i \leq 3)$**:** Without loss of generality, fix $i = 0$ and assume that $u_0u_1 \in E$. We consider two cases.

If $u_0u_3 \notin E$ and $u_1u_2 \notin E$ then let x be the third neighbor of u_0 and $G' = G - \{v_0, \ldots, v_3, u_0, u_1, x\}$. Clearly $G' \in \mathcal{G}$ and since $x \neq u_3$ (by the assumption that $u_0u_3 \notin E$) and $x \neq u_2$ (by (iv) above): $|E(G)| - |E(G')| \geq 13$, so G' has a forest induced by a set R' of size $\frac{29(n-7)-6(m-13)}{32} > \varphi - 4$. It's not hard to see that $R = R' \cup \{u_0, v_0, v_1, v_3\}$ induces a forest (of size $\geq \varphi$) in G, because from $\{u_0, v_0, v_1, v_3\}$ only v_3 can have (at most) one neighbor in R' (namely u_3).

Suppose that (at least) one of u_0u_3 or u_1u_2, say u_0u_3 is in E. If we set $G' = G - \{v_0, \ldots, v_3, u_0, \ldots, u_3\}$ then $|E(G)| - |E(G')| \geq 13$, and therefore it has a forest induced by a set R' of size $\frac{29(n-8)-6(m-13)}{32} > \varphi - 5$. It's not hard to see that $R = R' \cup \{u_0, u_1, v_0, v_3, v_2\}$ induces a forest in G. The reason is that among the vertices added to R' only u_1 can have (at most) one neighbor in R'.

-(vi) Graph $G + \{u_0u_1, u_1u_2, u_2u_3, u_3u_0\}$ **is triangle-free and planar:** By (ii) above, planarity is easy to see. To prove triangle-freeness, assume that $G + \{u_0u_1\}$ has a triangle. This means there exists a common neighbor of u_0 and u_1, call it x. Therefore, $C' = xu_0v_0v_1u_1$ is a 5-cycle, with four 3-vertices, u_0, v_0, v_1, u_1 (by (iii) all u_i's have degree 3). By Lemma 13 applied to C': $d(x) = 3$. Note that by (iv), x is distinct from u_2 and u_3. If we let $G' = G - \{v_0, \ldots, v_3, u_0, u_1, x\}$ then, G' has a forest induced by a set R' of size at least $\frac{29(n-7)-6(m-13)}{32} > \varphi - 4$. It's easy to see that $R = R' \cup \{x, v_0, v_1, v_2\}$ induces a forest (of size $\geq \varphi$) in G. Thus, adding u_0u_1 to G does not create any triangles. By symmetry we can add each of u_1u_2, u_2u_3, and u_3u_0, and by (iv), no two of these edges form a triangle. Therefore, $G + \{u_0u_1, u_1u_2, u_2u_3, u_3u_0\}$ is triangle-free, as wanted.

Now consider $G' = G + \{u_0u_1, u_1u_2, u_2u_3, u_3u_0\} - \{v_0, v_1, v_2, v_3\}$ which by (vi) is in $\mathcal{G}$. Therefore, it has a forest induced by a set R' of size $\frac{29(n-4)-6(m-4)}{32} > \varphi - 3$. Since $u_0u_1u_2u_3$ is a 4-cycle in G' at least one of $u_0, \ldots, u_3$ is not in R'. Without loss of generality assume that $u_0 \notin R'$. Then $R = R' \cup \{v_0, v_1, v_2\}$ induces a forest (of size $\geq \varphi$) in G, because u_1 and u_2 are in different connected components of $G_{R'}$ as u_1u_2 is in G' but not in G. □

Lemma 17. *G does not have a 4-cycle with exactly two 4-vertices.*

Proof. Let $C = v_0v_1v_2v_3$ be a 4-cycle with two 4-vertices. If the two 4-vertices of C are not adjacent, say $d(v_0) = d(v_2) = 4$ and $d(v_1) = d(v_3) = 3$, then the exact same proof of Lemma 15 works here. So suppose that the two 4-vertices of C are adjacent, say $d(v_0) = d(v_1) = 4$ and $d(v_2) = d(v_3) = 3$. Let $\{u_0, w_0\}$, $\{u_1, w_1\}$, $\{u_2\}$, and $\{u_3\}$ be the sets of neighbors of v_0, v_1, v_2, and v_3, respectively.

