

The Three Color Problem for Planar Graphs

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Abstract

In 1976 Steinberg conjectured that every planar graph without 4- and 5-cycles is 3-colorable. Borodin and independently Sanders and Zhao showed that every planar graph without any cycle of size in $\{4, 5, \dots, 9\}$ is 3-colorable. We improve this result by showing that every planar graph without any cycle of size in $\{4, 5, \dots, 8\}$ is 3-colorable. Our proof yields an $O(n^2)$ time algorithm for finding a 3-coloring for these graphs.

1 Introduction

Almost two decades before the Four Color Theorem was proved, in 1959, Grötsch [5] showed that every planar graph without 3-cycles is 3-colorable. In 1976, Steinberg [2, 8] conjectured that every planar graph without 4- and 5-cycles is 3-colorable. Both 4- and 5-cycles must be excluded. In fact there is an infinite family of 4-critical planar graphs that have only four 4-cycles and no 5-cycles, and there is an infinite family of 4-critical planar graphs that have no 4-cycles and have only six 5-cycles [1]. An equivalent formulation of this conjecture is that every 4-chromatic planar graph has a 4- or 5-cycle. This problem is also discussed in the monograph by Jensen and Toft [6] (problem 2.9).

In 1991, Erdős relaxed the conjecture of Steinberg by asking if there exists an integer $k \geq 5$ such that every planar graph without cycles of size i , for $4 \leq i \leq k$, is 3-colorable. An answer to the question of Erdős (and therefore a partial answer to the conjecture of Steinberg) was obtained by Abbott and Zhou [1], who showed that $k = 11$ is suitable, i.e any planar graph without i -cycles, $4 \leq i \leq 11$, is 3-colorable. Borodin [3] improved this result to $k = 10$. The best known answer to this question, which states that $k = 9$ is suitable, is due to Borodin [4] and independently to Sanders and Zhao [7].

Let $\mathcal{G}_8$ be the class of planar graphs without cycles of size in $\{4, \dots, 8\}$. The main result of this paper is:

Theorem 1.1 *Any graph in $\mathcal{G}_8$ is 3-colorable.*

Our proof also implies an $O(n^2)$ time algorithm for finding a 3-coloring of these planar graphs. We use the Discharging Method to prove Theorem 1.1. The general steps of our proof are similar to the other known proofs based on this method. By way of contradiction, we assume that the theorem is not true and among the graphs that are counter-examples to it we select one, call it G , that has minimum size. We prove some structural properties for this graph. These properties are described in terms of

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reducible configurations. A set of vertices and/or edges is called a *reducible configuration* if its existence in G guarantees the existence of a smaller counter-example to Theorem 1.1. By this definition, G cannot have any reducible configurations. Then we proceed by showing that G must have at least one of these configurations, which is an obvious contradiction. To prove this part we use the discharging method.

One key idea in the proof of Theorem 1.1 is as follows. To prove the reducibility of (some of) the configurations, we modify the configuration by removing some vertices and edges and by adding smaller number of vertices and edges, which will be called the “*gadget*”. This modification is designed carefully so that it enforces some properties that we need, without creating any i -cycles, $4 \leq i \leq 8$. Therefore the new graph will be in $\mathcal{G}_8$ and, since the original graph was a minimum counter-example, there must be a 3-coloring of this new graph. Then using the properties of the gadget we have added, we show that how this 3-coloring can be extended to a 3-coloring of the original graph. As we had assumed that the original graph is a counter-example to the theorem, this proves the reducibility of the configuration. These arguments will be clarified in section 3.

The total number of reducible configurations¹ is 74. We provide hand-checkable proofs for the (non-similar) configurations. We have also verified the reducibility of all the configurations using a simple and short computer program, which is available to the public to examine. The list of all 74 configurations is in Appendix A. The organization of the paper is as follows. The next section contains some preliminary definitions and notations. A hand-checkable proof of the reducible configurations is provided in section 3. The reader who is not interested in checking these configurations by hand may skip the proofs of Lemmas 3.2 to 3.13. Instead, he/she may find the program and the list of all reducible configurations at the URL <http://www.cs.toronto.edu/~mreza/3-color/index.html> in electronic form. Section 4 contains the discharging rules which also completes the proof of Theorem 1.1. There are only 7 discharging rules and they can be easily checked by hand. In section 5 we describe a quadratic time algorithm to find a 3-coloring. Finally, in section 6 we explain how the program for checking the reducibility of the configurations works.

2 Preliminaries

All graphs we consider are simple and finite. We denote the vertex set of a graph G by $V(G)$ and denote its edge set by $E(G)$. A k -vertex, $\leq k$ -vertex, or $\geq k$ -vertex, is a vertex which is incident with exactly k , at most k , or at least k edges, respectively. The notions of a k -face, $\leq k$ -face, and a $\geq k$ -face are defined similarly.

A 3-vertex is called a *simple* vertex if none of its incident faces is a triangle. A 3-vertex incident with a triangle is called a *bad* vertex. A 4-vertex incident with exactly zero, one, or two triangles is called a *type 0*, a *type 1*, or a *type 2* vertex, respectively. Note that every 4-vertex is one of these types, since G does not have any 4-cycle.

Let f be a 9-face which has eight bad vertices. Then f is called a simple, a type 0, a type 1, a type 2, or a type 5 face, if the ninth vertex of f is a simple, a type 0, a type 1, a type 2, or a 5-vertex, respectively. A semi-simple face is a 9-face having seven bad vertices and a type 1 vertex, whose ninth vertex is simple. Similarly, a semi-type 0, a semi-type 1, or a semi-type 2 face is a 9-face having seven bad vertices and a type 1 vertex, whose ninth vertex is type 0, type 1, or type 2, respectively. (see Figure 1 for some examples)

Let f_1 be a semi-type 0 face whose vertices are $v_1, v_2, \dots, v_9$, where v_1 is the type 0 vertex. Suppose

¹In fact, the number of reducible configurations is around 66 as the reducibility of some of them follows from the other ones. But the proof becomes much easier when based on 74 configurations.

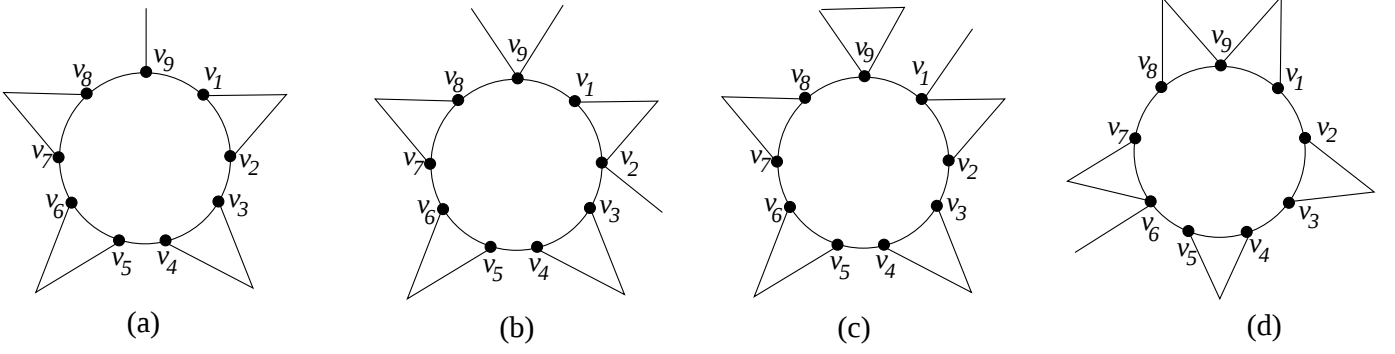


Figure 1: (a) A simple face, (b) a semi-type 0 face, (c) a semi-type 1 face, (d) a semi-type 2 face

that f_2 is a type 0 face whose type 0 vertex is v_1 . If v_i is the type 1 vertex of f_1 , for some $3 \leq i \leq 8$, and f_3 is a semi-simple face whose type 1 vertex is v_i , then we call this configuration a “*simple triple structure*”. If f_3 is a type 1 face whose type 1 vertex is v_i , then we call this configuration a “*triple structure of kind 1*”. Finally, if f_3 is a semi-type 2 face, then we call this configuration a “*triple structure of kind 2*”. See Figures 15(A1) and 17 for some sample graphs.

3 Reducible Configurations

From now on we assume that G is a minimum counter-example to Theorem 1.1. Trivially G is a connected graph. The first and the easiest reducible configuration is a ≤ 2 -vertex, which implies that the minimum vertex degree of G is 3. Otherwise, let v be a ≤ 2 -vertex. It is easy to see that $G - v \in \mathcal{G}_8$, and since G is a minimum counter-example, we can 3-color $G - v$ and then assign to v a color different from its (at most) two neighbors to get a 3-coloring of G .

The next reducible configuration is described in the following lemma. Although this is already proved in [7], we prove it here again for the sake of independence.

Lemma 3.1 *A $2k$ -face having (at least) $2k - 1$ bad vertices is reducible.*

Proof: Assume that f is a $2k$ -face whose vertices are $v_1, \dots, v_{2k}$, where $v_1, \dots, v_{2k-1}$ are bad vertices and v_{2i-1} and v_{2i} are incident with a triangle, $1 \leq i \leq k$. By minimality of G there is a 3-coloring of $G - (v_1, v_{2k})$, called C . Since G is not 3-colorable, $C(v_1) = C(v_{2k})$, which without loss of generality, we assume both are 1. We claim $C(v_3) = 1$, otherwise we could exchange $C(v_1)$ with $C(v_2)$ and get a 3-coloring of G , which would be an obvious contradiction. Using similar argument, we can show that $C(v_5) = 1$, and in general by induction, one can easily prove that $C(v_{2i+1}) = 1$, for $i = 0, 1, \dots, k - 1$. But $C(v_{2k-1})$ cannot be equal to 1, as it is adjacent to v_{2k} and $C(v_{2k}) = 1$. This contradiction completes the proof. ■

Lemma 3.2 *A simple face is reducible.*

Proof: Suppose that f is a simple face. Let's denote the bad vertices of f by $v_1, v_2, \dots, v_8$, in clockwise order, and call its simple vertex v_9 . We denote the vertex adjacent to both v_{2i-1} and v_{2i} by w_i , $1 \leq i \leq 4$. The third neighbor of v_9 is called w_5 . (see Figure 2(a)). We modify G in the following way: remove all $v_1, v_2, \dots, v_9$ and their incident edges from G . Then add six new vertices $u_1, u_2, \dots, u_6$. Make u_1, u_2, u_3

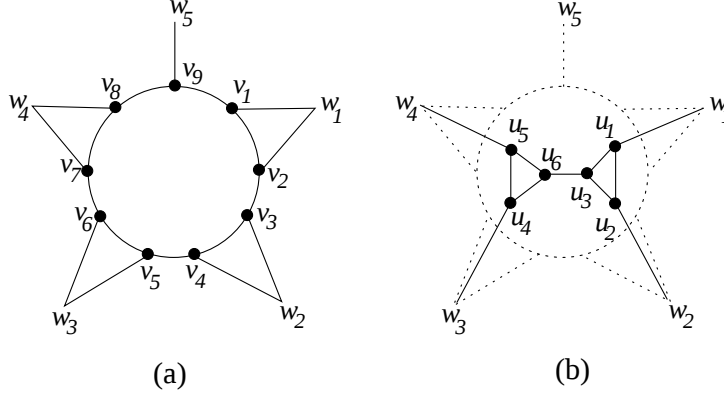


Figure 2: A simple face and the gadget added

and u_4, u_5, u_6 two triangles and add the following edges: $u_1w_1, u_2w_2, u_4w_3, u_5w_4, u_3u_6$. (see Figure 2(b)).

Call this graph G' and the new vertices and edges *the gadget*. Clearly G' is planar and it is straightforward to verify that the pairwise distances of $w_1, \dots, w_5$ in G' using only the vertices and the edges of the gadget is not less than their corresponding distances in G using only the vertices and the edges that are removed. So $G' \in \mathcal{G}_8$. Since the number of vertices of G' is smaller than those of G , therefore by minimality of G , there is a 3-coloring of G' which we call it C . A very useful property of the gadget is that $w_1, \dots, w_4$ cannot have all the same color in C . We can easily prove this by contradiction. Assume that they all have got the same color, say 1. Therefore, u_1, u_2, u_4 , and u_5 are all different from 1. Since u_1, u_2, u_3 and u_4, u_5, u_6 are triangles and we are using only three colors in C , both u_3 and u_6 (that are adjacent) should have been colored 1, which is impossible.

Consider coloring C induced on $G - \{v_1, \dots, v_9\}$. The only colored neighbor of v_9 is w_5 . So we can extend C to v_9 by assigning a color to it different from $C(w_5)$. Now the only two colored neighbors of v_8 are w_4 and v_9 , so there is a color available for v_8 . Using the same argument we can extend C by coloring $v_7, v_6, \dots, v_2$, greedily. By the time we get to v_1 this greedy algorithm will assign a color to v_1 different from $C(v_2)$ and $C(w_1)$. But since G is not 3-colorable, $C(v_1)$ must be equal to $C(v_9)$. Without loss of generality assume that $C(v_1) = C(v_9) = 1$. We could exchange $C(v_1)$ and $C(v_2)$ to resolve the conflict between $C(v_1)$ and $C(v_9)$, unless $C(v_3) = 1$. So assume that $C(v_3) = 1$. Similarly, we could exchange $C(v_3)$ and $C(v_4)$ to make $C(v_3) \neq 1$, unless $C(v_5) = 1$. So we must have $C(v_5) = 1$. By the same argument we can show that $C(v_7) = 1$.

Note: We have already used this technique in the proof of Lemma 3.1, and will use it frequently in the proofs of other lemmas. We call this argument the “*chaining argument*”.

On the other hand, without loss of generality, we can assume that $C(w_5) = 2$. Now if $C(v_8) = 2$ then we could simply assign $C(v_9) = 3$ and resolve the conflict between $C(v_9)$ and $C(v_1)$. Therefore $C(v_8) = 3$ and $C(w_4) = 2$. If $C(v_6) \neq 3$ then we could simply exchange $C(v_7)$ with $C(v_8)$ and set $C(v_9) = 3$. Therefore $C(v_6) = 3$ and $C(w_3) = 2$. Using the same argument $C(v_4) = C(v_2) = 3$ and $C(w_2) = C(w_1) = 2$. But this means that all $w_1, \dots, w_4$ have the same color in C , a contradiction. ■

The general idea of the proof of the other configurations is basically the same as above. In most of them we need to forbid some of the vertices from all having the same color. To do this, we remove some vertices and edges and add a gadget whose structure is similar to the one in the previous lemma. In all the cases the new graph does not have any i -cycles, for $4 \leq i \leq 8$, and is a smaller graph, therefore it is 3-colorable. Then we show that this 3-coloring induced on the original graph (which will be a partial

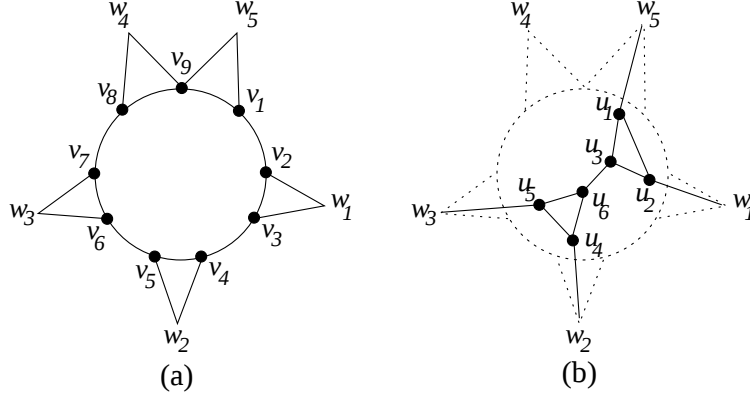


Figure 3: A type 2 face and the gadget added

3-coloring) can be extended to a 3-coloring of the whole graph, contradicting the assumption on G . This shows that the configuration is reducible.

