Frequency Channel Assignment on Planar Networks*

Michael Molloy ** and Mohammad R. Salavatipour ***

Department of Computer Science, University of Toronto,
10 King's College Rd., Toronto M5S 3G4, Canada
{molloy,mreza}@cs.toronto.edu

Abstract. For integers $p \geq q$, a $L(p,q)$-labeling of a network G is an integer labeling of the nodes of G such that adjacent nodes receive integers which differ by at least p, and nodes at distance two receive labels which differ by at least q. The minimum number of labels required in such labeling is $\lambda_q^p(G)$. This arises in the context of frequency channel assignment in mobile and wireless networks and often G is planar. We show that if G is planar then $\lambda_q^p(G) \leq \frac{5}{3}(2q-1)\Delta + 12p + 144q - 78$. We also provide an $O(n^2)$ time algorithm to find such a labeling. This provides a $(\frac{5}{3}+o(1))$-approximation algorithm for the interesting case of $q = 1$, improving the best previous approximation ratio of 2.

1 Introduction

The problem of *frequency assignment* arises when different radio transmitters which operate in the same geographical area interfere with each other when assigned the same or closely related frequency channels. This situation is common in a wide variety of real world applications related to mobile or general wireless networks and is best modeled using graph coloring where the vertices of a graph represent the transmitters and adjacencies indicate possible interferences.

There has been recently much interest in the $L(2,1)$-labeling problem, which is the problem of assigning radio frequencies (integers) to transmitters such that transmitters which are close (at distance 2 apart in the graph) to each other receive different frequencies and transmitters which are *very* close (adjacent in the graph) receive frequencies that are at least two apart. To keep the frequency bandwidth small, we are interested in minimizing the difference of the smallest and largest integers assigned as labels to the vertices of the graph. The minimum range of frequencies is called λ_1^2. In many applications, the differences between

* A full version of this paper is available with title "A Bound on the Chromatic Number of the Square of a Planar Graph" at http://www.cs.toronto.edu/~mreza/research.html

** Supported by NSERC, a Sloan Research Fellowship, and a Premier's Research Excellence Award.

*** Supported by Research Assistantship, Department of computer science, and University open fellowship, University of Toronto.

the frequency channels must be at least some specific given numbers. So we study $L(p, q)$-labelings, which are frequency assignments such that the transmitters that are adjacent in the graph get labels that are at least p apart and those at distance two get labels that are at least q apart; λ_q^p is defined similarly.

Several papers have studied this problem and different bounds and approximation algorithms for λ_1^2 have been obtained for various classes of graphs [2, 4–8, 10, 11, 14, 16], most of them based on Δ, the maximum degree of the graph. Much of this study has been focussed on planar graphs. For example, this problem is proved to be NP-complete for planar graphs in [2, 10]. Jonas [9] showed that for planar graphs $\lambda_1^2 \leq 8\Delta - 13$. This was the best known bound until recently, when Van den Heuvel and McGuinness [13] showed that $\lambda_q^p \leq (4q-2)\Delta + 10p + 38q - 24$. In this paper we improve this bound asymptotically by showing that:

Theorem 1. *For any planar graph G and positive integers $p \geq q$: $\lambda_q^p(G) \leq \frac{5}{3}(2q-1)\Delta + 12p + 144q - 78$.*

Although the proof of this result is lengthy and non-trivial, it yields an easy to implement $O(n^2)$ algorithm to find such a labeling.

For simplicity of exposition, we present the case $p = q = 1$. The proof of the general case is nearly identical. For this special case, we have the following longstanding conjecture:

Conjecture 1 (Wegner [15]). For a planar graph G:

$$\lambda_1^1(G) \leq \begin{cases} \Delta + 5 & \text{if } 4 \leq \Delta \leq 7, \\ \lceil \frac{3}{2}\Delta \rceil + 1 & \text{if } \Delta \geq 8. \end{cases}$$

This conjecture, if true, would be the best possible bound in terms of Δ, as shown by Wegner [15]. S.A. Wong [17] showed that $\lambda_1^1(G) \leq 3\Delta + 5$. Recently, Van den Heuvel and McGuinness [13] proved that $\lambda_1^1(G) \leq 2\Delta + 25$. For large values of Δ, Agnarsson and Halldórsson [1] have a better asymptotic bound for $\lambda_1^1(G)$. They prove that if G is a planar graph with $\Delta \geq 749$, then $\lambda_1^1(G) \leq \lceil \frac{9}{5}\Delta(G) \rceil + 1$, but as they noted, this is the best asymptotic bound that they can get via their approach. Recently, Borodin et al. [3] have been able to extend these results to $\lambda_1^1(G) \leq \lceil \frac{9}{5}\Delta(G) \rceil + 1$ for planar graphs with $\Delta(G) \geq 47$. We improve all these results asymptotically by showing that:

Theorem 2. *For a planar graph G, $\lambda_1^1(G) \leq \frac{5}{3}\Delta(G) + 78$.*

Remark 1. For planar graphs G with $\Delta \geq 241$ we can actually obtain $\lambda_1^1(G) \leq \frac{5}{3}\Delta(G) + 24$, but we do not present the proof for this result here, because of space limits.