-(i) C **is a 4-face:** Let $G' = G - \{v_0, \ldots, v_3\}$. Since $G' \in \mathcal{G}$, it has a forest induced by a set R' of size at least $\frac{29(n-4)-6(m-10)}{32} > \varphi - 2$. If u_2 and u_3 are on different sides of C in G, then u_2 and u_3 are in different connected components of $G_{R'}$, and therefore $R = R' \cup \{v_2, v_3\}$ induces a forest (of size $\geq \varphi$) in G. So, u_2 and u_3 are on the same side of C. If u_0 and w_0 are on different sides of C then u_0 and w_0 are in different connected components of $G_{R'}$, and therefore $R = R' \cup \{v_0, v_2\}$ induces a forest in G. So both u_0 and w_0 are on the same side of C. Using a similar argument it follows that u_1 and w_1 are on the same side of C. Thus, if C is not a 4-face, u_2 and u_3 are on one side of C, while both of u_0 and w_0, or both of u_1 and w_1 are on the other side. Without loss of generality, assume that u_0 and w_0 are inside of C while u_2 and u_3 are outside of C. Define $G' = G - \{v_1, v_2, v_3, u_2\}$. Since $d(u_2) \geq 3$: $|E(G)| - |E(G')| \geq 10$. So there is a set R' of size at least $\frac{29(n-4)-6(m-10)}{32} > \varphi - 2$ that induces a forest in G'. Consider the subgraph of G induced by $R = R' \cup \{v_2, v_3\}$. From v_2 and v_3, only v_3 can have neighbors in R', and since neighbors of v_0 (i.e. u_0, w_0) and u_3 are on different sides of C, u_3 and v_0 are in different connected components of $G_{R'}$. Therefore, G_R is a forest (of size $\geq \varphi$). Thus all $u_0, w_0, u_1, w_1, u_2, u_3$ are on the same side of C, i.e. C is a face.

-(ii) Vertices $u_0, w_0, u_1, w_1, u_2, u_3$ **are distinct:** It is easily seen that since G is triangle-free, u_0, w_0, u_1, w_1 are distinct and $u_2 \neq u_3$. Without loss of generality, assume that $u_0 = u_2$ and consider the separating cycle $C' = u_0v_0v_3v_2$ (with v_1 and u_3 being on different sides of C'). First suppose that w_0 and v_1 are on different sides of C' (one inside and one outside) and let $G' = G - \{v_0, v_2, v_3, u_0\}$. Since $d(u_0) \geq 3$: $|E(G)| - |E(G')| \geq 9$. So G' has a forest induced by a set R' of size at least $\frac{29(n-4)-6(m-9)}{32} > \varphi - 2$. We claim that $R = R' \cup \{v_0, v_2\}$ induces a forest (of size $\geq \varphi$) in G: because v_2 has only one possible neighbor (namely v_1) in R', and since C' is a separating cycle w_0 and v_1 are in different connected components of $G'_{R'}$, that is the possible neighbors of v_0 in R' (i.e. w_0 and v_1) are in different components of $G'_{R'}$. So assume that w_0 and v_1 are on the same side of C' and let $G' = G - \{v_0, v_1, v_2, u_0\}$. Again, G' has a forest induced by a set R' of size at least $\varphi - 2$. Now $R = R' \cup \{v_0, v_2\}$ induces a forest in G (using a similar argument and the fact that w_1 and v_3 are on different sides of $u_0v_0v_1v_2$). Thus $u_0 \neq u_2$ (and also $w_0 \neq u_2$) and by symmetry u_1, w_1 are distinct from u_3.