Lemma 3.3 *A type 2 face is reducible.*

Proof: Suppose that f is a type 2 face. Let's denote the bad vertices of f by $v_1, v_2, \dots, v_8$, in clockwise order, and call its type 2 vertex v_9 . We denote the vertex adjacent to both v_{2i} and v_{2i+1} by w_i , $1 \leq i \leq 4$. The common neighbor of v_9 and v_1 is w_5 . (see Figure 3(a)). We modify G in a way similar to that of Lemma 3.2: remove $v_1, \dots, v_9$ and add a gadget similar to that of Lemma 3.2 by creating triangles u_1, u_2, u_3 and u_4, u_5, u_6 , and adding $u_1w_5, u_2w_1, u_4w_2, u_5w_3$, and u_3u_6 . (see Figure 3(b)).

It is straightforward to verify that the new graph G' is in $\mathcal{G}_8$. By minimality of G , there exists a 3-coloring of G' , say C . Note that by the same argument as we had in Lemma 3.2 we cannot have all w_5, w_1, w_2, w_3 colored with the same color in C . Consider C induced on G . Since the only colored neighbors of v_9 are w_4 and w_5 , we can extend C to v_9 . Assign a color different from $C(v_9)$ and $C(w_5)$ to v_1 . Also, starting from v_8 and moving around f towards v_2 in counterclockwise order, we can extend C by coloring $v_8, \dots, v_3$ greedily. Using this greedy algorithm, v_2 will get the same color as v_1 , say 1, since G is not 3-colorable. By the chaining argument similar to the one we had in Lemma 3.2, $C(v_4) = C(v_6) = C(v_8) = 1$. Without loss of generality we assume $C(v_9) = 2$ which yields $C(w_4) = C(w_5) = 3$. If $C(v_7) \neq 2$ then we could set $C(v_8) = 2$, $C(v_9) = 1$, and $C(v_1) = 2$ and get a 3-coloring of G . So $C(v_7) = 2$ and $C(w_3) = 3$. By the chaining argument $C(v_5) = C(v_3) = 2$. This means that $C(w_1) = C(w_2) = C(w_3) = C(w_5) = 3$, which is a contradiction. ■

Lemma 3.4 *Two type 0 faces sharing a type 0 vertex is reducible.*

Proof: Suppose that f_1 and f_2 are two type 0 faces sharing a type 0 vertex. There are two possibilities. We consider each case separately:

Configuration of Figure 4(a): First we remove $v_1, \dots, v_9$ and $u_2, \dots, u_8$ and all the incident edges. Then add four new triangles and connect them together and to the rest of the vertices of G as in Figure 4(b). Call this new graph G' . It is straightforward to verify that: (i) $G' \in \mathcal{G}_8$ (ii) because of minimality of G there is a 3-coloring of G' , say C , and (iii) $w_1, \dots, w_6$ cannot all have the same color in C .

Now consider this 3-coloring induced on G . We can extend C to v_1 easily, since only one neighbor of v_1 , which is u_1 , is colored. Similarly, we can extend C by coloring $v_9, \dots, v_3$ greedily. Also, starting from u_2 and moving around f_2 in clockwise order, we can color $u_3, \dots, u_8$, greedily. Now assign a

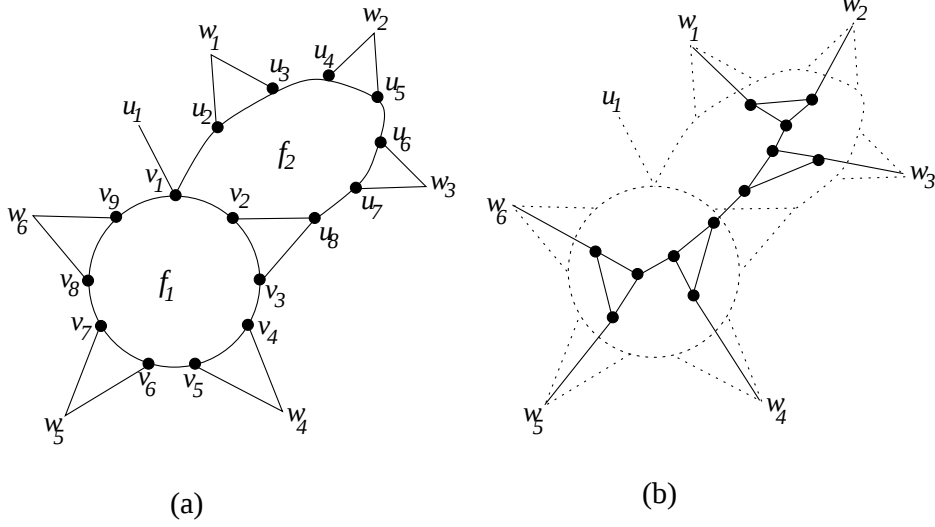


Figure 4: Two type 0 faces sharing a type 0 vertex

color different from $C(v_3)$ and $C(u_8)$ to v_2 , which will be equal to $C(v_1)$. Without loss of generality, assume that $C(v_1) = C(v_2) = 1$. By the chaining argument starting from v_2 and going around f_1 : $C(v_4) = C(v_6) = C(v_8) = 1$. Similarly, by the same argument for the vertices around f_2 : $C(u_7) = C(u_5) = C(u_3) = 1$.

Without loss of generality assume $C(u_1) = 3$. Suppose that $C(u_2) = 3$. First exchange $C(v_3)$ with $C(u_8)$ (if needed) so that $C(v_3) \neq C(v_5)$. Now exchange $C(v_9)$ with $C(v_8)$, $C(v_7)$ with $C(v_6)$, and $C(v_5)$ with $C(v_4)$, and set $C(v_1) = 2$. This gives a 3-coloring of G which is a contradiction. Thus $C(u_2) = 2$ and by the chaining argument $C(u_4) = C(u_6) = C(u_8) = 2$. Using exactly the same argument we can show that $C(v_9) = 2$ and by the chaining argument $C(v_7) = C(v_5) = 2$. But this means that $w_1, \dots, w_6$ all have color 3 in C , contradicting property (iii) mentioned for C .

Configuration of Figure 5(a): First remove $v_1, \dots, v_9$ and $u_1, \dots, u_8$ and all the incident edges. Then add four new triangles and connect them together and to the rest of the vertices of G as in Figure 5(b). Call this new graph G' . Again, it is straightforward to verify that: (i) $G' \in \mathcal{G}$, (ii) because of minimality of G there is a 3-coloring of G' , say C , (iii) $w_1, \dots, w_4$ cannot all have the same color in C . Similarly, $t_1, \dots, t_4$ cannot all have the same color in C .

Now consider this 3-coloring induced on G . We extend C by coloring the uncolored vertices of G greedily in the following order: $u_8, u_7, \dots, u_1, v_1, v_9, v_8, \dots, v_3$, since at each step there are at most two colors in the neighborhood of the vertex we want to color. We can also assign a color different from $C(w_1)$ and $C(v_3)$ to v_2 . By definition of G , $C(v_1) = C(v_2)$, which we can assume is equal to 1. By the chaining argument $C(v_4) = C(v_6) = C(v_8) = 1$.

Without loss of generality, assume that $C(v_9) = 3$. We exchange $C(v_9)$ with $C(v_8)$. If $C(v_7) = 3$ exchange $C(v_7)$ with $C(v_6)$ and then if $C(v_5) = 3$ exchange $C(v_5)$ with $C(v_4)$. In this case $C(v_3)$ cannot be 3, otherwise all $w_1, \dots, w_4$ have color 2, contradicting (iii). Note that now $C(v_9) = C(v_1) = C(v_2) = 1$. If $C(u_1) = C(u_8)$ then there are only two colors in the neighborhood of v_1 and so we can assign a different color to it and get a 3-coloring of G . So let's assume that $C(u_1) = 2$ and $C(u_8) = 3$ (the other case is symmetric). Since $C(u_7)$ is either 1 or 2, if we could exchange $C(u_8)$ with $C(u_7)$ then there would be only two colors in the neighborhood of v_1 and we could use the same argument as mentioned above to extend C to G . Therefore, $C(u_6) = 3$. Similarly, by the chaining argument, $C(u_4) = C(u_2) = 3$. Using

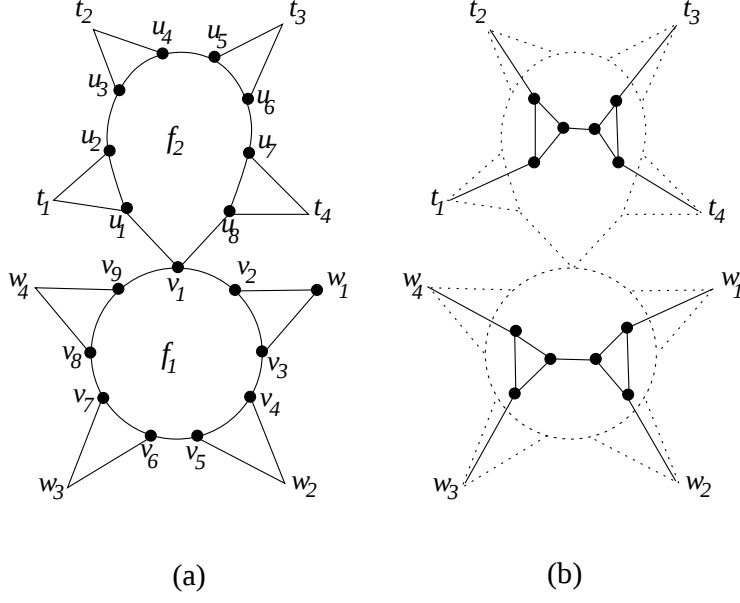


Figure 5: Two type 0 faces sharing a type 0 vertex

the same argument and starting from $C(u_1)$ we can show that $C(u_1) = C(u_3) = C(u_5) = C(u_7) = 2$. But this yields $C(t_1) = C(t_2) = C(t_3) = C(t_4) = 1$, which again is a contradiction. ■

Lemma 3.5 *Three type 5 faces sharing a 5-vertex is reducible.*

Proof: There are two possible non-symmetric configurations, which are shown in Figures 6(a) and 7(a). We consider each case separately:

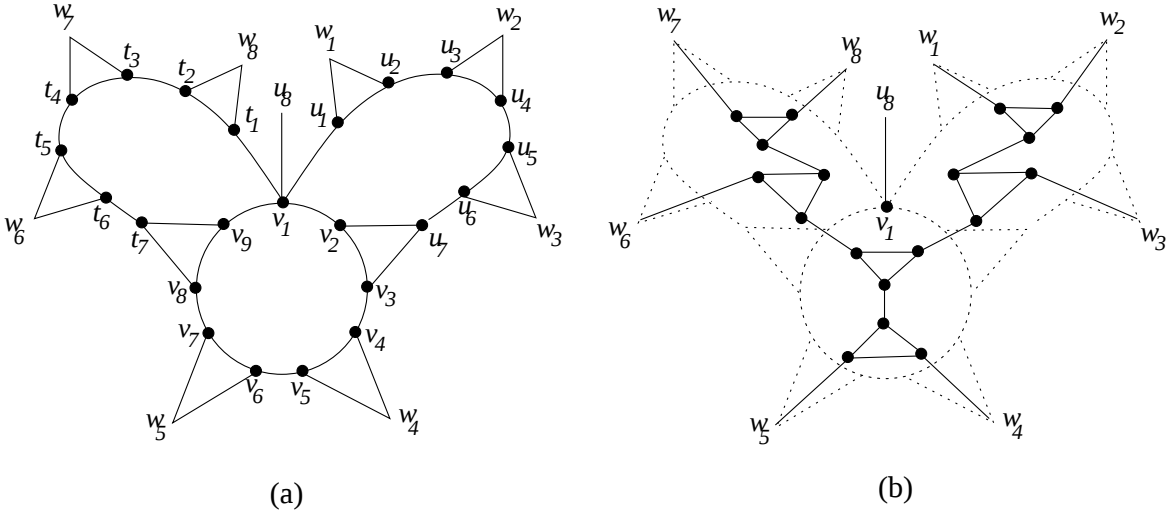


Figure 6: Three type 5 faces sharing a 5-vertex

Configuration of Figure 6(a): First we remove $u_1, \dots, u_7, v_2, \dots, v_9, t_1, \dots, t_7$, and all the incident edges. Then add six new triangles and connect them together and to the rest of the vertices of G as in Figure 6(b). Call this new graph G' . Again, it is straightforward to verify that: (i) $G' \in \mathcal{G}_8$, (ii)

because of minimality of G there is a 3-coloring of G' , say C , and (iii) $w_1, \dots, w_8$ cannot all have the same color in C .

Consider this 3-coloring induced on G . We extend C by coloring the uncolored vertices of G greedily in the following order: $t_1, t_2, \dots, t_7, v_9, v_8, \dots, v_2, u_7, u_6, \dots, u_2$, since at each step there are at most two colors in the neighborhood of the vertex we want to color. We also assign a color different from $C(u_2)$ and $C(w_1)$ to u_1 . Since G is not 3-colorable, $C(u_1) = C(v_1)$, which we can assume is equal to 1. By the chaining argument, $C(u_3) = 1$, and also all $u_5, u_7, v_4, v_6, v_8, t_6, t_4$, and t_2 must have been colored 1. First we show that $C(t_1) \neq C(u_8)$. Assume that they are both equal, say 2. We can exchange $C(t_7)$ with $C(v_9)$ (if needed) so that $C(v_9) = 2$, too. Similarly, we can exchange $C(v_2)$ with $C(v_3)$ if needed to set $C(v_2) = 2$. Then we can set $C(v_1) = 3$ and get a 3-coloring of G .

So we can assume that $C(t_1) = 3$ and $C(u_8) = 2$. If we could exchange $C(t_1)$ with $C(t_2)$ by an argument similar to the previous case, we can set $C(v_9) = C(v_2) = 2$ and set $C(v_1) = 3$. This shows that we cannot exchange $C(t_1)$ with $C(t_2)$, because $C(t_3) = 3$. By the chaining argument $C(t_5) = 3$, too. Now, if $C(v_7) = 2$ then we could set $C(v_8) = 3$, $C(v_9) = 1$, $C(t_7) = 2$, and exchange $C(t_6)$ with $C(t_5)$, $C(t_4)$ with $C(t_3)$, and $C(t_2)$ with $C(t_1)$, and set $C(v_1) = 3$. This shows that $C(v_7) = 3$. By the chaining argument $C(v_5) = 3$, and by a similar argument we can show that $C(u_6) = C(u_4) = C(u_2) = 3$. All these shows that $w_1, \dots, w_8$ are all colored with 2 which is contradicting property (iii) mentioned above for C .

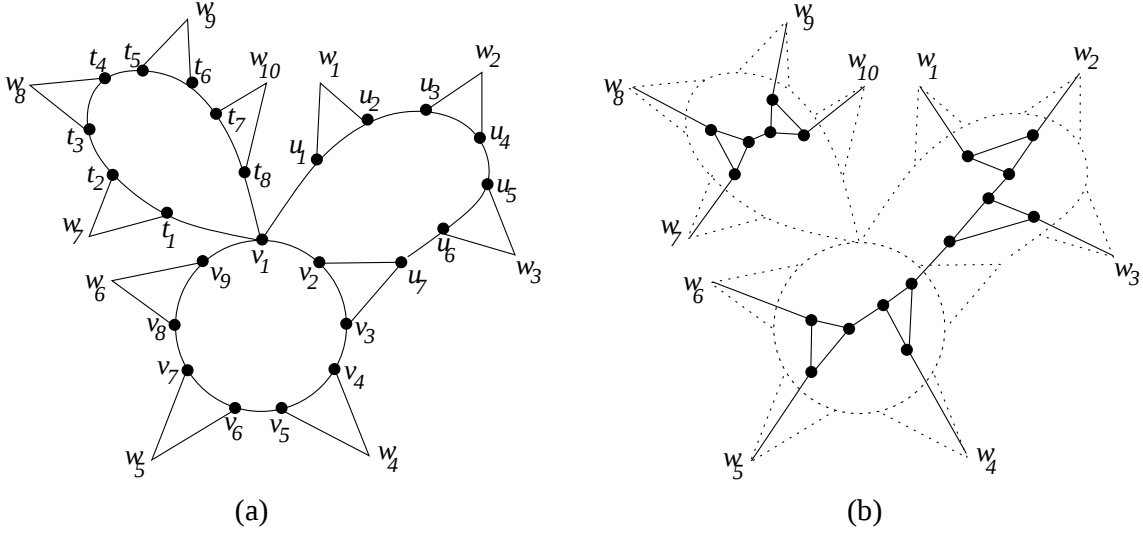


Figure 7: Three type 5 faces sharing a 5-vertex

Configuration of Figure 7(a): First remove $v_1, \dots, v_9, t_1, \dots, t_8, u_1, \dots, u_7$ and all the incident edges. Then add six new triangles and connect them together and to the rest of the vertices of G as in Figure 7(b). Call this new graph G' . It is straightforward to verify that: (i) $G' \in \mathcal{G}_8$, (ii) because of minimality of G there is a 3-coloring of G' , say C , and (iii) $w_1, \dots, w_6$ cannot all have the same color in C . Also, $w_7, \dots, w_{10}$ cannot all have the same color in C .