In section 5 we explain how to modify the proof of Theorem 2 to prove Theorem 1. The technique we use is inspired by that used by Sanders and Zhao [12] to obtain a similar bound on the cyclic chromatic number of planar graphs.

In [2] Bodlaender et al. have given approximation algorithms to compute λ_1^2 for some classes of graphs and noted that the result of Jonas [9] yields an

8-approximation algorithm for planar graphs. In [5] Fotakis et al. use the result of [13] to obtain a $(2 + o(1))$-approximation algorithm for λ_1^1 on planar graphs. They state that a major open problem is to get a polynomial time approximation algorithm of approximation ratio < 2. Agnarsson and Halldórsson [1] also give a 2-approximation. The results of this paper yield a $(\frac{5}{3} + o(1))$-approximation algorithm for λ_1^p and in general a $(\frac{5}{3q}(2q-1) + o(1))$-approximation algorithm for λ_q^p, for planar graphs. The reason for this is that for a planar graph G with maximum degree Δ: $\lambda_q^p(G) \geq q\Delta + p$. So, the algorithm we obtain for Theorem 1 will have approximation ratio of $\frac{5}{3q}(2q-1) + o(1)$ for λ_q^p.

The organization of the paper is as follows: In the next section we give an overview of the algorithms we obtain. In Sections 3 and 4 we give some preliminary definitions and (the sketch of) the proof of Theorem 2. Section 5 contains (the sketch of) the proof of Theorem 1. Finally, in Section 6 we explain the algorithm and talk about the asymptotic tightness of the results.

2 Overview of the algorithms

We use G^2 to denote the *square* of G, i.e. the graph formed by joining all pairs of vertices which are at distance at most 2 in G. It is convenient to note that $\lambda_1^1(G) = \chi(G^2)$. Thus, our proof of Theorem 2 is simply a proof that the square of any planar graph can be colored with at most $\frac{5}{3}\Delta + 78$ colors.

An edge (u, v) of G is *reducible* if it has the following properties:

(i) H, the graph obtained from G by contracting (u, v) has maximum degree $\Delta(G)$.
(ii) Any $(\frac{5}{3}\Delta(G) + 78)$-coloring of H^2 can *"easily"* be extended to a coloring of G^2.

The exact meaning of "easily" is made clear in the proofs of Lemmas 8 and 9. For now, it suffices to say that this extension can be done in $O(\Delta(G))$ time. We prove that every planar graph has a reducible edge. Furthermore, that edge can be found in $O(n)$ time. This yields an $O(n^2)$ recursive algorithm for finding the coloring. We elaborate more on this in section 6.

We find a reducible edge using the Discharging Method, which was first used to prove the Four Color Theorem. We start by assigning an initial charge (that will be defined in the proof) to each vertex such that the sum of the charges is negative. Then we move the charges among the vertices based on the 12 discharging rules given in Section 4. This process preserves the sum of the charges, and so at least one vertex will have negative charge. We will show that any vertex with negative charge will have a reducible edge in its neighborhood. Applying the charges and discharging rules and then searching for a negative charge vertex and then finding the associated reducible edge can be done in $O(n)$, as required. We can use exactly the same procedure to develop an algorithm for Theorem 1.

3 Preliminaries

We assume that the given graph is a simple connected planar graph with at least 8 vertices. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The length of a path between two vertices is the number of edges on that path. We define the distance between two vertices to be the length of the shortest path between them. The square of a graph G, denoted by G^2, is a graph on the same vertex set such that two vertices are adjacent in G^2 iff their distance in G is at most 2. The degree of a vertex v is the number of edges incident with v and is denoted by $d_G(v)$ or simply $d(v)$ if it is not confusing. We denote the maximum degree of a graph G by $\Delta(G)$ or simply Δ. If the degree of v is i, at least i, or at most i we call it an i-vertex, a $\geq i$-vertex, or a $\leq i$-vertex, respectively. By $N_G(v)$, we mean the open neighborhood of v in G, which contains all those vertices that are adjacent to v in G. The closed neighborhood of v, which is denoted by $N_G[v]$, is $N_G(v) \cup \{v\}$. We usually use $N(v)$ and $N[v]$ instead of $N_G(v)$ and $N_G[v]$, respectively.