-(iii) $u_2u_3 \in E$**, and** $u_1u_2 \in E$ **or** $w_1u_2 \in E$**, and** $u_0u_3 \in E$ **or** $w_0u_3 \in E$**:** By way of contradiction, suppose that $u_2u_3 \notin E$. Let $G' = G - \{v_0, v_3\} + \{u_3v_2\}$. Clearly G' is planar and the only possible triangle it can have is $u_3v_2u_2$, but $u_2u_3 \notin E$. Thus $G' \in \mathcal{G}$ and has a forest induced by a set R' of size $\frac{29(n-2)-6(m-5)}{32} > \varphi - 1$. We claim that $R = R' \cup \{v_3\}$ induces a forest (of size $\geq \varphi$) in G. The claim is trivial if at most one of u_3 and v_2 are in R'. If both of them are in R' then, because $v_2u_3 \in G'$ and $v_2u_3 \notin G$, v_2 and u_3 are in different connected components of $G_{R'}$. So adding v_3 does not create a cycle.

If $u_0u_3 \notin E$ and $w_0u_3 \notin E$ then let $G' = G - \{v_1, v_2, v_3, u_2\} + \{v_0u_3\}$. The only case in which G' has a triangle is when one of u_0 and w_0 is connected to u_3, which is not the case by our assumption.. Thus, G' is triangle-free (and trivially planar). Note that because $d(u_2) \geq 3$: $|E(G)| - |E(G')| \geq 9$. So G' has a forest induced by a set R' of size at least $\frac{29(n-4)-6(m-9)}{32} > \varphi - 2$. Let $R = R' \cup \{v_2, v_3\}$. Note that v_2 has degree 1 in G_R. As in the previous paragraph, if at most one of v_0 and u_3 is in R' then clearly R induces a forest. If both $v_0, u_3 \in R'$ then, because $v_0u_3 \in G'$ and $v_0u_3 \notin G$, v_0 and u_3 are in different connected components of $G_{R'}$. Thus R induces a forest in G. This implies that $u_0u_3 \in E$ or $w_0u_3 \in E$. By symmetry, $u_1u_2 \in E$ or $w_1u_2 \in E$.

-(iv) One of $G_1 = G - \{v_0, \ldots, v_3, u_1, u_2\} + \{u_3w_1\}$ **or** $G_2 = G - \{v_0, \ldots, v_3, u_0, u_3\} + \{u_2w_0\}$ **is in** $\mathcal{G}$**:** It is easy to see that both G_1 and G_2 are planar. Assume that both have triangles. So in G_1, there is a vertex x such that $xu_3 \in E(G_1)$ and $xw_1 \in E(G_1)$, and in G_2 there is a vertex y such that $yu_2 \in E(G_2)$ and $yw_0 \in E(G_2)$. Thus xu_3, xw_1, yu_2, yw_0 are in G as well. Because of planarity of G, this implies that $x = y$. But in this case, since $u_2u_3 \in E(G)$ too, xu_2u_3 forms a triangle in G, a contradiction. So at least one of G_1 or G_2 is in $\mathcal{G}$.

Without loss of generality, assume that G_1 as defined above is in $\mathcal{G}$. Since in the 4-cycle $v_2u_2u_3v_3$, vertices v_2 and v_3 have degree 3, by Lemmas 15 and 16: $d(u_2) \geq 4$. Thus $|E(G)| - |E(G_1)| \geq 13$. This, together with the assumption that $G_1 \in \mathcal{G}$, imply that G_1 has a forest induced by a set R' of size at least $\frac{29(n-6)-6(m-13)}{32} = \varphi - 3$. We claim that $R = R' \cup \{v_1, v_2, v_3\}$ induces a forest (of size $\geq \varphi$) in G. First note that the only possible neighbors of v_1 and v_3 in R' are w_1 and u_3, respectively. Since $u_3w_1 \in G_1$ and $u_3w_1 \notin G$, u_3 and w_1 are in different connected components of $G_{R'}$. This shows that G_R is a forest. $\square$

Lemma 18. *G does not have a 4-face with exactly three 4-vertices.*

Proof. Suppose that $f = v_0v_1v_2v_3$ is a 4-face with $d(v_0) = 3$ and $d(v_1) = d(v_2) = d(v_3) = 4$. Let u_0 be the third neighbor of v_0.