Consider this 3-coloring induced on G . We extend C by coloring the uncolored vertices of G greedily in the following order: $t_8, t_7, \dots, t_1, v_1, v_9, v_8, \dots, v_2, u_7, u_6, \dots, u_2$, since at each step there are at most two colors in the neighborhood of the vertex we want to color. We also assign a color different from $C(u_2)$ and $C(w_1)$ to u_1 . Since G is not 3-colorable, $C(u_1) = C(v_1)$, which we can assume is equal to 1. By the chaining argument, $C(u_3) = 1 = C(u_5) = C(u_7) = C(v_4) = C(v_6) = C(v_8)$.

First we show that $C(t_1) \neq C(t_8)$. By contradiction assume that they are equal to 2. So $C(v_9) = 3$, otherwise we could simply set $C(v_1) = 3$ and exchange $C(v_2)$ with $C(v_3)$ if needed. By the chaining argument $C(v_7) = C(v_5) = 3$. By the jumping and chaining arguments, $C(u_6) = 3 = C(u_4) = C(u_2)$. But this requires that all $w_1, \dots, w_6$ be colored 2, which is contradicting property (iii) mentioned above.

So we can assume that $C(t_1) = 2$ and $C(t_8) = 3$. If we could exchange $C(t_8)$ with $C(t_7)$ then we could use the same argument as in the previous paragraph to modify C so that there are only colors 1 and 2 in the neighborhood of v_1 and set $C(v_1) = 3$ to get a 3-coloring of G . This contradiction shows that $C(t_6) = 3$, and by the chaining argument $C(t_4) = C(t_2) = 3$. We can do a very similar argument to show that $C(t_3) = 2$ and by the chaining argument $C(t_5) = C(t_7) = 2$. But then we have to have $C(w_7) = C(w_8) = C(w_9) = C(w_{10}) = 1$ which is contradicting property (iii) we mentioned. ■

Lemma 3.6 *Two semi-simple faces sharing a type 1 vertex is reducible.*

Proof: Instead, we prove that the four configurations shown in Figures 8(A), (B), (C), and (D), are reducible. Each of these configurations contains a semi-simple face f_1 , in which the both neighbors of its type 1 vertex which are not incident with f_1 are a 3-vertex. The proof of lemma follows easily. We first give the proof for the configuration of Figure 8(A): By minimality of G , there is a 3-coloring of

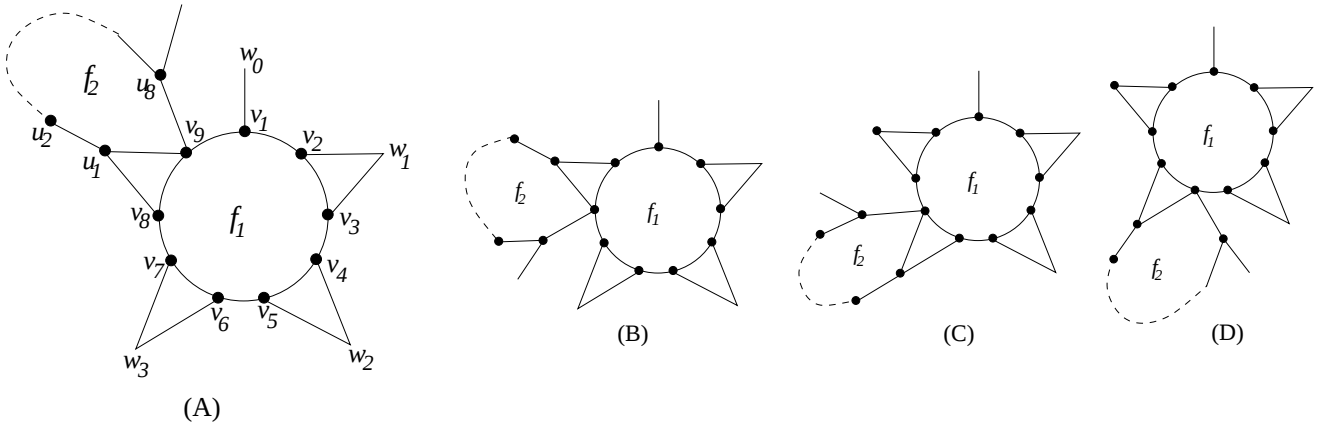


Figure 8: Two semi-simple faces sharing their type 1 vertex

$G' = G - v_1v_2$, called C . So $C(v_1) = C(v_2)$, which we can assume is equal to 1. Consider this coloring induced on G . By the chaining argument $C(v_4) = C(v_6) = C(v_8) = C(u_2) = 1$, otherwise we could 3-color G . Without loss of generality, assume $C(w_0) = 2$. So $C(v_9) = 3$ and $C(u_1) = 2$, otherwise we could set $C(v_1) = 3$. If $C(u_8) = 1$ then we could exchange $C(u_1)$ with $C(v_9)$ and set $C(v_1) = 3$. Therefore $C(u_8) = 2$. Now set $C(v_1) = 3$, $C(v_9) = 1$ and assign a color different from 1 and $C(v_7)$ (which is either 2 or 3) to v_8 and give a color different from $C(v_9)$ (which is 1) and $C(v_8)$ to u_1 (we can do this because $C(v_9) = C(u_2) = 1$). This gives a 3-coloring of G , which is a contradiction.

Using very similar arguments, we can show that the configurations of Figures 8(B), (C), and (D) are reducible. ■

Lemma 3.7 *Two semi-type 2 faces sharing a type 1 vertex is reducible.*

Proof: Suppose that f_1 and f_2 are two semi-type 2 faces sharing a type 1 vertex. There are eight possible configurations of this type up to isomorphism, we consider each one separately. Assume that $v_1, \dots, v_9$ are the vertices of f_1 , where v_9 is the type 2 vertex. In the first two cases we assume that v_1

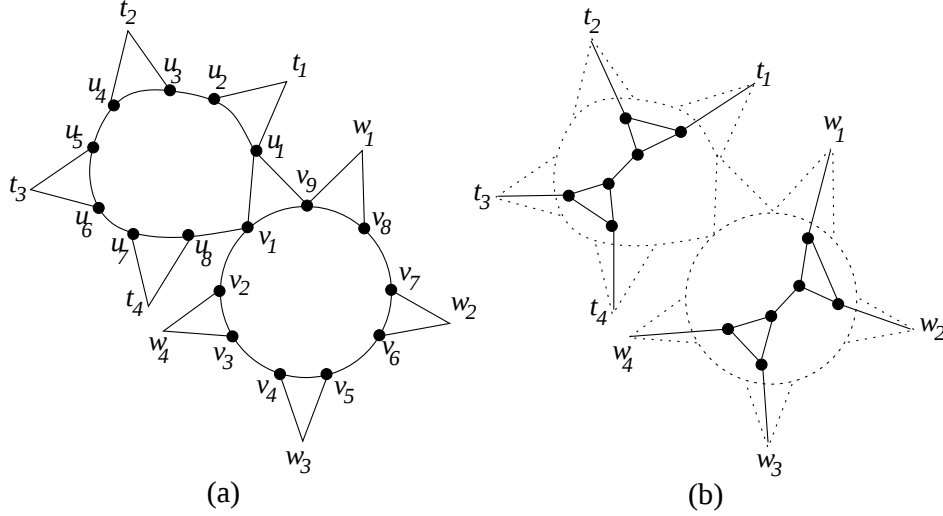


Figure 9: Two semi-type 2 faces sharing their type 2 vertex

is the type 2 vertex of f_1 (Figures 9(a) and 10(a)). The other cases are based on v_2 , v_3 , or v_4 being the type 2 vertex of f_1 , shown in Figure 11.

Configuration of Figure 9(a): In this case u_1 is the type 2 vertex of f_2 . First we remove some vertices and edges and add two gadgets each similar to the one in lemma 3.2. The vertices to be removed are $v_1, \dots, v_9$ and $u_1, \dots, u_8$, and the new graph G' after adding the gadgets is shown in Figure 9(b). It is straightforward to verify that: (i) $G' \in \mathcal{G}_8$, (ii) because of minimality of G there is a 3-coloring of G' , say C , and (iii) $w_1, \dots, w_4$ cannot all have the same color in C . Also, $t_1, \dots, t_4$ cannot all have the same color in C .

Consider this 3-coloring induced on G . First we show that $C(w_1) \neq C(t_1)$. By contradiction, assume that $C(w_1) = C(t_1) = 3$. Now we can extend C to a new coloring C' in this way: for all common vertices of G and G' , C' and C are equal. Then assign $C'(v_1) = 3$, and color $u_8, u_7, \dots, u_1$ greedily. Note that by the time we reach to u_1 it has three colored neighbors but two of them (v_1 and t_1) have the same color. Assume that $C'(u_1) = 2$. Set $C'(v_9) = 1$, $C'(v_8) = 2$, and color $v_2, v_3, \dots, v_6$ greedily. Finally, assign a color different from $C'(v_6)$ and $C'(w_2)$ to v_7 . By minimality of G , both v_6 and v_7 have the same color, which is 2. By the chaining argument we must have $C'(v_5) = C'(v_3) = C'(v_1) = 2$, but $C'(v_1) = 3$. This contradiction shows that $C(w_1) \neq C(t_1)$.

Now we extend C to color the uncolored vertices of G in a different way. Assume that $C(w_1) = 3$. Since $C(t_1) \neq C(w_1)$ we can assign $C(u_1) = 3$ and color the uncolored vertices of G greedily in the following order: $u_2, \dots, u_8$, $v_1, v_9, v_8, v_2, v_3, \dots, v_6$. Note that by the time we want to color v_9 there are two neighbors of it (u_1 and w_1) that have the same color and so we can find a color for v_9 . We also assign a color different from $C(v_6)$ and $C(w_2)$ to v_7 . By definition of G , $C(v_8) = C(v_7)$, which we can assume is equal to 1. By the chaining argument $C(v_5) = C(v_3) = C(v_1) = 1$, and so $C(v_9) = 2$.

Suppose that $C(u_8) \neq 2$. We can set $C(v_1) = 2$, $C(v_9) = 1$, and $C(v_8) = 2$, unless $C(v_2) = 2$ and by the chaining argument $C(v_2) = C(v_4) = C(v_6) = 2$. But this means that all $w_1, \dots, w_4$ have color 3, which contradicts property (iii).

Now assume that $C(u_8) = 2$. If we could exchange $C(u_8)$ and $C(u_7)$ then $C(u_8)$ becomes different from 2 and we can use the argument of the previous paragraph. This shows that $C(u_6) = 2$ and by the chaining argument $C(u_4) = C(u_2) = 2$. If $C(u_3) \neq 3$ then we can modify C in the following way: set $C(u_2) = 3$, $C(u_1) = 2$, $C(v_1) = 3$, $C(v_9) = 1$, $C(v_8) = 2$, exchange $C(v_2)$ with $C(v_3)$ if $C(v_2) = 3$,

exchange $C(v_4)$ with $C(v_5)$ if $C(v_4) = 3$, and finally exchange $C(v_6)$ with $C(v_7)$ if $C(v_6) = 3$, which yields a 3-coloring of G . Therefore, $C(u_3) = 3$ and by the chaining argument $C(u_5) = C(u_7) = 3$. But this means that all $t_1, \dots, t_4$ have color 1, again contradicting (iii).

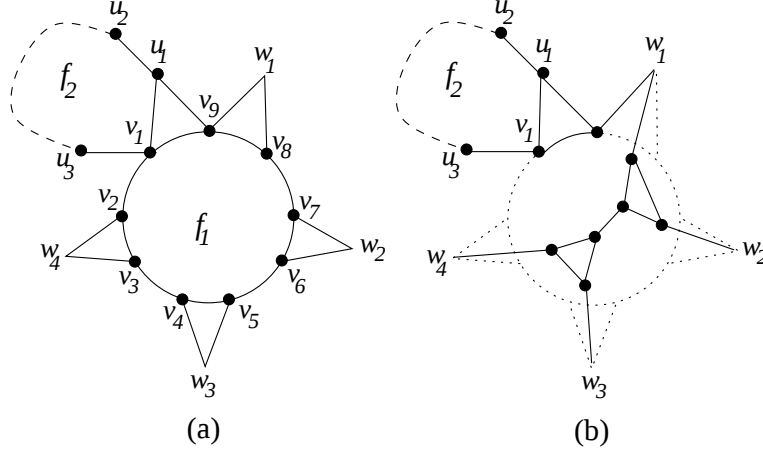


Figure 10: Two semi-type 2 faces sharing their type 2 vertex

Configuration of Figure 10(a): In this case u_1 is a 3-vertex in f_2 . First we remove $v_2, \dots, v_8$ and add a gadget similar to that of Lemma 3.2. The new graph G' is shown in Figure 10(b). It can be easily shown that: (i) $G' \in \mathcal{G}_8$, (ii) because of minimality of G there is a 3-coloring of G' , say C , and (iii) $w_1, \dots, w_4$ cannot all have the same color in C .

Consider this 3-coloring induced on G . We extend C by coloring the uncolored vertices of G greedily in the following order: $v_8, v_2, \dots, v_6$. Then assign a color different from $C(v_6)$ and $C(w_2)$ to v_7 . By minimality of G , $C(v_7) = C(v_8)$ which we can assume both are 1. By the chaining argument $C(v_5) = C(v_3) = C(v_1) = 1$. Without loss of generality, assume that $C(v_9) = 2$ and so $C(u_1) = C(w_1) = 3$.

If $C(u_3) = 3$ then we could set $C(v_8) = 2$, $C(v_9) = 1$, $C(v_1) = 2$, then exchange $C(v_2)$ with $C(v_3)$ if $C(v_2) = 2$, and then exchange $C(v_5)$ with $C(v_4)$ if $C(v_4) = 2$. In this case $C(v_6) \neq 2$, otherwise $w_1, \dots, w_4$ all are colored 3, a contradiction.

So assume that $C(u_3) = 2$. If $C(u_2) = 2$ then we can exchange $C(v_1)$ with $C(u_1)$, $C(v_2)$ with $C(v_3)$, $C(v_4)$ with $C(v_5)$, and $C(v_6)$ with $C(v_7)$, which gives a 3-coloring of G . If $C(u_2) = 1$ then we set $C(u_1) = 2$, $C(v_1) = 3$, $C(v_9) = 1$, and $C(v_8) = 2$. Then we can exchange $C(v_2)$ with $C(v_3)$ if $C(v_2) = 3$, then exchange $C(v_4)$ with $C(v_5)$ if $C(v_4) = 3$, and finally exchange $C(v_6)$ with $C(v_7)$ if $C(v_6) = 3$. So we get a 3-coloring of G , which again is a contradiction.

Configurations of Figure 11: The other possibilities, up to isomorphism, for two semi-type 2 faces to sharing their type 2 vertex are shown in Figure 11. Here we only give the proof for configuration of Figure 11(A). The proof for the other configurations is almost the same.

By minimality of G , there is a 3-coloring of $G - (v_7, v_8)$, called C . Consider this coloring induced on G in which both v_7 and v_8 have the same color. Without loss of generality, assume that $C(v_7) = C(v_8) = 1$. By the chaining argument $C(v_5) = C(v_3) = C(u_7) = C(u_5) = C(u_3) = 1$. So $C(v_2) \neq 1$.

First assume that both u_1 and v_1 have the same color different from 1, say 2. Then we can exchange $C(v_2)$ with $C(v_3)$, $C(v_4)$ with $C(v_5)$, and $C(v_6)$ with $C(v_7)$, which yields a 3-coloring of G , a contradiction. Also, $\{C(v_1), C(u_1)\} \neq \{2, 3\}$, since $C(v_2) \neq 1$. So at least one of $C(v_1)$ or $C(u_1)$ is 1.