A vertex k-coloring of a graph G is a mapping $C : V \longrightarrow \{1, \ldots, k\}$ such that any two adjacent vertices u and v are mapped to different integers. The minimum k for which a coloring exists is called the chromatic number of G and is denoted by $\chi(G)$. A vertex v is called *big* if $d_G(v) \geq 47$, otherwise we call it a *small* vertex.

From now on assume that G is a minimum counter-example to Theorem 2.

Lemma 1. *For every vertex v of G, if there exists a vertex $u \in N(v)$, such that $d_G(v) + d_G(u) \leq \Delta(G) + 2$ then $d_{G^2}(v) \geq \frac{5}{3}\Delta(G) + 78$.*

Proof. Assume that v is such a vertex. Contract v on edge (v, u). The resulting graph has maximum degree at most $\Delta(G)$ and because G was a minimum counter-example, the new graph can be colored with $\frac{5}{3}\Delta(G) + 78$ colors. Now consider this coloring induced on G, in which every vertex other than v is colored. If $d_{G^2}(v) < \frac{5}{3}\Delta(G) + 78$ then we can assign a color to v to extend the coloring to v, which contradicts the definition of G. □

If we define H to be the graph obtained from G by contracting (u, v), the above proof actually shows how to extend a coloring of H^2 to a coloring of G^2. Therefore, we have the following:

Type 1 reducible edge: *An edge (u, v) where $d_G(u) + d_G(v) \leq \Delta(G) + 2$ and $d_{G^2}(v) < \frac{5}{3}\Delta(G) + 78$.*

As we mentioned before, Van den Heuvel and McGuinness [13] showed that $\chi(G^2) \leq 2\Delta + 25$. So:

Observation 1. *We can assume that $\Delta(G) \geq 160$, otherwise $2\Delta(G) + 25 \leq \frac{5}{3}\Delta(G) + 78$.*

Lemma 2. *Every ≤ 5-vertex in G must be adjacent to at least two big vertices.*

Corollary 1. *Every vertex of G is a ≥ 2-vertex.*

The proof of Theorem 2 becomes significantly simpler if we can assume that the underlying graph is triangulated, i.e. all faces are triangles, and has minimum degree at least 4. To be able to make this assumption, we modify graph G in two phases and get a contradiction from the assumption on G. In the first phase we make a (simple) triangulated graph G', by adding edges to every non-triangle face of G.

Observation 2. *For every vertex v, $N_G(v) \subseteq N_{G'}(v)$.*

It is also easy to verify that:

Lemma 3. *All vertices of G' are $\geq$3-vertices.*

Lemma 4. *Each $\geq$4-vertex v in G' can have at most $\frac{1}{2}d(v)$ neighbors which are 3-vertices.*

In the second phase we transform graph G' into another triangulated graph G'', whose minimum degree is at least 4. Initially G'' is equal to G'. As long as there is any 3-vertex v we do the following *switching* operation: let x, y, z be the three neighbors of v. At least two of them, say x and y, are big in G' by Lemma 2 and Observation 2. Remove edge (x, y). Since G' (and also G'') is triangulated this leaves a face of size 4, say $\{x, v, y, t\}$. Add edge (v, t) to G''. This way, the graph is still triangulated.

Observation 3. *If v is a small vertex in G then $N_G(v) \subseteq N_{G''}(v)$.*

Lemma 5. *If v is a big vertex in G then $d_{G''}(v) \geq 24$.*

So a big vertex v in G will not be a $\leq$23-vertex in G''. Let v be a big vertex in G and $x_0, x_2, \ldots, x_{d_{G''}(v)-1}$ be the neighbors of v in G'' in clockwise order. We call $x_a, \ldots, x_{a+b}$ (where addition is in mod $d_{G''}(v)$) a *sparse segment* in G'' iff:

- $b \geq 2$,
- Each x_i is a 4-vertex.

In the next two lemmas let's assume that $x_a, \ldots, x_{a+b}$ is a maximal sparse segment of v in G'', which is not equal to all the neighborhood of v. Also assume that x_{a-1} and x_{a+b+1} are the neighbors of v right before x_a and right after x_{a+b}, respectively.

Lemma 6. *There is a big vertex other than v, that is connected to all the vertices of $x_a, \ldots, x_{a+b}$.*

We use u to denote the big vertex, other than v, that is connected to all $x_a, \ldots, x_{a+b}$.