-(i) $u_0v_2 \notin E$**:** Assume, for a contradiction, that vertex $u_0v_2 \in E$. Let x be the fourth neighbor of v_2. If x and v_3 are on different sides of the separating cycle $C = v_0u_0v_2v_1$, then let $G' = G - \{v_0, v_1, v_2, u_0\}$. Since $d(u_0) \geq 3$: $|E(G)| - |E(G')| \geq 10$. Therefore, G' has a forest induced by a set R' of size $\frac{29(n-4)-6(m-10)}{32} > \varphi - 2$. We claim that $R = R' \cup \{v_0, v_2\}$ induces a forest (of size $\geq \varphi$) in G. Note that v_0 has at most one neighbor (namely v_3) in R'. Vertex v_2 may have two neighbors in R'; v_3 and x, but since they are on different sides of C, v_3 and x are in different connected components of $G'_{R'}$. Thus G_R is acyclic.

If x and v_3 are both inside or outside of C then we set $G' = G - \{v_0, v_2, v_3, u_0\}$. A similar argument shows that there is a set $R' \subseteq V(G')$ that induces a forest of size $\varphi - 2$ in G' and $R = R' \cup \{v_0, v_2\}$ induces a forest (of size $\geq \varphi$) in G.

-(ii) Other than v_0, there is a common neighbor of u_0 and v_1, and a common neighbor of u_0 and v_3: By way of contradiction, suppose that v_0 is the only common neighbor of u_0 and v_1. Let $G' = G - \{v_0, v_3\} + \{u_0v_1\}$. Clearly, G' is planar, and because by assumption there is no vertex adjacent to both v_1 and u_0, G' is triangle-free. Therefore, it has a forest induced by a set R' of size $\frac{29(n-2)-6(m-5)}{32} > \varphi - 1$. We claim $R = R' \cup \{v_0\}$ induces a forest in G. The reason is that $v_3 \notin R$ and because $u_0v_1 \in G'$ and $u_0v_1 \notin G$, u_0 and v_1 are in different connected components of $G_{R'}$. Thus u_0 and v_1 must have a common neighbor other than v_0. By symmetry, u_0 and v_3 have a common neighbor other than v_0.

Assume that $x \neq v_0$ is connected to both u_0 and v_1 and $y \neq v_0$ is connected to both u_0 and v_3.

(iii) $x \neq y$**:** By way of contradiction, assume that $x = y$, i.e. x is adjacent to u_0, v_1, and v_3 (see Figure 1). So $v_0u_0xv_1$ is a 4-cycle with at least one 3-vertex; v_0. By Lemmas 15 and 17 (and Lemma 1) u_0 and x are 4-vertices. Let u_1 be the fourth neighbor of v_1 (other than v_0, v_2, x) and u_3 be the fourth neighbor of v_3 (other than v_0, v_2, x). Note that $u_1 = u_3$ is possible. If $C_1 = v_0u_0xv_1$ is a 4-face then, since $d(v_0) = 3$ and $d(u_0) = d(x) = d(v_1) = 4$, it violates part **(i)** above (because of the edge xv_3). Therefore, C_1 is a separating cycle. Similar argument applied to $C_2 = v_0u_0xv_3$ shows that C_2 is a separating cycle, too. Let z be the fourth neighbor of x (other than v_1, u_0, and v_3). If z and v_2 are on different sides of C_1 (i.e. when z is inside C_1) then define $G' = G - \{v_0, v_1, v_3, u_0, x, u_1\}$. As $G' \in \mathcal{G}$ and