Assume that $C(v_1) = 1$ and $C(u_1) = 2$. So $C(v_2) = 3$. If $C(v_9) = 2$ we can set $C(v_1) = C(v_8) = 2$ and $C(v_9) = 1$ which gives a 3-coloring of G . On the other hand, if $C(v_9) = 3$ we can modify C in this way: set $C(v_2) = 1$, $C(v_1) = 3$, $C(v_9) = 1$, $C(v_8) = 3$, assign a color different from $C(v_4)$ and 1 to v_3 .

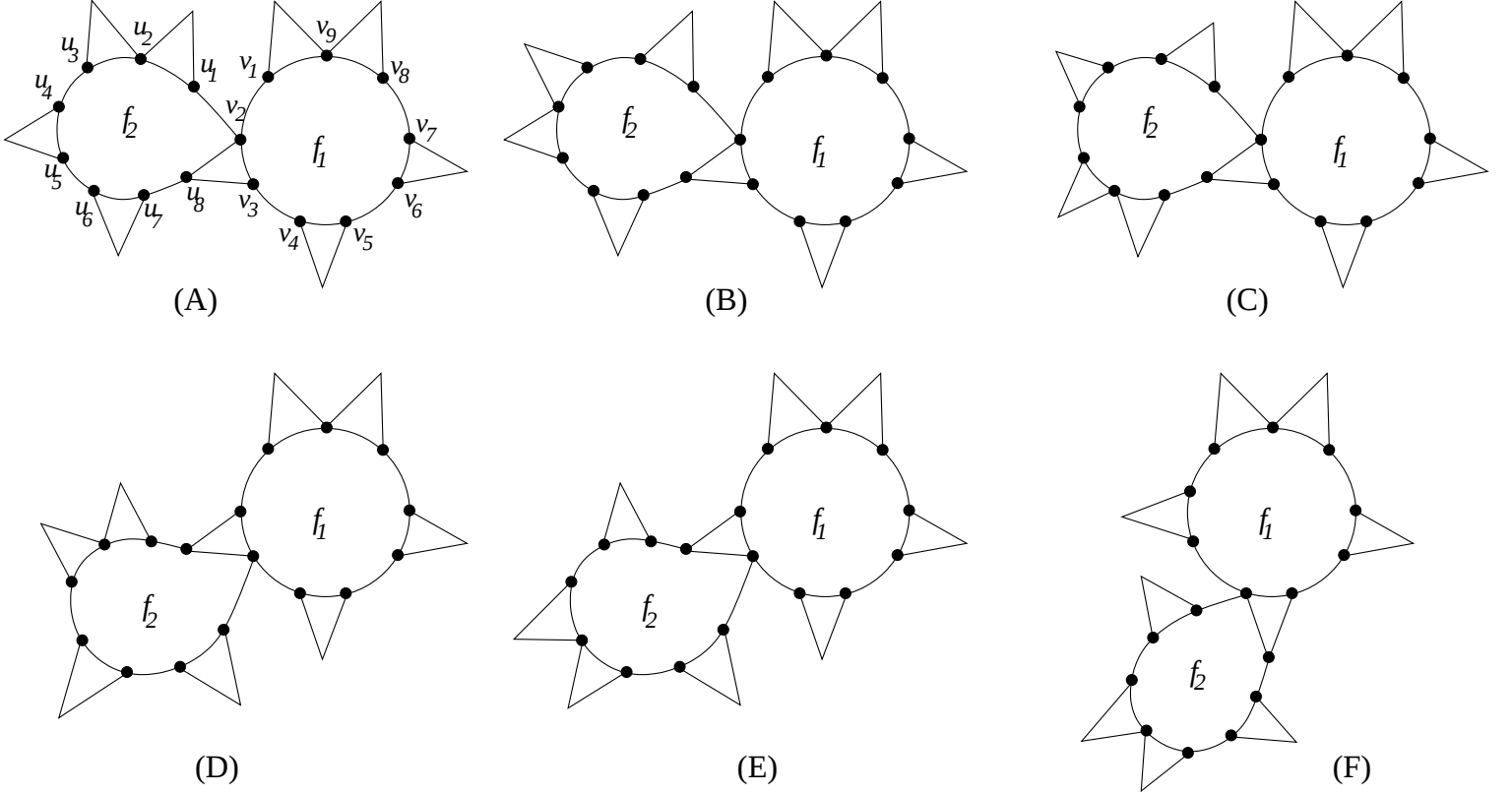


Figure 11: Two semi-type 2 faces sharing their type 2 vertex

Now since $C(v_2) = C(u_7) = 1$, we can assign a color different from 1 and $C(v_3)$ to u_8 . This gives a 3-coloring of G , an obvious contradiction.

Now, let's assume that $C(u_1) = 1$ and $C(v_1) = 2$. So $C(v_2) = 3$ and $C(u_8) = 2$. If $C(u_2) = 2$ then set $C(u_1) = 2$, $C(u_2) = 1$, $C(u_3) = 2$, exchange $C(u_4)$ with $C(u_5)$, $C(u_6)$ with $C(u_7)$, $C(u_8)$ with $C(v_3)$, $C(v_4)$ with $C(v_5)$, and $C(v_6)$ with $C(v_7)$, which yields a 3-coloring of G . If $C(u_2) = 3$ then set $C(u_1) = C(u_3) = 3$, $C(u_2) = 1$, $C(v_2) = 1$, exchange $C(u_4)$ with $C(u_5)$, and $C(u_6)$ with $C(u_7)$. Assign a color different from $C(v_2)$ (which is 1) and $C(u_7)$ to u_8 . Then assign a color different from 1 and $C(u_8)$ to v_3 . Now exchange $C(v_4)$ with $C(v_5)$ and $C(v_6)$ with $C(v_7)$. This again is a 3-coloring of G .

Finally, assume that $C(v_1) = C(u_1) = 1$. Without loss of generality, assume that $C(v_9) = 2$. If $C(v_2) = 2$ we exchange it with $C(u_8)$ so that $C(v_2) \neq C(v_9)$. Now set $C(v_1) = 2$, $C(v_9) = 1$, and $C(v_8) = 2$. This yields a 3-coloring of G , which is a contradiction. ■

Lemma 3.8 *A semi-type 2 face sharing its type 1 vertex with a type 1 face is reducible.*

Proof: There are four possible configurations of this type up to isomorphism, shown in Figures 12(a), 13(A1), 13(B1), and 13(C1). We consider each one separately:

Configuration of Figure 12(a): First remove $v_2, v_3, \dots, v_8$ and all the incident edges and create the graph G' as in Figure 12(b) by adding a gadget. It is straightforward to verify that: (i) $G' \in \mathcal{G}_8$, (ii) because of minimality of G there is a 3-coloring of G' , say C , and (iii) $w_1, \dots, w_4$ cannot all have the same color in C .

Consider this 3-coloring induced on G . We extend C by coloring the uncolored vertices of G greedily in the following order: $v_2, v_8, v_7, \dots, v_4$. We also assign a color different from $C(v_2)$ and $C(w_4)$ to v_3 .

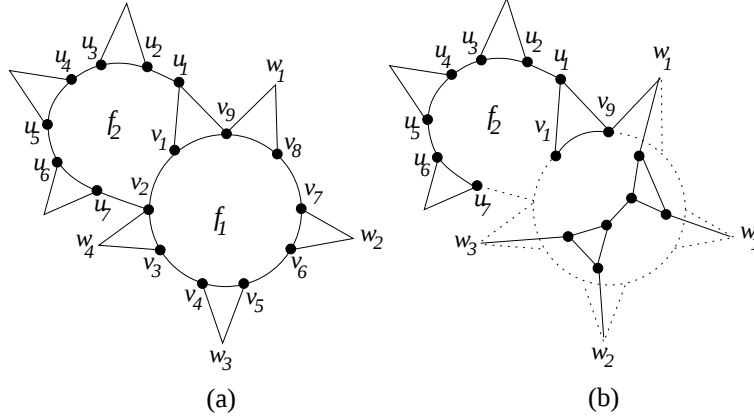


Figure 12: A semi-type 2 face sharing a type 1 vertex with a type 1 face

By definition of G , $C(v_3) = C(v_4)$, which we can assume is equal to 1, and by the chaining argument $C(v_6) = C(v_8) = 1$ and at least one of $C(v_1)$ or $C(u_7)$ must be 1.

First assume that $C(u_7) = 1$ and $C(v_1) \neq 1$. By the chaining argument $C(u_5) = C(u_3) = C(u_1) = 1$. Without loss of generality assume that $C(v_9) = 2$ and so $C(w_1) = 3$. Now set $C(v_9) = 1$ and $C(u_1) = C(v_8) = 2$, exchange $C(u_2)$ with $C(u_3)$, $C(u_4)$ with $C(u_5)$, $C(u_6)$ with $C(u_7)$, and $C(v_3)$ with $C(v_2)$. The only conflict we may have is between $C(v_8)$ and $C(v_7)$, which happens if $C(v_7) = 2$. We can exchange $C(v_7)$ with $C(v_6)$, unless $C(v_5) = 2$. In this case we can exchange $C(v_5)$ with $C(v_4)$, unless $C(v_3) = 2$. But this means that all $w_1, \dots, w_4$ have been colored 3, which contradicts (iii).

Now assume that $C(v_1) = 1$ and $C(u_7) \neq 1$. By the chaining argument $C(u_2) = C(u_4) = C(u_6) = 1$. Assume that $C(v_9) = 2$. Set $C(v_9) = 1$, $C(v_1) = C(v_8) = 2$, and exchange $C(v_2)$ with $C(v_3)$. Similar to the previous case we can solve the possible conflict between $C(v_8)$ and $C(v_7)$, unless all $w_1, \dots, w_4$ have color 3, which is impossible, according to (iii).

Finally, assume that $C(v_1) = C(u_7) = 1$. If we could modify $C(v_1)$ or $C(u_7)$ then we would reduce to the one of the two cases we just considered. Therefore, by the chaining argument and starting from u_7 : $C(u_5) = C(u_3) = C(u_1) = 1$, which is impossible, since $C(v_1) = 1$. This completes the proof of this configuration.

The other three possible configuration of this kind, up to isomorphism, are shown in Figure 13(A1), (B1), and (C1). First consider the configuration of Figure 13(A1).

Remove $v_1, v_2, \dots, v_9$ and $u_1, \dots, u_7$ and all the incident edges and create the graph G' as in Figure 13(A2). It is straightforward to verify that: (i) $G' \in \mathcal{G}_8$, (ii) because of minimality of G there is a 3-coloring of G' , say C , and (iii) $w_1, \dots, w_6$ cannot all have the same color in C .

Consider this 3-coloring induced on G . We extend C by coloring the uncolored vertices of G greedily in the following order: $v_9, v_1, v_8, v_7, \dots, v_3, u_1, u_2, \dots, u_7$. We also assign a color different from $C(v_3)$ and $C(u_7)$ to v_2 . By definition of G , $C(v_1) = C(v_2)$, which we can assume is equal to 1, and by the chaining argument $C(v_4) = C(v_6) = C(v_8) = C(u_6) = C(u_4) = C(u_2) = 1$. Without loss of generality assume that $C(v_9) = 2$.

If $C(u_1) = 3$ then we can set $C(v_1) = C(v_8) = 2$, $C(v_9) = 1$, then exchange $C(v_7)$ with $C(v_6)$ if $C(v_7) = 2$, then exchange $C(v_5)$ with $C(v_4)$ if $C(v_5) = 2$, and finally exchange $C(v_3)$ with $C(u_7)$ if $C(v_3) = 2$. This yields a 3-coloring of G .

So we can assume that $C(u_1) = 2$. If we could exchange $C(u_1)$ with $C(u_2)$ we could use the argument of the previous paragraph. So by the chaining argument $C(u_3) = C(u_5) = 2$. We could use the same argument as in the previous case unless $C(v_7) = C(v_5) = 2$. This means that all $w_1, \dots, w_6$ have

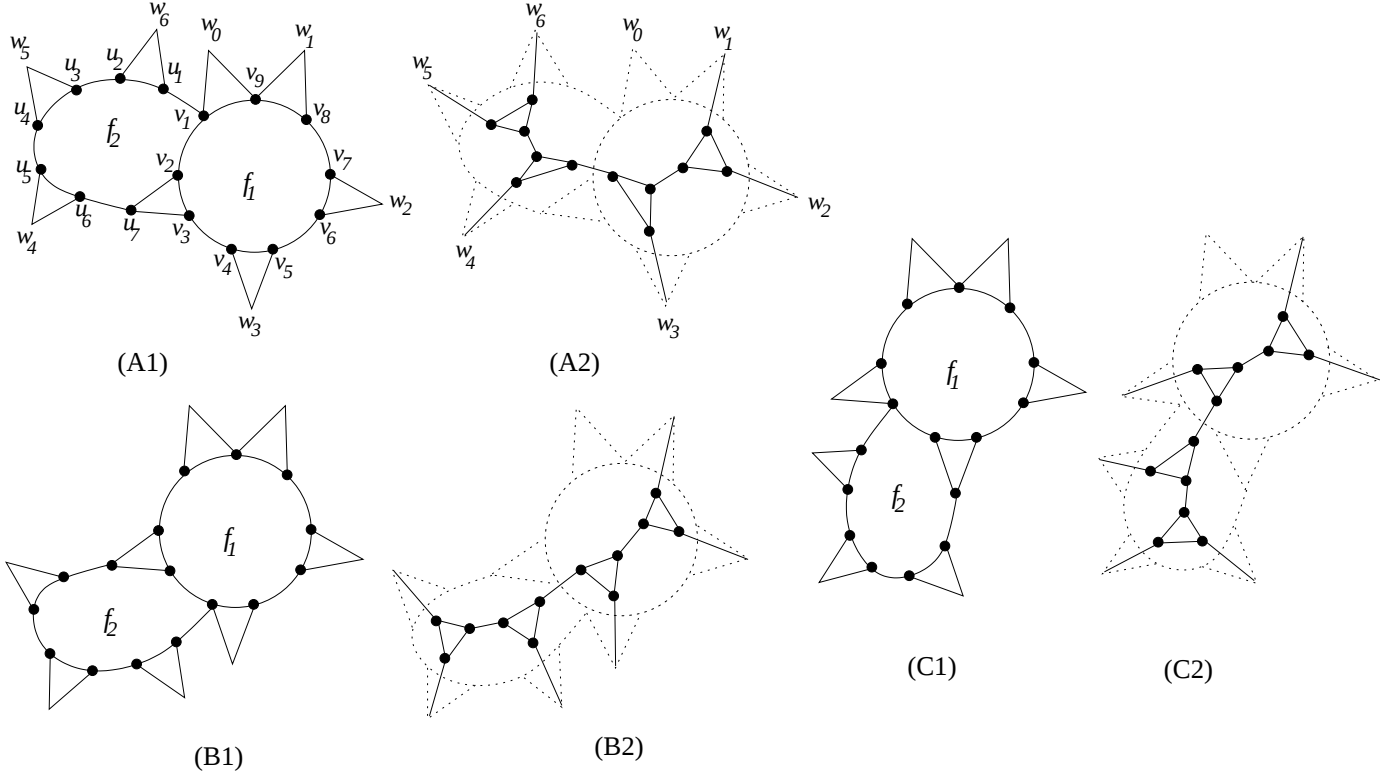


Figure 13: A semi-type 2 face sharing a type 1 vertex with a type 1 face

been colored 3 in C , contradicting property (iii) we just mentioned. This completes the proof of this configuration.

Using a very similar argument, we can prove the reducibility of configurations of Figure 13(B1) and (C1). The gadget we have to add in each case is shown in Figures 13(B2) and (C2), respectively. ■

Lemma 3.9 *A semi-type 2 face sharing its type 1 vertex with a semi-simple face is reducible.*

Proof: It is straightforward to check that there are five possible configurations of this type up to isomorphism. One of them is the same as the configuration of Figure 10(a), and the other four ones are equivalent to configurations of Figures 8(A1), 8(B1), 8(C1), and 8(D1). Each of these configurations are already proved to be reducible. ■

Lemma 3.10 *A semi-simple face sharing its type 1 vertex with a type 1 face is reducible.*

Proof: There are three possible configuration up to isomorphism, shown in Figure 14(A1), (B1), and (C1). Let's consider (A1).