Lemma 7. *All the vertices $x_{a+1}, \ldots, x_{a+b-1}$ are connected to both u and v in G. If x_{a-1} is not big in G then x_a is connected to both u and v in G. Otherwise it is connected to at least one of them. Similarly, if x_{a+b+1} is not big in G then x_b is connected to both u and v in G, and otherwise it is connected to at least one of them.*

We call $x_{a+1}, \ldots, x_{a+b-1}$ the *inner* vertices of the sparse segment, and x_a and x_{a+b} the *end* vertices of the sparse segment. Consider vertex v and let's call the maximal sparse segments of it $Q_1, Q_2, \ldots, Q_m$ in clockwise order, where $Q_i = q_{i,1}, q_{i,2}, q_{i,3}, \ldots$. The next two lemmas are the key lemmas in the proof of Theorem 1.

Lemma 8. *$|Q_i| \leq d_G(v) - \frac{2}{3}\Delta - 73$, for $1 \leq i \leq m$.*

Proof. We prove this by contradiction. Assume that for some i, $|Q_i| \geq d_G(v) - \frac{2}{3}\Delta - 72$. Let u_i be the big vertex that is adjacent to all the inner vertices of Q_i (in both G and G''). (See Figure 1).

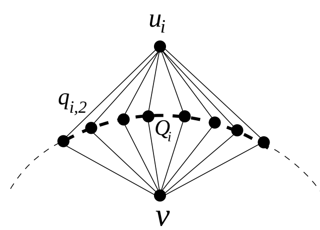

Fig. 1. The configuration of lemma 8

For an inner vertex of Q_i, say $q_{i,2}$, we have:

$$\begin{aligned} d_{G^2}(q_{i,2}) &\leq d_G(u_i) + d_G(v) + 2 - (|Q_i| - 3) \\ &\leq \Delta(G) + d_G(v) - |Q_i| + 5 \\ &\leq \tfrac{5}{3}\Delta(G) + 77. \end{aligned}$$

If $q_{i,2}$ is adjacent to $q_{i,1}$ or $q_{i,3}$ in G then it is contradicting Lemma 1. Otherwise it is only adjacent to v and u_i in G, therefore has degree 2, and so along with v or u_i contradicts Lemma 1. □

Therefore, if lemma 8 fails for some Q_i, then there must be a type 1 reducible edge with an end point in Q_i.

Lemma 9. *Consider G and suppose that u_i and u_{i+1} are the big vertices adjacent to all the inner vertices of Q_i and Q_{i+1}, respectively. Furthermore assume that t is a vertex adjacent to both u_i and u_{i+1} but not adjacent to v (see Figure 2) and there is a vertex $w \in N_G(t)$ such that $d_G(t) + d_G(w) \leq \Delta(G) + 2$. Let $X(t)$ be the set of vertices at distance at most 2 of t that are not in $N_G[u_i] \cup N_G[u_{i+1}]$. If $|X(t)| \leq 4$ then:*

$$|Q_i| + |Q_{i+1}| \leq \frac{1}{3}\Delta(G) - 69.$$

Proof. Again we use contradiction. Assume that $|Q_i| + |Q_{i+1}| \geq \frac{1}{3}\Delta(G) - 68$. Using the argument of the proof of Lemma 1, by contracting (t, w), we can color

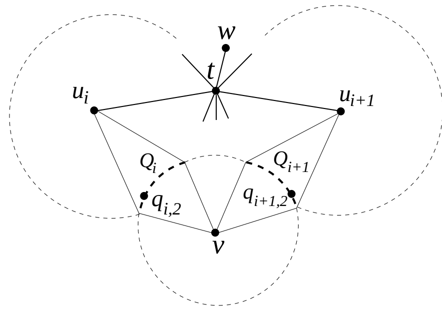

Fig. 2. The configuration of lemma 9

every vertex of G other than t. Note that $d_{G^2}(t) \le d_G(u_i) + d_G(u_{i+1}) + |X(t)| \le 2\Delta(G) + 4$. If all the colors of the inner vertices of Q_i have appeared on the vertices of $N_G[u_{i+1}] \cup X(t) - Q_{i+1}$ and all the colors of inner vertices of Q_{i+1} have appeared on the vertices of $N_G[u_i] \cup X(t) - Q_i$ then there are at least $|Q_i| - 2 + |Q_{i+1}| - 2$ repeated colors at $N_{G^2}(t)$. So the number of colors at $N_{G^2}(t)$ is at most $2\Delta(G) + 4 - |Q_i| - |Q_{i+1}| + 4$, which is at most $\frac{5}{3}\Delta(G) + 76$ and so there is still one color available for t, which is a contradiction.