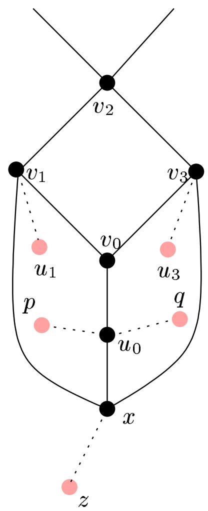

Fig. 1. part (iii) in the proof of Lemma 18

$|E(G)| - |E(G')| \geq 14$, there is a set $R' \subseteq V(G')$ of size at least $\frac{29(n-6)-6(m-14)}{32} > \varphi - 3$, which induces a forest in G'. It is easy to see that $R = R' \cup \{v_0, v_1, x\}$ induces a forest (of size $\geq \varphi$) in G, since the only possible neighbor of v_1 in R' (i.e. v_2) and the only possible neighbor of x in R' (i.e. z) are in different connected components of G' (because of the separating cycle C_1). We get a similar contradiction if z and v_2 are on different sides of C_2. Thus we can assume that z is outside of C_1 and C_2 (i.e. the same side as v_2 is). If none of the neighbors of u_0 is inside C_1 then C_1 is indeed a 4-face unless u_1 is inside of C_1. But in this case v_1u_1 will be a bridge, which in turn contradicts Lemma 2. Thus at least one neighbor of u_0, call it p, is inside C_1 (note that $p = u_1$ is possible). Similar arguments show that at least one neighbor of u_0, call it q, is inside C_2 (note that $q = u_3$ is possible). If u_1 is not inside C_1 then u_0p will be a bridge, contradicting Lemma 2. So u_1 is inside C_1, too. Similarly, u_3 is inside C_2 (or else u_0q is a bridge). Now define $G' = G - \{v_0, \ldots, v_3, u_0, x\}$. Since $G' \in \mathcal{G}$ and $|E(G)| - |E(G')| \geq 15$, there is a set $R' \subseteq V(G')$ of size at least $\frac{29(n-6)-6(m-15)}{32} > \varphi - 3$, which induces a forest in G'. Then $R = R' \cup \{v_0, v_1, x\}$ induces a forest of size φ in G because u_1 and z are in different connected components of G' (as they are on different sides of C_1). This contradiction implies that $x \neq y$.

So v_1 is adjacent to v_0, v_2, x, u_1 and v_3 is adjacent to v_0, v_2, y, and u_3 (recall that $u_1 = u_3$ is possible). As in the previous paragraph, by Lemmas 15 and 17 applied to 4-cycles $v_0u_0xv_1$ and $v_0u_0yv_3$: $d(u_0) = d(x) = d(y) = 4$. Let z be the fourth neighbor of u_0. Since G is triangle-free, among the six vertices v_1, v_2, v_3, u_0, x, y, the only possible edges are $e_1 = u_0v_2$ (i.e. when $z = v_2$), $e_2 = xv_3$ (i.e. when $x = u_3$), and $e_3 = yv_1$ (i.e. when $y = u_1$). By part **(i)** proved above: $e_1 \notin E$. Furthermore, because of planarity of G, at any time, at most one of e_2 or e_3 exists.

- If $e_2 \notin E$, and $e_3 \notin E$: Define $G' = G - \{v_0, \ldots, v_3, u_0, x, y\}$. In this case, $|E(G)| - |E(G')| \geq 18$, and so G' has a forest induced by a set R' of size at least $\frac{29(n-7)-6(m-18)}{32} > \varphi - 3$. It is easy to see that $R = R' \cup \{v_1, v_3, u_0\}$ induces a forest (of size $\geq \varphi$) in G, since they are non-adjacent and each has at most one neighbor in R'.
- If $e_2 \in E(G)$ or $e_3 \in E(G)$: Suppose that $e_2 \in E(G)$ (the argument for case $e_3 \in E(G)$ is symmetric). In this case, we define $G' = G - \{v_0, v_1, v_3, u_0, x, y\}$. Therefore $|E(G)| - |E(G')| \geq 15$ and so G' has a forest induced by a set R' of size at least $\varphi - 3$. Let $R = R' \cup \{v_0, v_3, u_0\}$. The only case in which R induces a cycle in G is when the

possible neighbors of u_0 and v_3 in R' (namely z and v_2, respectively), are in the same connected component of $G'_{R'}$. But this cannot happen, because z and v_2 are on different sides of the separating cycle $C' = v_0v_1xv_3$. Therefore G_R is a forest.

□

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