First remove $v_1, \dots, v_9$ and $u_1, \dots, u_7$, and all the incident edges and create the graph G' as in Figure 14(A2). It is straightforward to verify that: (i) $G' \in \mathcal{G}_8$, (ii) because of minimality of G there is a 3-coloring of G' , say C , and (iii) $w_1, \dots, w_6$ cannot all have the same color in C .

Consider this coloring induced on G and extend it by coloring the uncolored vertices of G in the following order: $v_1, v_9, v_8, u_1, \dots, u_7, v_7, v_6, \dots, v_3$. Also, assign a color different from $C(v_3)$ and $C(w_1)$ to $C(v_2)$. By minimality of G , $C(v_1) = C(v_2)$, which we can assume is 1. By the chaining argument

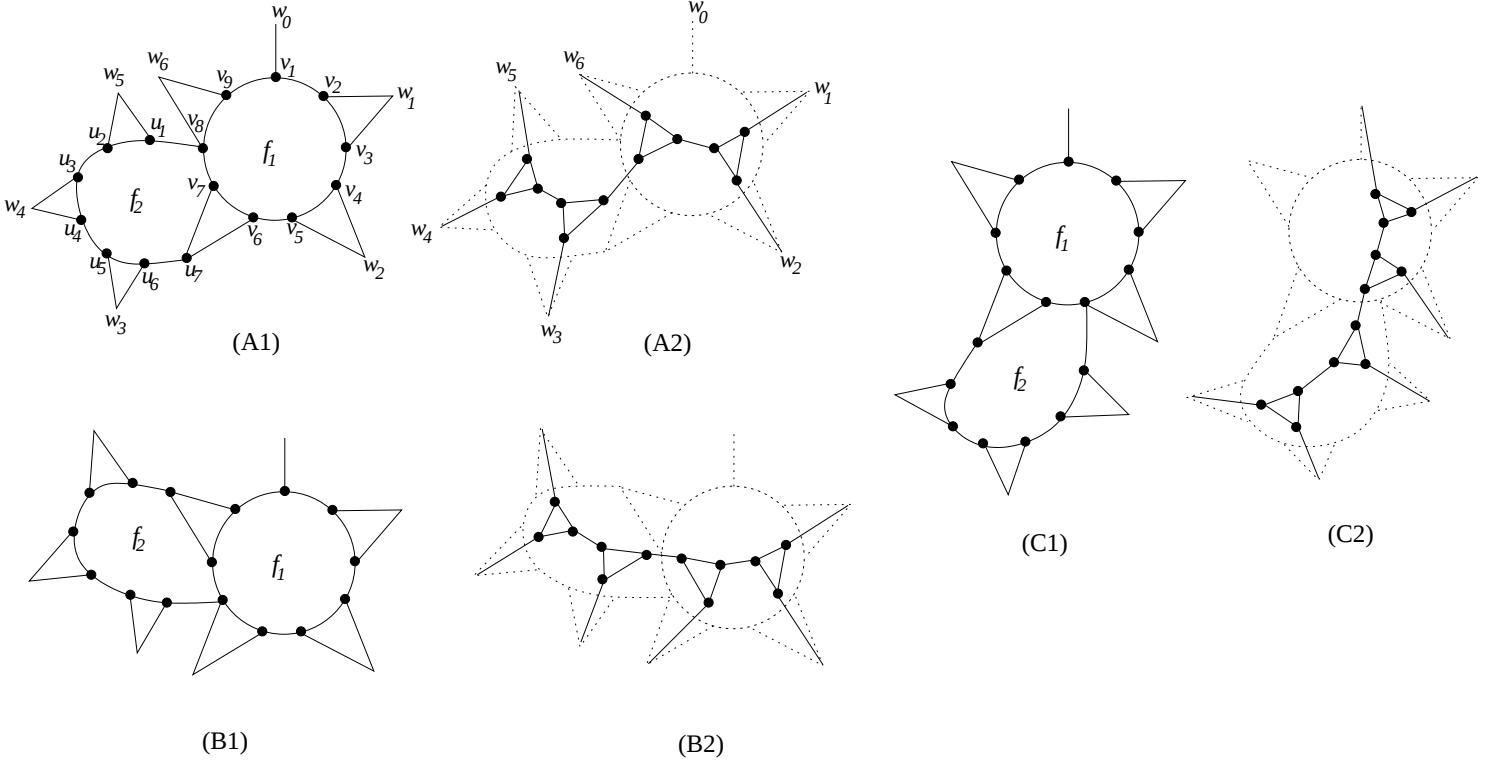


Figure 14: A semi-simple face sharing a type 1 vertex with a type 1 face

$C(v_4) = C(v_6) = C(u_6) = C(u_4) = C(u_2) = C(v_8) = 1$. Without loss of generality assume that $C(w_0) = 2$. So $C(v_9) = 3$, otherwise we could set $C(v_1) = 3$. Note that we can safely exchange $C(v_7)$ with $C(u_7)$. If $C(u_1) \neq 3$ we can exchange $C(v_9)$ with $C(v_8)$ and set $C(v_1) = 3$. So $C(u_1) = 3$ and by the chaining and jumping argument $C(u_3) = C(u_5) = C(v_5) = C(v_3) = 3$. But this means that all $w_1, \dots, w_6$ have color 3, contradicting property (iii).

Using a very similar argument, we can prove the reducibility of configurations of Figures 14(B1) and (C1). The gadget we have to add in each case is shown in parts (B2) and (C2), respectively. ■

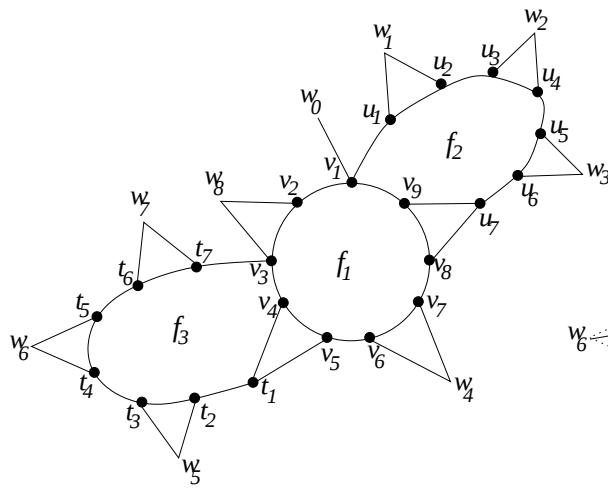
Lemma 3.11 *A simple triple structure is reducible.*

Proof: The reducibility of this structure follows from the arguments we had in Lemma 3.6, if we consider f_3 as the semi-simple face f_1 of configurations of Figure 8(A1), 8(B1), 8(C1), or 8(D1), depending on the position of v_i in face f_3 . ■

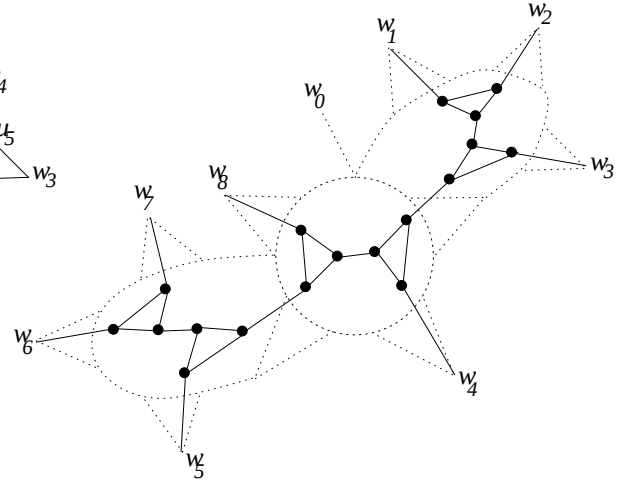
Lemma 3.12 *A triple structure of kind 1 is reducible.*

Proof: There are nine configurations of this kind, up to isomorphism, depending on whether f_2 has an edge in common with f_1 or not, and on the value of i , which defines the position of v_i . Six of these are shown in Figure 15(A1), (B1), (C1), (D1), (E1), and (F1), the other three are in Figure 16(A1), (B1) and (C1). First consider the configuration of Figure 15(A1):

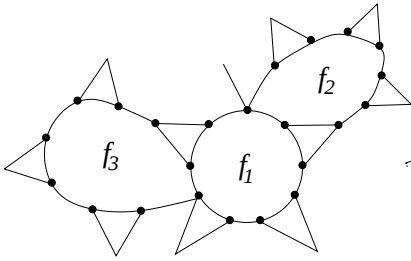
Remove $u_1, \dots, u_7, v_1, \dots, v_9, t_1, \dots, t_7$ and all the incident edges. Add a gadget as in Figure 15(A2). Call this new graph G' . Since G' doesn't have any i -cycle, $4 \leq i \leq 8$, by minimality of G there is a 3-coloring of G' , called C . Note that $w_1, w_2, \dots, w_8$ cannot all have the same color in C . Induce C on G and



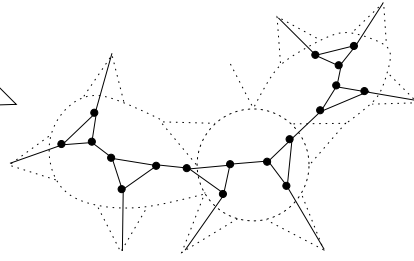
(A1)



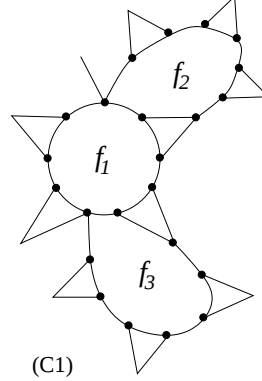
(A2)



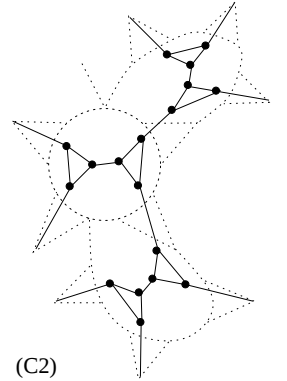
(B1)



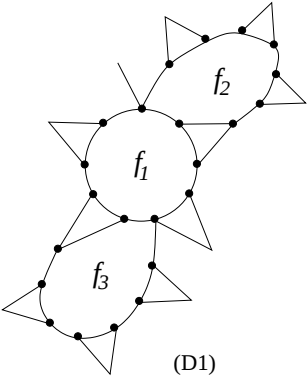
(B2)



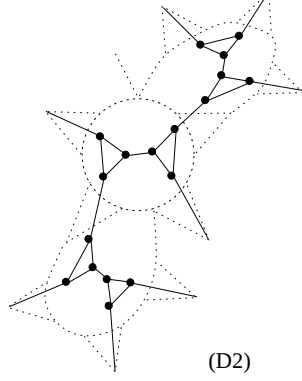
(C1)



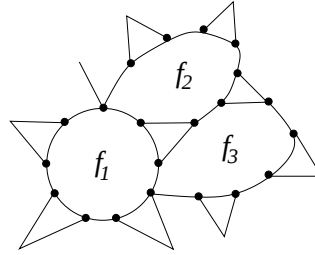
(C2)



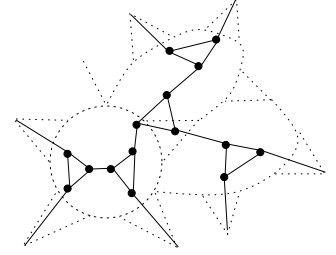
(D1)



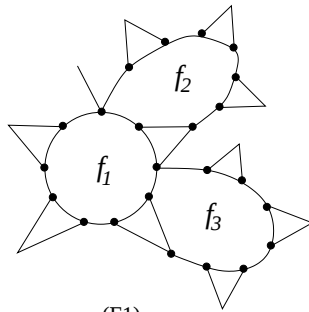
(D2)



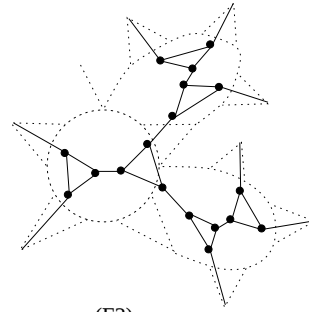
(E1)



(E2)



(F1)



(F2)

Figure 15: Triple structure of kind 1

extend it greedily to the uncolored vertices of G in the following order: $v_1, u_1, u_2, \dots, u_7, v_9, v_8, \dots, v_5, v_2, v_3, t_7, t_6, \dots, t_1$. Finally assign a color different from $C(v_5)$ and $C(t_1)$ to v_4 . By minimality of G we can assume that $C(v_3) = C(v_4) = 1$ and therefore by the chaining argument $C(t_2) = C(t_4) = C(t_6) = 1$ and $C(v_6) = C(v_8) = C(u_6) = C(u_4) = C(u_2) = C(v_1) = 1$.

First we show that $C(u_1) \neq C(w_0)$. By contradiction assume that both are 2. Note that by (possibly) exchanging $C(v_9)$ and $C(u_7)$ we can make $C(v_9) = 2$. Similarly, we can exchange $C(t_1)$ with $C(v_5)$ (if needed) so that $C(t_1) \neq C(t_3)$. Now exchange $C(v_2)$ with $C(v_3)$, $C(t_7)$ with $C(t_6)$, $C(t_5)$ with $C(t_4)$, $C(t_3)$ with $C(t_2)$, and set $C(v_1) = 3$. This gives a 3-coloring of G .

So let's assume that $C(u_1) = 2$ and $C(w_0) = 3$. If $C(v_2) = 3$ we exchange $C(u_7)$ with $C(v_9)$ (if needed) so that $C(v_9) = 3$. Then we can set $C(v_1) = 2$ and exchange $C(u_1)$ with $C(u_2)$, $C(u_3)$ with $C(u_4)$, $C(u_5)$ with $C(u_6)$, $C(u_7)$ with $C(v_8)$, $C(v_7)$ with $C(v_6)$, and finally $C(v_5)$ with $C(v_4)$. This is a 3-coloring of G . Therefore, we can assume that $C(v_2) = 2$. In this case we can use the same argument, then exchange $C(t_1)$ with $C(v_4)$ if $C(v_4) = 2$, and then exchange $C(v_2)$ with $C(v_3)$, unless $C(t_7) = 2$, and by the chaining argument $C(t_5) = C(t_3) = 2$. But this means that all $w_1, w_2, \dots, w_8$ have color 3, which is a contradiction.

Using very similar arguments, we can prove the reducibility of the other five configurations shown in Figure 15. For each case, the gadget that we have to add is shown in the figure next to it.

Now, let's consider the configuration of Figure 16(A1). Remove $u_1, \dots, u_8, v_1, \dots, v_9, t_1, \dots, t_7$ and all the incident edges. Add two gadgets as in Figure 16(A2). Call this new graph G' . Since G' doesn't have any i -cycle, $4 \leq i \leq 8$, by minimality of G there is a 3-coloring of G' , called C . Note that $w_1, \dots, w_4$ cannot all have the same color in C . Similarly, $w_5, \dots, w_9$ cannot all have the same color in C . Induce C on G and extend it greedily to the uncolored vertices of G in the following order: $u_1, u_2, \dots, u_8, v_1, v_9, v_8, \dots, v_5, v_2, v_3, t_7, t_6, \dots, t_1$. Finally assign a color different from $C(v_5)$ and $C(t_1)$ to v_4 . By minimality of G we can assume that $C(v_3) = C(v_4) = 1$ and therefore by the chaining argument $C(t_2) = C(t_4) = C(t_6) = 1$ and $C(v_6) = C(v_8) = C(v_1) = 1$.

First we show that $C(u_1) \neq C(u_8)$. By contradiction assume that both are 2. First exchange $C(v_5)$ with $C(t_1)$ (if needed) so that $C(v_5) = 2$. If $C(v_2) = 3$ then exchange $C(v_2)$ with $C(v_3)$, then exchange $C(t_7)$ with $C(t_6)$ if $C(t_7) = 3$, exchange $C(t_5)$ with $C(t_4)$ if $C(t_4) = 3$, exchange $C(t_3)$ with $C(t_2)$ if $C(t_3) = 3$, and finally exchange $C(t_1)$ with $C(v_5)$. Set $C(v_1) = 3$ and exchange $C(v_9)$ with $C(v_8)$ if $C(v_9) = 3$, then exchange $C(v_7)$ with $C(v_6)$ if $C(v_7) = 3$. The only conflict that we may have is when $C(v_6) = 3$ and $C(v_5) = 3$. This happens only if the previous exchanging process was extended up to exchanging $C(v_5)$ with $C(t_1)$. But in this case all $w_4, \dots, w_9$ have color 2, which is a contradiction.