Therefore, without loss of generality, there exists an inner vertex of Q_{i+1}, say $q_{i+1,2}$, whose color is not in $N_G[u_i] \cup X(t) - Q_i$. If there are less than $\frac{5}{3}\Delta(G)+77$ colors at $N_{G^2}(q_{i+1,2})$ then we could assign a new color to $q_{i+1,2}$ and assign the old color of it to t and get a coloring for G. So there must be $\frac{5}{3}\Delta(G) + 77$ different colors at $N_{G^2}(q_{i+1,2})$. ¿From the definition of a sparse segment, we have: $N_G(q_{i+1,2}) \subseteq \{v, u_{i+1}, q_{i+1,1}, q_{i+1,3}\}$. There are at most $d_G(u_{i+1})+5$ colors, called the *smaller* colors, at $N_G[u_{i+1}] \cup X(t) \cup N_G[q_{i+1,1}] \cup N_G[q_{i+1,3}] - \{v\} - \{q_{i+1,2}\}$ (note that t is not colored). So there must be at least $\frac{2}{3}\Delta(G)+72$ different colors, called the *larger* colors, at $N_G[v] - Q_{i+1}$. Since $|N_G[v]| - |Q_i| - |Q_{i+1}| \le \Delta(G) + 1 - \frac{1}{3}\Delta(G) + 68 \le \frac{2}{3}\Delta(G) + 69$, one of these *larger* colors must be on an inner vertex of Q_i, which without loss of generality, we can assume is $q_{i,2}$. Because t is not colored, we must have all the $\frac{5}{3}\Delta(G) + 78$ colors at $N_{G^2}(t)$. Otherwise we could assign a color to t. As there are at most $\Delta(G) + 4$ colors, all from the *smaller* colors, at $N_G[u_{i+1}] \cup X(t)$, all the *larger* colors must be in $N_G[u_i]$ too. Therefore the larger colors are in both $N_G[v]$ and $N_G[u_i]$. Note that $|N_{G^2}(q_{i,2})| \le |N_G[v]|+|N_G[u_i]| \le 2\Delta(G)$. So there are at most $2\Delta(G)-\frac{2}{3}\Delta(G)-72 = \frac{4}{3}\Delta(G) - 72$ different colors at $N_{G^2}(q_{i,2})$ and so we can assign a new color to $q_{i,2}$ and assign the old color of $q_{i,2}$, which is one of the *larger* colors and is not in $N_{G^2}(t) - \{q_{i,2}\}$, to t and extend the coloring to G, a contradiction. □

Another way of looking at the proof of Lemma 9 is that we showed the following is a reducible edge:
Type 2 reducible edge: (t, w), *under the assumptions of Lemma 9.*

4 Discharging Rules

We give an initial charge of $d_{G''}(v) - 6$ units to each vertex v. Using the Euler formula, $|V| - |E| + |F| = 2$, and noting that $3|F(G'')| = 2|E(G'')|$:

$$\sum_{v \in V} (d_{G''}(v) - 6) = 2|E(G'')| - 6|V| + 4|E(G'')| - 6|F(G'')| = -12 \qquad (1)$$

By these initial charges, the only vertices that have negative charges are 4- and 5-vertices, which have charges -2 and -1, respectively. The goal is to show that, based on the assumption that G is a minimum counter-example, we can send charges from other vertices to $\leq$5-vertices such that all the vertices have non-negative charge, which is of course a contradiction since the total charge must be negative by equation (1).

We call a vertex v *pseudo-big* (in G'') if v is big (in G) and $d_{G''}(v) \geq d_G(v) - 11$. Note that a pseudo-big vertex is also a big vertex, but a big vertex might or might not be a pseudo-big vertex. Before explaining the discharging rules, we need a few more notations.

Suppose that $v, x_1, x_2, \ldots, x_k, u$ is a sequence of vertices such that v is adjacent to x_1, x_i is adjacent to x_{i+1}, $1 \leq i < k$, and x_k is adjacent to u.

Definition: By *"v sends c units of charge through $x_1, \ldots, x_k$ to u"* we mean v sends c units of charge to x_1, it passes the charge to x_2... etc, and finally x_k passes the charge to u. In this case, we also say *"v sends c units of charge through x_1"* and *"u gets c units of charge through x_k"*.

Note that in order to simplify the calculations of the total charges on vertex x_i, $1 \leq i \leq k$, we do not take into account the charges that only pass through x_i. We say v *saves* k units of charge on a set of size l of its neighbors, if the total charge sent from v to or through them minus the total charge sent from or through them to v is at most $l - k$ units. So, for example, if v is sending nothing to u and is getting $\frac{1}{2}$ through u then v saves $\frac{3}{2}$ on u.

In discharging phase, a big vertex v of G:

1) Sends 1 unit of charge to each 4-vertex u in $N_{G''}(v)$.
2) Sends $\frac{1}{2}$ unit of charge to each 5-vertex u in $N_{G''}(v)$.