So let's assume that $C(u_1) = 2$ and $C(u_8) = 3$. If we could exchange $C(u_8)$ with $C(u_7)$ then we could set $C(v_1) = 3$ and use the same argument as in the previous paragraph. So $C(u_6) = 3$ and by the chaining argument $C(u_4) = C(u_2) = 3$. By the same argument and starting from u_1 , we can show that $C(u_3) = C(u_5) = C(u_7) = 2$. But this implies that all $w_1, \dots, w_4$ have color 1, which again is a contradiction.

Very similar arguments show the reducibility of configurations of Figures 16(B1) and (C1). The gadget that we have to add is shown in Figure 16(B2) and (C2), respectively. ■

Lemma 3.13 *A triple structure of kind 2 is reducible.*

Proof: *Configuration of Figure 17(a) or 17(b):* If we consider f_3 in these two configurations as the face f_1 in the configuration of figure 10 then exactly the same argument shows the reducibility of these two cases.

Configuration of Figure 18: By minimality of G , there is a 3-coloring of $G - (t_1, t_2)$, called C . Consider C induced on G . We can assume that $C(t_1) = C(t_2) = 1$ and by the chaining argument $C(v_1) = 1$.

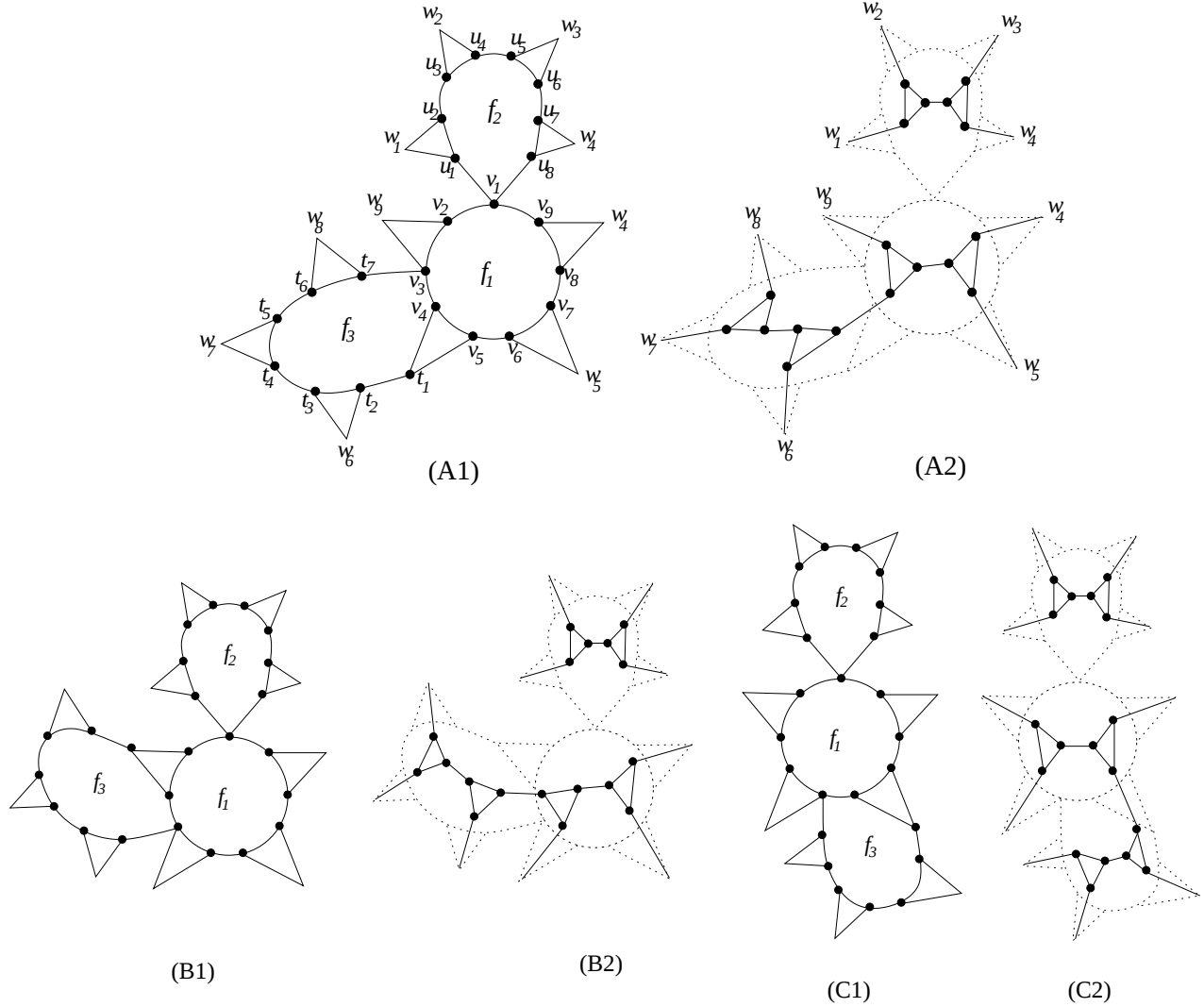


Figure 16: Triple structure of kind 1

If both $C(v_4)$ and $C(t_8)$ are different from 1 then we can simply exchange $C(t_1)$ with $C(v_3)$ and get a 3-coloring of G . So at least one of $C(v_4)$ and $C(t_8)$ is equal to 1.

First assume that $C(v_4) = 2$ and $C(t_8) = 1$. So by the chaining argument $C(t_6) = C(t_4) = 1$. Now assign $C(t_3)$ to both t_2 and t_4 , and set $C(t_3) = 1$, exchange $C(v_5)$ with $C(v_6)$, and $C(v_7)$ with $C(v_8)$, set $C(v_3) = 1$, assign a color different from 1 and $C(t_2)$ to t_1 . Now since $C(v_1) = C(v_3) = 1$, there is a color available for v_2 . This yields a 3-coloring of G .

Now assume that $C(v_4) = 1$ and $C(t_8) = 2$. So by the chaining argument $C(v_6) = C(v_8) = C(u_6) = C(u_4) = C(u_2) = 1$. Without loss of generality assume that $C(w) = 3$. Exchange $C(v_9)$ with $C(u_7)$ (if needed) so that $C(v_9) = 3$. Now set $C(v_1) = 2$, exchange $C(u_1)$ with $C(u_2)$, $C(u_3)$ with $C(u_4)$, $C(u_5)$ with $C(u_6)$, $C(u_7)$ with $C(v_8)$, $C(v_7)$ with $C(v_6)$, $C(v_5)$ with $C(v_4)$, set $C(v_3) = 1$, $C(v_2) = 3$, and $C(t_1) = 2$. This is a 3-coloring of G .

Finally, assume that $C(v_4) = C(t_8) = 1$. So by the chaining argument $C(t_6) = C(t_4) = 1$. Assign $C(t_3)$ to both t_2 and t_4 , set $C(t_3) = 1$, exchange $C(t_5)$ with $C(t_6)$, $C(t_7)$ with $C(t_8)$, assign a color

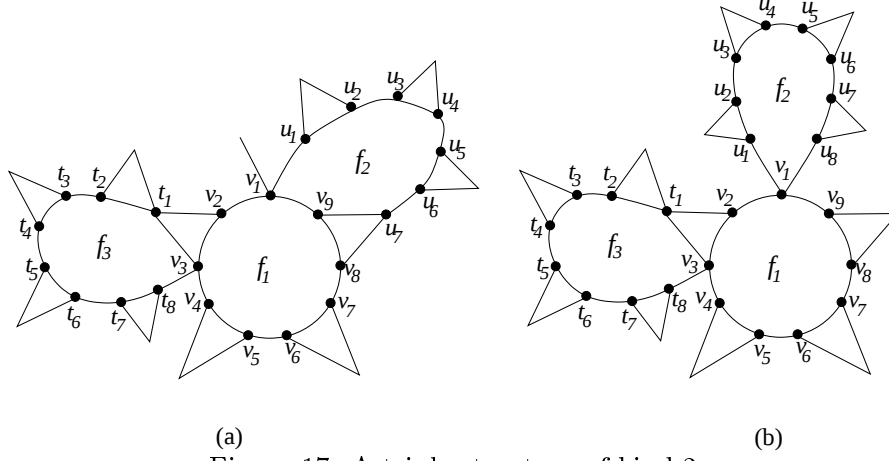


Figure 17: A triple structure of kind 2

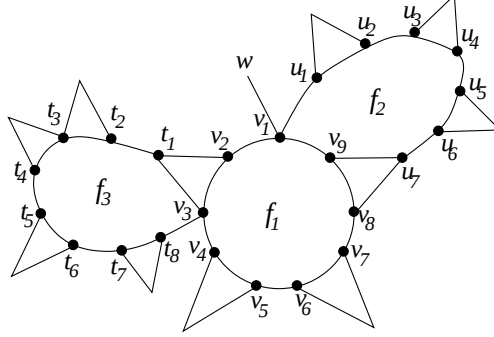


Figure 18: A triple structure of kind 2

different from $C(t_8)$ and 1 to v_3 , and a color different from $C(v_3)$ and 1 to v_2 . This 3-coloring of G , which is an obvious contradiction, completes the proof of this configuration.

Almost identical arguments work for the cases that one of $v_4, \dots, v_8$ is the type 2 vertex of f_1 , instead of v_3 . Also, in each case, including the configuration of Figure 18, very similar arguments work for the cases that one of $t_4, \dots, t_7$ is the type 2 vertex of f_3 , instead of t_3 . We do not repeat the proof for these cases.

Configuration of Figure 19: Remove $u_1, \dots, u_8$ and all the incident edges and create a new graph G' as in Figure 19(b). By minimality of G , there is a 3-coloring of G' , called C . By property of the gadget added, $w_1, \dots, w_4$ cannot all have the same color in G' . Consider C induced on G and extend it to the uncolored vertices of G in this way: color $u_1, \dots, u_8$ greedily. We show that we can assign a color to v_1 , after which we can color v_2 , and finally assign a color different from $C(v_2)$ and $C(v_3)$ (but equal to $C(t_2)$) to t_1 . The reason that we can color v_1 is that if, for example, $C(v_9) = 1$ then the only case that we cannot assign a color to v_1 is when $C(u_1) = 2$ and $C(u_8) = 3$. In this case, by the chaining argument $C(u_3) = C(u_5) = C(u_7) = 2$ and $C(u_6) = C(u_4) = C(u_2) = 3$, but this implies that all $w_1, \dots, w_4$ have color 1, which is impossible. So we can extend C in the way we described above, to get a (improper) 3-coloring of G with the only conflict between $C(t_1)$ and $C(t_2)$.

Assume that $C(t_1) = C(t_2) = 1$ and by the chaining argument $C(v_1) = 1$. If both $C(v_4)$ and $C(t_8)$ are different from 1 then we can simply exchange $C(t_1)$ with $C(v_3)$ and get a 3-coloring of G . So at least one of $C(v_4)$ and $C(t_8)$ is equal to 1.

The case that $C(v_4) = 2$ and $C(t_8) = 1$ is identical to the one in the second paragraph of proof of

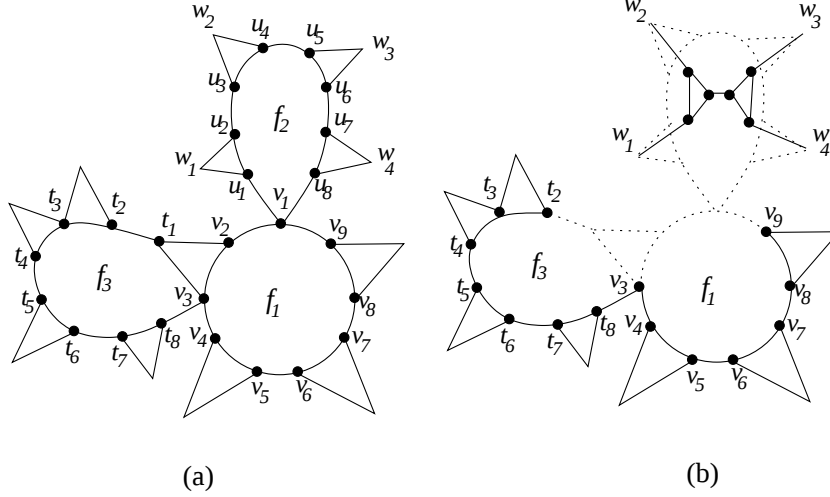


Figure 19: A triple structure of kind 2

the previous configuration.

So let's assume that $C(v_4) = 1$ and $C(t_8) = 2$. So by the chaining argument $C(v_6) = C(v_8) = 1$. If $C(u_1)$ and $C(u_8)$ are both equal, to for example 2, then we can set $C(v_1) = 3$, exchange $C(v_8)$ with $C(v_9)$, $C(v_7)$ with $C(v_6)$, $C(v_5)$ with $C(v_4)$, assign 1 to v_3 , set $C(v_2) = 2$ and $C(t_1) = 3$. This yields a 3-coloring of G . So $C(u_1) = 2$ and $C(u_8) = 3$, and by the chaining argument $C(u_3) = C(u_5) = C(u_7) = 2$ and $C(u_6) = C(u_4) = C(u_2) = 3$, which means all $w_1, \dots, w_4$ have the same color, another contradiction.

The proof of the case that $C(v_4) = C(t_8) = 1$ is identical to that of configuration of Figure 18. This completes the proof of this configuration.

Almost identical arguments work for the cases that one of $v_4, \dots, v_8$ is the type 2 vertex of f_1 , instead of v_3 . Also, in each of these case, including the configuration of Figure 19, very similar arguments work for the cases that one of $t_4, \dots, t_7$ is the type 2 vertex of f_3 , instead of t_3 . We do not repeat the proof for these cases. ■

4 The Discharging Rules

We give an initial charge of $d(v) - 6$ units to each vertex v and $2|f| - 6$ units to each face f . Using the Euler formula:

$$\begin{aligned} \sum_{v \in V} (d(v) - 6) + \sum_{f \in F} (2|f| - 6) &= 2|E| - 6|V| + 4|E| - 6|F| \\ &= -12. \end{aligned} \tag{1}$$

It is easy to see that the only elements with negative charges are 3-, 4-, and 5-vertices, which have charges -3 , -2 , and -1 respectively. The goal is to show that, based on the assumption that G is a minimum counter-example, we can send charges from the faces to ≤ 5 -vertices such that all the vertices and faces have non-negative charge, which is of course a contradiction since the total charge must be negative by equation (1).

The discharging rules are as follows:

1. Each ≥ 9 -face sends $\frac{3}{2}$ units to each of its bad vertices and 1 unit to each of its simple vertices.

2. Each ≥ 10 -face sends 1 unit to each of its 4-vertices, and $\frac{1}{2}$ units to each of its 5-vertices.
3. If f is semi-simple then it sends $\frac{1}{2}$ units to its type 1 vertex.
4. If f is semi-type 2 then it sends $\frac{1}{2}$ units to its type 1 vertex and 1 unit to its type 2 vertex.
5. If f is semi-type 1 then it sends 1 unit to its type 1 vertex which is incident to a triangle that shares an edge with f , and sends $\frac{1}{2}$ to its other type 1 vertex.
6. If f is a semi-type 0 face whose type 0 vertex is incident with a type 0 face then f sends $\frac{3}{4}$ to each of its 4-vertices. Otherwise, it sends $\frac{1}{2}$ to its type 0 vertex and 1 unit to its type 1 vertex.
7. If f is a 9-face but not simple, semi-simple, type 0, semi-type 0, type 1, semi-type 1, type 2, or semi-type 2, then it sends $\frac{1}{2}$ to each of its 5-vertices and 1 to each of its 4-vertices.