In addition, if v is a big vertex and u_0, u_1, u_2, u_3, u_4 are consecutive neighbors of v in clockwise or counter-clockwise order, where $d_{G''}(u_0) = 4$, then:

3) If $d_{G''}(u_1) = 5$, u_2 is big, $d_{G''}(u_3) = 4$, $d_{G''}(u_4) \geq 5$, and the neighbors of u_1 in clockwise or counter-clockwise order are v, u_0, x_1, x_2, u_2 then v sends $\frac{1}{2}$ to x_1 through u_2, u_1.
4) If $d_{G''}(u_1) = 5$, $5 \leq d_{G''}(u_2) \leq 6$, $d_{G''}(u_3) \geq 7$, and the neighbors of u_1 in clockwise or counter-clockwise order are v, u_0, x_1, x_2, u_2 then v sends $\frac{1}{2}$ to x_1 through u_3, u_2, u_1.
5) If $d_{G''}(u_1) = 5$, u_2 is big, $d_{G''}(u_3) \geq 5$, and the neighbors of u_1 in clockwise or counter-clockwise order are v, u_0, x_1, x_2, u_2 then v sends $\frac{1}{4}$ to x_1 through u_2, u_1.

6) If $d_{G''}(u_1) = 6$, $d_{G''}(u_2) \leq 5$, $d_{G''}(u_3) \geq 7$, and the neighbors of u_1 in clockwise or counter-clockwise order are $v, u_0, x_1, x_2, x_3, u_2$ then v sends $\frac{1}{2}$ to x_1 through u_1.
7) if $d_{G''}(u_1) = 6$, $d_{G''}(u_2) \geq 6$, and the neighbors of u_1 in clockwise or counter-clockwise order are $v, u_0, x_1, x_2, x_3, u_2$ then v sends $\frac{1}{4}$ to x_1 through u_1.

if $7 \leq d_{G''}(v) < 12$ then:

8) If u is a big vertex and $u_0, u_1, u_2, v, u_3, u_4, u_5$ are consecutive neighbors of u where all $u_0, u_1, u_2, u_3, u_4, u_5$ are 4-vertices then v sends $\frac{1}{2}$ to u.
9) if u_0, u_1, u_2, u_3 are consecutive neighbors of v, such that $d_{G''}(u_1) = d_{G''}(u_2) = 5$, u_0 and u_3 are big, and t is the other common neighbor of u_1 and u_2 (other than v), then v sends $\frac{1}{2}$ to t.

Every $\geq$12-vertex v of G'' that was not big in G:

10) Sends $\frac{1}{2}$ to each of its neighbors.

A $\leq$5-vertex v sends charges as follows:

11) if $d_{G''}(v) = 4$ and its neighbors in clockwise order are u_0, u_1, u_2, u_3, such that u_0, u_1, u_2 are big in G and u_3 is small, then v sends $\frac{1}{2}$ to each of u_0 and u_2 through u_1.
12) If $d_{G''}(v) = 5$ and its neighbors in clockwise order are u_0, u_1, u_2, u_3, u_4, such that $d_{G''}(u_0) \leq 11$, $d_{G''}(u_1) \geq 12$, $d_{G''}(u_2) \geq 12$, $d_{G''}(u_3) \leq 11$, and u_4 is big, then v sends $\frac{1}{2}$ to u_4.

It can be proved that after applying the discharging rules:

Lemma 10. *Every vertex v that is not big in G will have non-negative charge.*

Lemma 11. *Every big vertex v that is not pseudo-big will have non-negative charge.*

Lemma 12. *Every pseudo-big vertex v has non-negative charge.*

We omit the proofs of Lemmas 10, 11, and 12 because of a lack of space.

Proof of Theorem 2: By Lemmas 10, 11, and 12 every vertex of G'' will have non-negative charge, after applying the discharging rules. Therefore the total charge over all the vertices of G'' will be non-negative, but this is contradicting equation (1). This disproves the existence of G, a minimum counter-example to the theorem.

5 Generalization to Bound λ_q^p

The general steps of the proof of Theorem 1 are very similar to those of Theorem 2. The case $q = 0$ reduces to the Four Color Theorem. So let's assume that $q \geq 1$. Let G be a planar graph which is a minimum counter-example to Theorem 1. Recall that a vertex is big, if $d_G(v) \geq 47$. The proof of the following lemmas and observations are very similar to the corresponding ones for Theorem 2.

Lemma 13. *Suppose that v is a $\leq$5-vertex in G. If there exists a vertex $u \in N(v)$, such that $d_G(v) + d_G(u) \leq \Delta + 2$ then $d_{G^2}(v) \geq d_G(v) + \frac{5}{3}\Delta + 73$.*

Since the bound $2(2q-1)\Delta + 10p + 38q - 24$ is already proved in [13]:

Observation 4. *We can assume that $\Delta \geq 160$.*

Lemma 14. *Every $\leq$5-vertex must be adjacent to at least 2 big vertices.*

Now construct graph G' from G and then G'' from G' in the same way we did in the proof of Theorem 2. Also, we define the sparse segments in the same way. Consider vertex v and let's call the maximal sparse segments of it $Q_1, Q_2, \ldots, Q_m$ in clockwise order, where $Q_i = q_{i,1}, q_{i,2}, q_{i,3}, \ldots$.

Lemma 15. $|Q_i| \leq d_G(v) - \frac{2}{3}\Delta - 70$.

Lemma 16. *Suppose that u_i and u_{i+1} are the big vertices adjacent to all the vertices of Q_i and Q_{i+1}, respectively. Furthermore assume that t is a $\leq$6-vertex adjacent to both u_i and u_{i+1} but not adjacent to v (see Figure 2) and there is a vertex $w \in N(t)$ such that $d_G(t) + d_G(w) \leq \Delta(G) + 2$. Let $X(t)$ be the set of vertices at distance at most 2 of t that are not in $N[u_i] \cup N[u_{i+1}]$. If $|X(t)| \leq 4$ then:*

$$|Q_i| + |Q_{i+1}| \leq \frac{1}{3}\Delta(G) - 69. \tag{2}$$

The rest of the proof is almost identical to that of Theorem 2. We apply the same initial charges and discharging rules, and use Lemmas 14, 15, and 16, instead of Lemmas 2, 8, and 9, respectively.

6 The Algorithm and the Asymptotic Tightness of the Result

Now we describe an algorithm that can be used for each of Theorems 1 or 2 to find such colorings. Consider a planar graph G. One iteration of the algorithm is to reduce the size of the problem by finding a reducible edge in G, contracting it, coloring the new smaller graph recursively, and then extending the coloring to G. At each iteration, first we check to see if every $\leq$5-vertex is adjacent to at least 2 big vertices or not. If not, then that vertex along with one of its small neighbors will be a type 1 reducible edge by Lemma 2. Otherwise, we construct the triangulated graph G'' and apply the initial charges and the discharging rules. As the total charge is negative, we can find a big vertex with negative charge. This vertex must have at least one of the configurations of Lemmas 8 and 9 or 15 and 16 for the cases of Theorem 2 or 1, respectively. If it has the configuration of Lemma 8 then one of the inner vertices of the sparse segment along with one of its two big neighbors will be a type 1 reducible edge. Otherwise, if it has the configuration of Lemma 9 then (t, w) will be a type 2 reducible edge (recall t and w from Lemma 9). In either of these cases we reduce the size of the graph by one. Thus, we can iterate until the size of the graph is small enough, i.e constant. In that case we can find the required coloring in constant time.

To see if there is a $\leq$5-vertex with less than 2 big neighbors we spend at most $O(n)$ time, where n is the number of vertices in G. Also, applying the initial charges and discharging rules takes $O(n)$ time. After finding a vertex with negative charge, finding the suitable edge and then contracting it can be done in $O(n)$. Since there are $O(n)$ iterations of the main procedure, the total running time of the algorithm would be $O(n^2)$.

Now we show that these theorems are asymptotically tight, if we use this proof technique. The results of [1] and [3] are essentially based on showing that in a planar graph G, there exists a vertex v such that $d_{G^2}(v) \leq \lfloor \frac{9}{5}\Delta(G) \rfloor + 1$. This also leads to a greedy algorithm for coloring G^2. However, as pointed out in [1], this is the best possible bound. That is, there are planar graphs in which every vertex v satisfies $d_{G^2}(v) \geq \lceil \frac{9}{5}\Delta(G) \rceil$. See [1] for an example. For the moment, let's just focus on the asymptotic order of the bounds and denote the additive constants by C. The reducible configuration in Lemma 8, after modifying the coefficient from $\frac{5}{3}$ to $\frac{9}{5}$, is the only configuration needed in obtaining the bound $\chi(G^2) \leq \frac{9}{5}\Delta(G) + C$. The extremal graph of [1] is actually an extremal graph for this lemma, and this is the reason that we need another reducible structure, like the one in Lemma 9, to improve previously known results asymptotically. But there are graphs that are extremal for both of these lemmas. For an odd value of k, one of these graphs is shown in Figure 3. In this graph G, which is obtained based on a tetrahedron, we have to join the three copies of v_8 and remove the multiple edges (we draw the graph in this way for clarity). The dashed lines represent sequences of consecutive 4-vertices. Around each of $v_1, \ldots, v_4$ there are $3k-5$ of such vertices. It is easy to see that G does not have the configuration of Lemma 9. So the minimum degree of the vertices in G^2 is of order of $\frac{5}{3}\Delta(G)$ and G does not have the configuration of Lemma 9. One can easily check that if we are going to use only $\frac{3}{2}\Delta(G) + C$ colors, we can get a variation of Lemma 9 in which the coefficient of Δ is $\frac{3}{2}$, instead of $\frac{5}{3}$. But even that configuration does not appear in the graph of Figure 3. Therefore, using these two reducible configurations the best asymptotic bound that we can achieve is $\frac{5}{3}\Delta(G)$, and we probably need another reducible configuration to improve this result asymptotically.