Lemma 4.1 *After the discharging phase, each face f will have non-negative charge.*

Proof: If $|f| = 3$ then its initial charge is zero and it sends nothing during the discharging phase. If $|f| \geq 12$ then it sends at most $\frac{3}{2}|f|$ to its vertices, which is at most $2|f| - 6$, the value of its initial charge. If $|f| = 11$ then it can have at most 10 bad vertices and so sends $\frac{3}{2} \times 10 = 15$ units of charge to them and sends at most 1 unit to its 11'th vertex. So the total charge sent out would be at most 16 which is equal to its initial charge. By Lemma 3.1, if $|f| = 10$ then it has at most 8 bad vertices. Therefore, it sends at most $8 \times \frac{3}{2} + 2$ units of charge to its vertices, which is equal to the value of its initial charge.

The last case is when $|f| = 9$. Note that in this case the initial charge of f is 12. If it has 8 bad vertices then it only sends $8 \times \frac{3}{2}$ units to these bad vertices, and it can be checked that it sends nothing to its other vertex and therefore will have non-negative charge. If f has at most 6 bad vertices then it sends at most $6 \times \frac{3}{2}$ to these bad vertices and at most 3 units to the other vertices, for a total of 12.

So let's assume that it has exactly 7 bad vertices. It sends $7 \times \frac{3}{2}$ to the bad vertices and so still has $\frac{3}{2}$ units of charge. If one of the non-bad vertices is a ≥ 5 -vertex then it sends at most $\frac{1}{2}$ to it and sends at most 1 unit to the other one and will have non-negative charge. The only remaining case is that both of its non-bad vertices are ≤ 4 -vertices. This implies that f is either a semi-simple, a semi-type 0, a semi-type 1, or a semi-type 2 face. In each of these cases, by rules 3, 4, 5, and 6 it sends at most another $\frac{3}{2}$ units of charge and therefore will have non-negative charge. ■

Lemma 4.2 *Every 3-vertex has non-negative charge after the discharging phase.*

Proof: By rule 1 each 3-vertex receives 3 units of charge and therefore will have non-negative charge. ■

Lemma 4.3 *Every 4-vertex v has non-negative charge after the discharging phase.*

Proof: Since the initial charge of v is -2 it is enough to show that during the discharging phase v gets at least 2 units of charge.

Case 1: First assume that v is a type 2 vertex. So there are two non-triangle faces incident with it, called f_1 and f_2 . If $|f_1| \geq 10$ then by rule 2 it sends 1 unit of charge to v . If $|f_1| = 9$ and it is a semi-type 2 it sends 1 unit of charge to v by rule 4. By Lemma 3.3, f_1 cannot be a type 2 face. In all other cases, it sends 1 unit of charge to v by rule 7. Therefore, in each case f_1 sends 1 unit to v . Using the same argument, we can show that f_2 sends 1 unit of charge to v and therefore v will have non-negative charge.

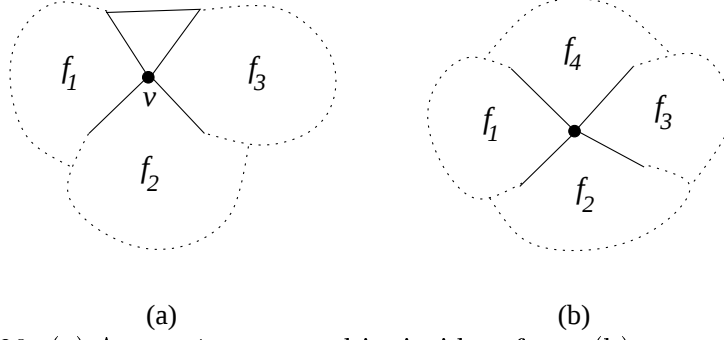


Figure 20: (a) A type 1 vertex and its incident faces, (b) a type 0 vertex

Case 2: Now assume that v is a type 0 vertex and the non-triangle faces incident with it are f_1 , f_2 , f_3 , and f_4 . (see Figure 20(b)). If none of these faces is type 0 then they each sends at least $\frac{1}{2}$ to v , for a total of 2. By Lemma 3.4, at most one of these faces can be type 0. Without loss of generality, assume that f_1 is type 0. Then by rule 2, 6, or 7 each of f_2 , f_3 , and f_4 sends at least $\frac{3}{4}$ to v and therefore v gets at least 2 units of charge.

Case 3: Finally, assume that v is a type 1 vertex and the non-triangle faces incident with it are f_1 , f_2 , and f_3 . (see Figure 20(a)). For each of these faces, if it is a ≥ 10 -face then it sends 1 unit of charge to v . If at least two of them are each sending 1 unit of charge to v then v will have non-negative charge. So let's assume that at least two of them are sending less than 1 unit of charge. This implies that at least one of f_1 or f_3 is sending less than 1 unit of charge to v . Without loss of generality, assume it is f_1 (by symmetry, the same arguments work for f_3). Note that f_1 cannot be simple (because v is a 4-vertex), type 0 (because v is a type 1 vertex), type 1 (since v is incident with a triangle that shares an edge with f_1), or type 2 (because v is a type 1 vertex). Also, it can't be semi-type 1, by the assumption that it is sending less than 1 unit of charge to v , and by rule 5. Therefore, f_1 is either semi-simple, semi-type 0, or semi-type 2:

Case 3.1: Let f_1 be semi-simple which means it sends $\frac{1}{2}$ to v . So f_2 cannot be of type 1 by Lemma 3.10, and therefore sends at least $\frac{1}{2}$ to v , by rules 2, 5, or 7. If f_3 is sending 1 unit of charge then v gets 2 units of charge and we are done. We show that in fact this is the case. Since we assumed that f_1 is semi-simple, by Lemmas 3.6, 3.9, and 3.11 f_3 cannot be a semi-simple, a semi-type 2, or a semi-type 0 face which is sending less than 1 unit of charge to v , respectively. Therefore f_3 will be sending at least 1 unit of charge to v by rules 2, 5, 6, or 7.

Case 3.2: Assume f_1 is semi-type 0 and so it sends $\frac{3}{4}$ to v (since we assumed it is sending less than 1 unit). This implies that it is adjacent to a type 0 face. Thus by Lemma 3.12 f_2 cannot be of type 1, and therefore sends at least $\frac{1}{2}$ to v . Also, f_3 cannot be a type 2 face (because v is a type 1 vertex) and by Lemmas 3.11 and 3.13 it cannot be semi-simple or semi-type 2 either. If it is a semi-type 1 it sends 1 unit of charge by rule 5, and if it is semi-type 0 then it sends at least $\frac{3}{4}$ to v . Overall, v gets at least 2 units of charge.

Case 3.3: Now, let f_1 be semi-type 2. Thus it sends $\frac{1}{2}$ to v . By Lemma 3.8 f_2 cannot be of type 1, and therefore, sends at least $\frac{1}{2}$ to v . Also, by Lemmas 3.9 and 3.7 f_3 cannot be semi-simple, or semi-type 2, because f_1 is semi-type 2. It cannot be type 1 either, since v is a 4-vertex incident with a triangle that shares an edge with it. If f_3 is a semi-type 0 then it cannot be of a kind that sends $\frac{3}{4}$ to v by Lemma 3.13 because f_1 is semi-type 2. If it is semi-type 1 it sends 1 unit to v by rule 5. In all other cases, f_3 sends 1 unit of charge to v by rules 2 or 7, and therefore, v has non-negative charge. ■

Lemma 4.4 *Every ≥ 5 -vertex has non-negative charge, after the discharging phase.*

Proof: If $d(v) \geq 6$ then its initial charge is non-negative and it doesn't lose any charges in the discharging phase. Assume that $d(v) = 5$ and the faces incident with v are f_1, f_2, f_3, f_4, f_5 . If none of $f_1, \dots, f_5$ is triangle then by Lemma 3.5 at least three of them are not type 5 and therefore by rules 2 or 7 each of them sends $\frac{1}{2}$ to v and so v will have non-negative charge. Assume that exactly one of $f_1, \dots, f_5$, say f_1 , is a triangle. Then f_2 and f_5 are not type 5 and so each sends at $\frac{1}{2}$ to v , by rules 2 or 7. Finally, assume that exactly two of $f_1, \dots, f_5$ are triangles. (note that if more than two of them are triangles then G will have a 4-cycle). Without loss of generality, assume that f_1 and f_3 are triangles. Therefore f_4 and f_5 cannot be of type 5, and so each of them sends $\frac{1}{2}$ to v by rule 2 or 7. ■

Proof of Theorem 1.1: By Lemmas 4.1, 4.2, 4.3, and 4.4 all the elements of G will have non-negative charge, after applying the discharging rules. Therefore, the total charge will be non-negative, which is contradicting equation 1. This disproves the existence of G , a minimal counter example to the theorem.

5 The 3-Coloring Algorithm

The proof of Theorem 1.1 easily implies a relatively simple algorithm for finding a 3-coloring. We say a planar graph has property Π , if it doesn't have any face of size in $\{4, \dots, 8\}$ and has no two triangles that have an edge in common. Consider a planar graph G that has property Π . One iteration of the algorithm is to reduce the size of the problem by removing some of the vertices and/or edges and possibly adding a smaller number of vertices and edges to G while preserving property Π , coloring the new graph recursively, and then extending the coloring to G . We keep doing this, until the size of the graph is $O(1)$, when we can find a 3-coloring in constant time. Each iteration consists of the following steps:

1. First check to see whether the current graph has any ≤ 2 -vertex. If there exists such a vertex v , then v along with its incident edges are removed from G . A 3-coloring of the new graph can easily be extended to v .
2. Else, check if there exists a $2k$ -face with at least $2k - 1$ bad vertices, for some $k \geq 5$. Suppose that $f = v_1, v_2, \dots, v_{2k}$ is such a face, where $v_1, \dots, v_{2k-1}$ are bad and v_{2i-1} and v_{2i} are incident with a triangle, $1 \leq i \leq k$. Remove (v_1, v_{2k}) from G . By the argument of lemma 3.1 any 3-coloring of the new graph is also a 3-coloring of G .
3. If neither of the first two cases happens then we apply the initial charges and the discharging rules. As the total charge is negative, there must be some element (face or vertex) with negative charge. If it is a face, by the proof of Lemma 4.1 it must be a simple or a type 2 face. If the element is a vertex, call it v , then by the proofs of Lemmas 4.3 and 4.4 v must be a vertex of one of the configurations of Lemmas 3.2 to 3.13. In any of these two cases, we find one of the reducible configurations described in section 3. For each configuration, based on the proof of the corresponding lemma, we construct the graph G' , which is either the graph obtained by removing some vertices and edges and adding a gadget, or by removing a single edge. Then we find a 3-coloring of G' recursively. Again based on the proof of the lemma corresponding to the reducible configuration, we know how to extend this 3-coloring to a 3-coloring of G .

At each iteration, the number of vertices or edges is reduced by at least one. Therefore the number of iterations of the algorithm will be at most $O(|E|)$. Applying the initial charges and the discharging rules takes at most $O(|E|)$, and once we've done that, we can find a reducible configuration in linear time.

Constructing the graph G' from G takes constant time. So the overall running time of the algorithm will be $O(|E|^2)$. Since for a planar graph $|E| \leq 3|V| - 6$, we can say the running time of the algorithm is $O(n^2)$, where n is the size of the input.

6 The Automated Proof of Reducible Configurations

For any reducible configuration R , a vertex v which is not in R but has a neighbor in R is called a boundary vertex. For example $w_1, \dots, w_5$ in Figure 2 are boundary vertices. For some configurations, such as a simple configuration, we have to forbid some of the boundary vertices from all having the same color. We do this by adding a gadget. We call this set of boundary vertices a *constrained group*. For some reducible configurations, such as configuration of Figure 5, we have two groups of constrained vertices. A 3-coloring of the boundary vertices of a reducible configuration is called *valid* if it satisfies the requirements of its constrained groups. That is, not all the vertices in the same constrained group have the same color. The program reads the configurations one by one and the corresponding constrained group(s) of vertices. For each configuration and for all the possible valid 3-colorings of its boundary vertices, the program checks whether the 3-coloring is extensible to a 3-coloring of the uncolored vertices of the configuration or not. If *all* the valid 3-colorings of the boundary vertices are extensible, then the configuration is a reducible one. We didn't attempt to make any optimizations in the program, since this simple straightforward implementation checks all the reducible configurations relatively quickly, on a desktop computer, and further optimizations would be at the cost of losing its readability.

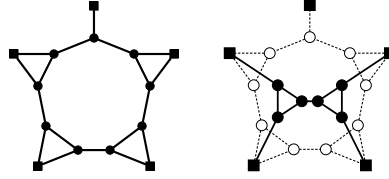
Acknowledgment: I am grateful to my supervisor, Mike Molloy, for introducing me to the problem and for many helpful discussions we had.

References

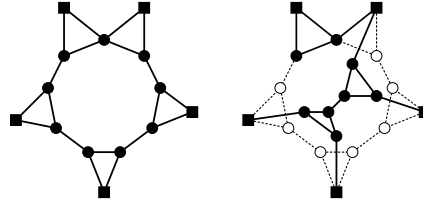
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A The List of Reducible Configurations

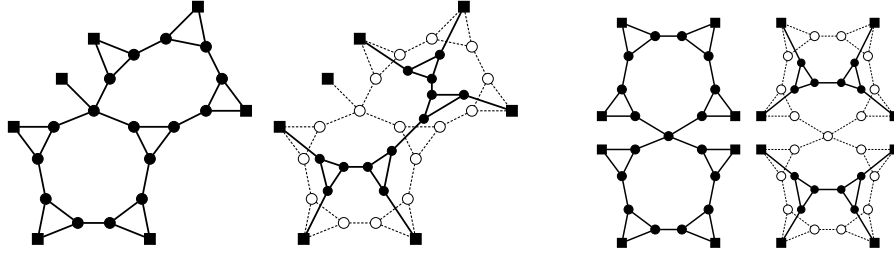
Here is the list of all 74 reducible configurations. Each graph that has white vertices and dotted edges is the “*modified*” version (by removing some vertices and edges and adding a gadget) of the graph to its left. The vertices and the edges that have been removed are the white vertices and the dotted edges, respectively.