References

1. G. Agnarsson and M.M. Halldórsson, Coloring powers of planar graphs, *To appear in SIAM J. Disc. Math.*, Earlier version appeared in *Proc. of the 11th annual ACM-SIAM Symp. on Disc. Alg.*, pages 654-662, 2000.
2. H.L. Bodlaender, T. Kloks, R.B. Tan, and J. van Leeuwen, Approximations for λ-Coloring of Graphs, In *Proc. of 17th Annual Symp. on Theo. Aspc. of Comp. Sci.* pages 395-406, Springer 2000.
3. O. Borodin, H.J. Broersma, A. Glebov, and J. Van Den Heuvel, Colouring at distance two in planar graphs, *In preparation* 2001.
4. J. Chang and Kuo, The $L(2,1)$-labeling problem on graphs, *SIAM J. Disc. Math.* 9:309-316, 1996.

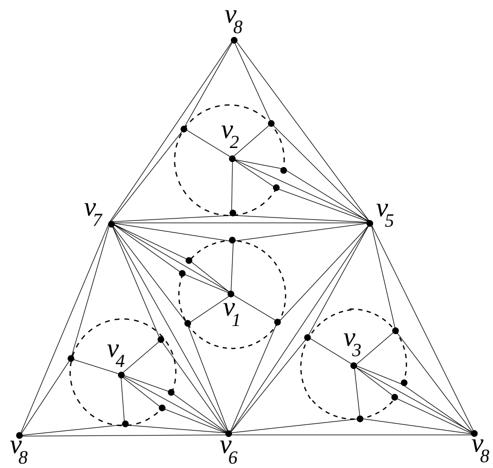

Fig. 3. The extremal graph for Theorem 2

5. D.A. Fotakis, S.E. Nikoletseas, V.G. Papadopoulou, and P.G. Spirakis, Hardness results and efficient approximations for frequency assignment problems: radio labeling and radio coloring, *J. of Computers and Artificial intelligence*, 20(2):121-180, 2001.
6. J.P. Georges and D.W. Mauro, On the size of graphs labeled with a condition at distance two, *J. Graph Theory*, 22:47-57, 1996.
7. J.P. Georges and D. W. Mauro, Some results on λ^i_j-numbers of the products of complete graphs, *Congr. Numer.*, 140:141-160, 1999.
8. J.R. Griggs and R.K. Yeh, Labeling graphs with a condition at distance 2, *SIAM J. Disc. Math.*, 5:586-595, 1992.
9. T.K. Jonas, Graph coloring analogues with a condition at distance two: $L(2,1)$-labelings and list λ-labelings, Ph.D. Thesis, University of South Carolina, 1993.
10. S. Ramanathan and E.L. Lloyd, On the complexity of distance-2 coloring, In *Proc. 4th Int. Conf. Comput. and Inform.* pages 71-74, 1992.
11. S. Ramanathan and E.L. Lloyd, Scheduling algorithms for multi-hop radio networks, *IEEE/ACM Trans. on Networking*, 1(2):166-172, 1993.
12. D.P. Sanders and Y. Zhao, A new bound on the cyclic chromatic number, *J. of Comb. Theory Series B* 83:102-111, 2001.
13. J. Van Den Heuvel and S McGuinness, Colouring the Square of a Planar Graph, *Preprint*.
14. J. Van Den Heuvel, R.A. Leese, and M.A. Shepherd, Graph labeling and radio channel assignment, *J. Graph Theory*, 29:263-283, 1998.
15. G. Wegner, Graphs with given diameter and a coloring problem, Technical report, University of Dortmond, 1977.
16. A. Whittlesey, J.P. Georges, and D.W. Mauro, On the λ-number of Q_n and related graphs, *SIAM J. Disc. Math.*, 8:499-506, 1995.
17. S.A. Wong, Colouring Graphs with Respect to Distance, M.Sc. Thesis, Department of Combinatorics and Optimization, University of Waterloo, 1996.