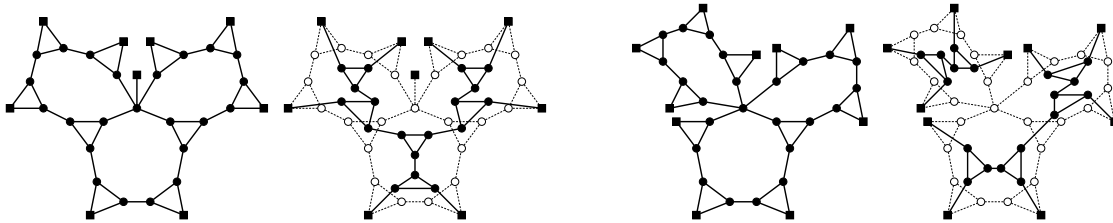
Simple face (Lemma 3.2)



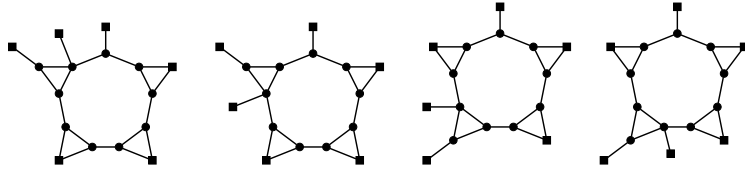
Type 2 face (Lemma 3.3)



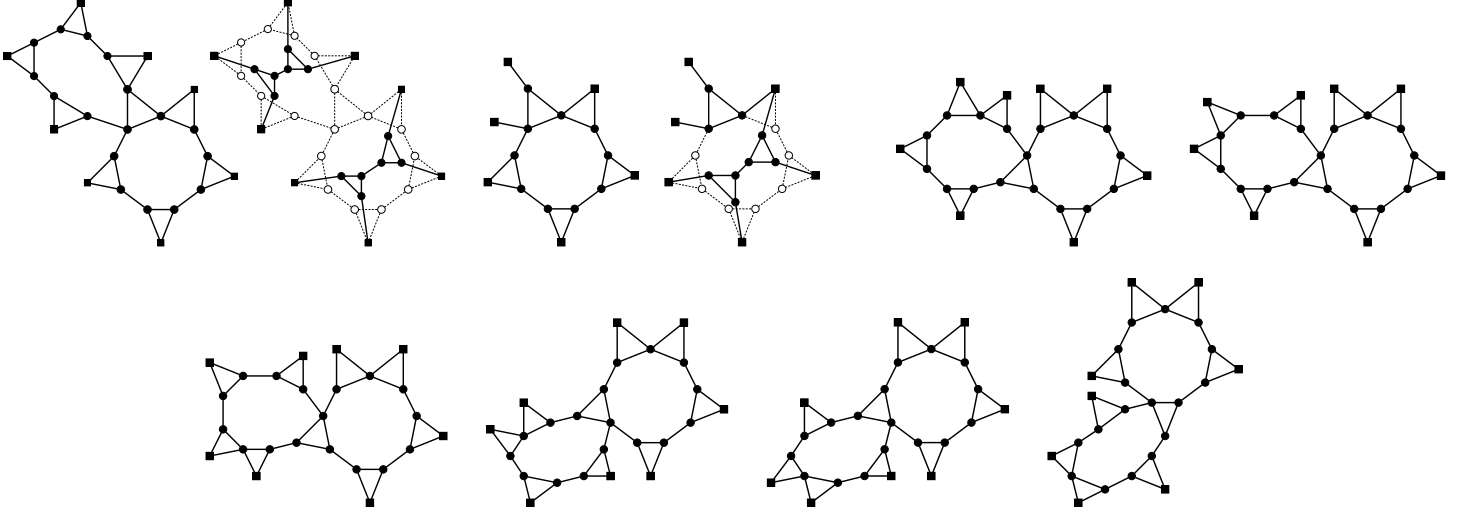
Two type 0 faces sharing their type 0 vertex (Lemma 3.4)



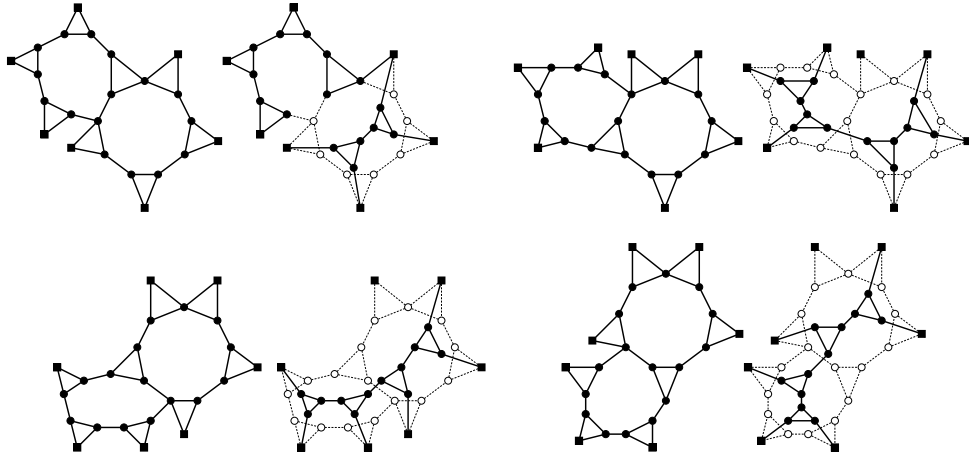
Three type 5 faces sharing a 5-vertex (Lemma 3.5)



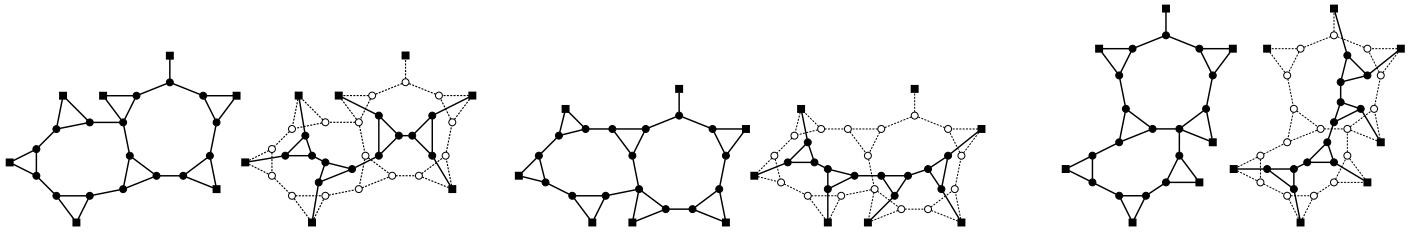
Two semi-simple faces sharing a type 1 vertex (Lemma 3.6)



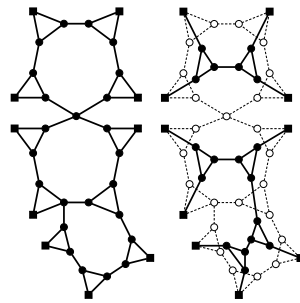
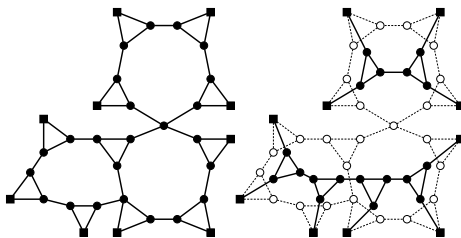
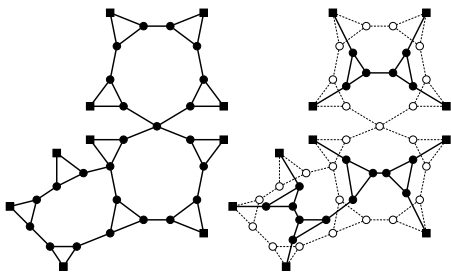
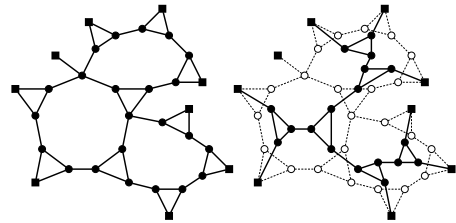
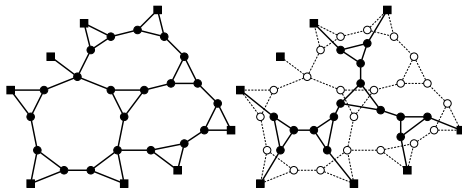
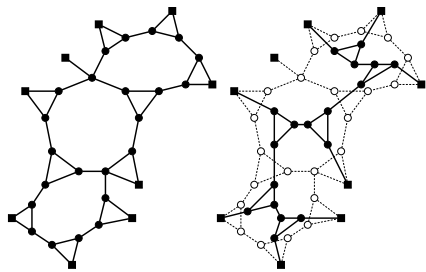
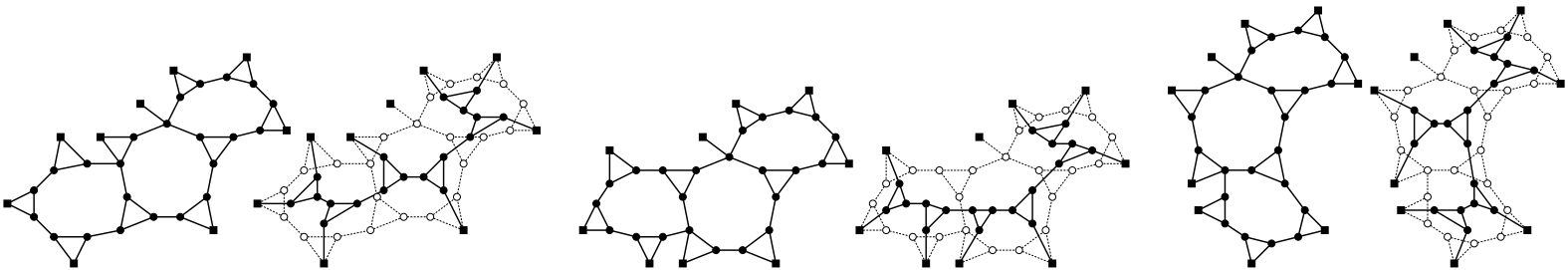
Two semi-type 2 faces sharing a type 1 vertex (Lemma 3.7)



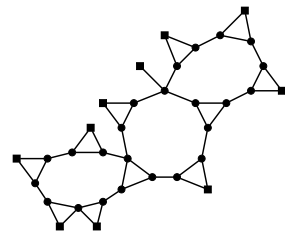
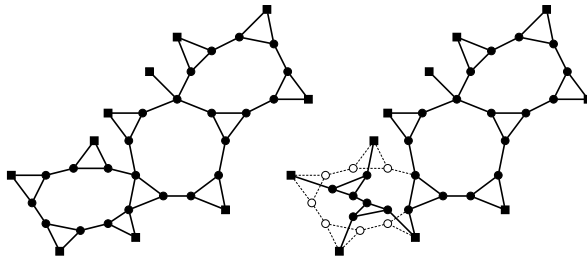
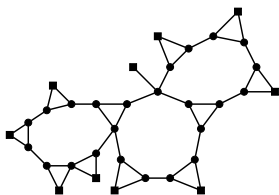
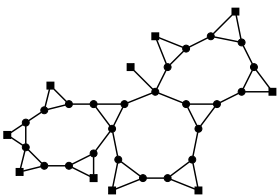
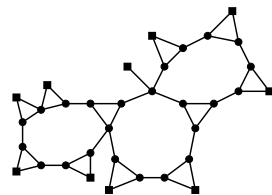
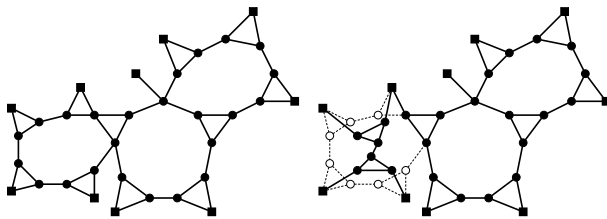
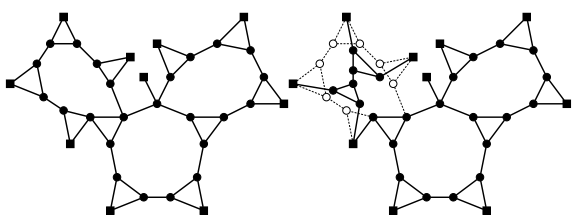
A semi-type 2 face sharing its type 1 vertex with a type 1 face (Lemma 3.8)

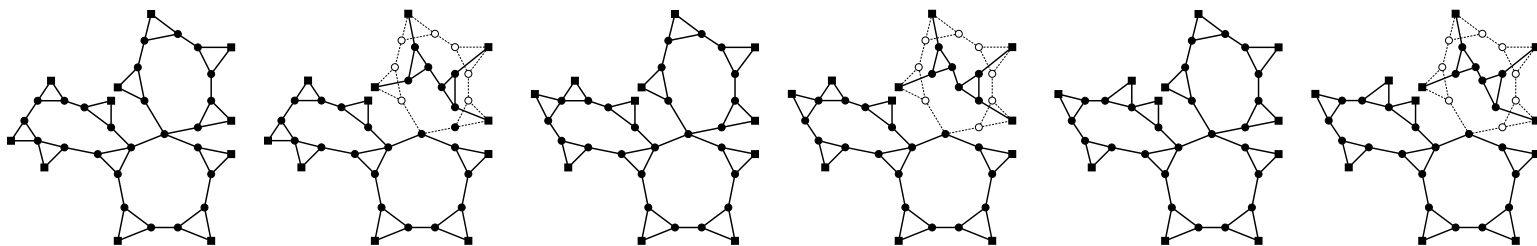
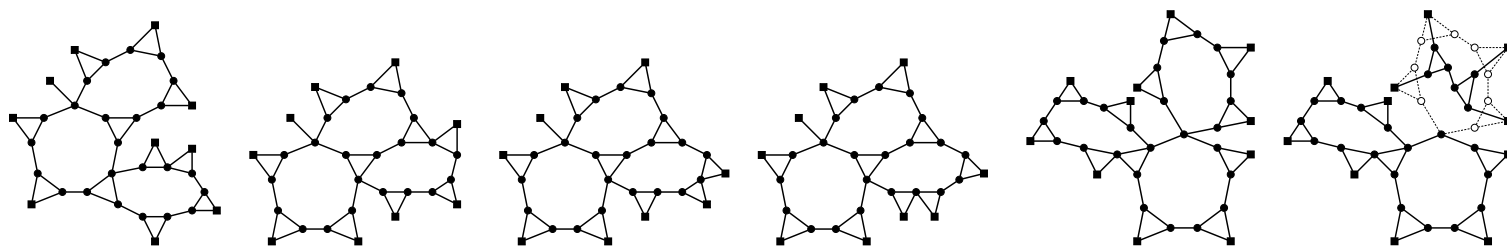
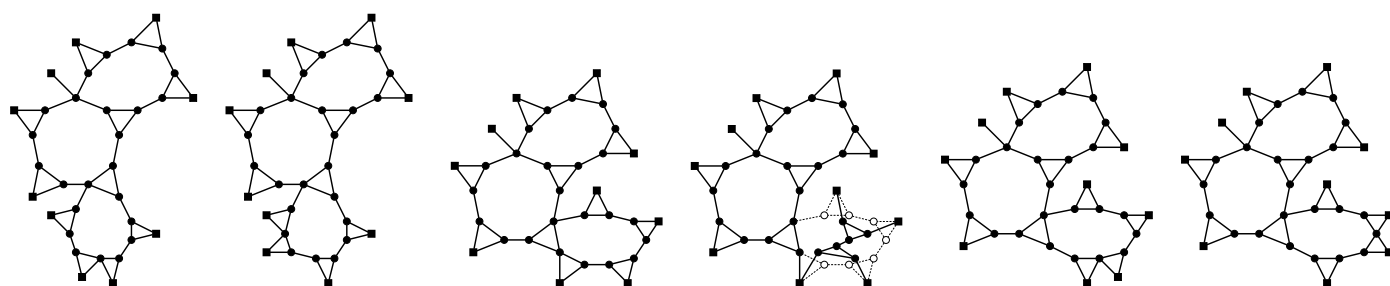
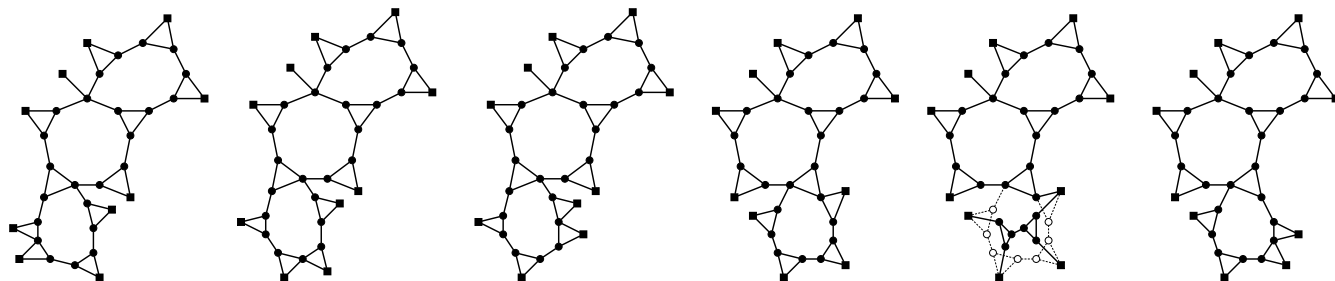
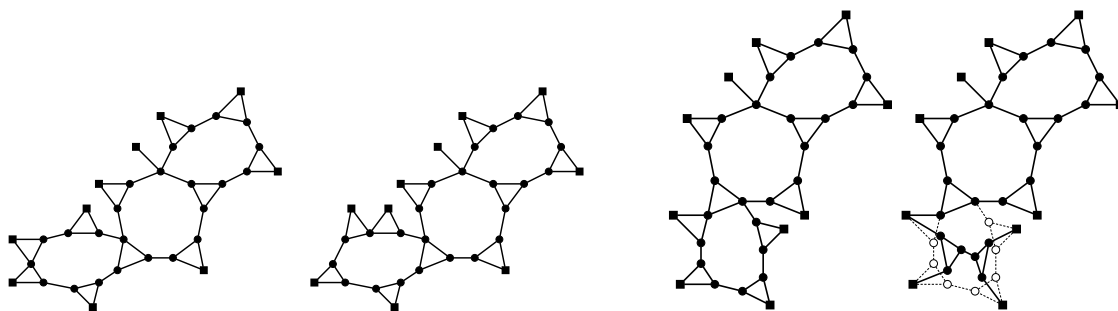


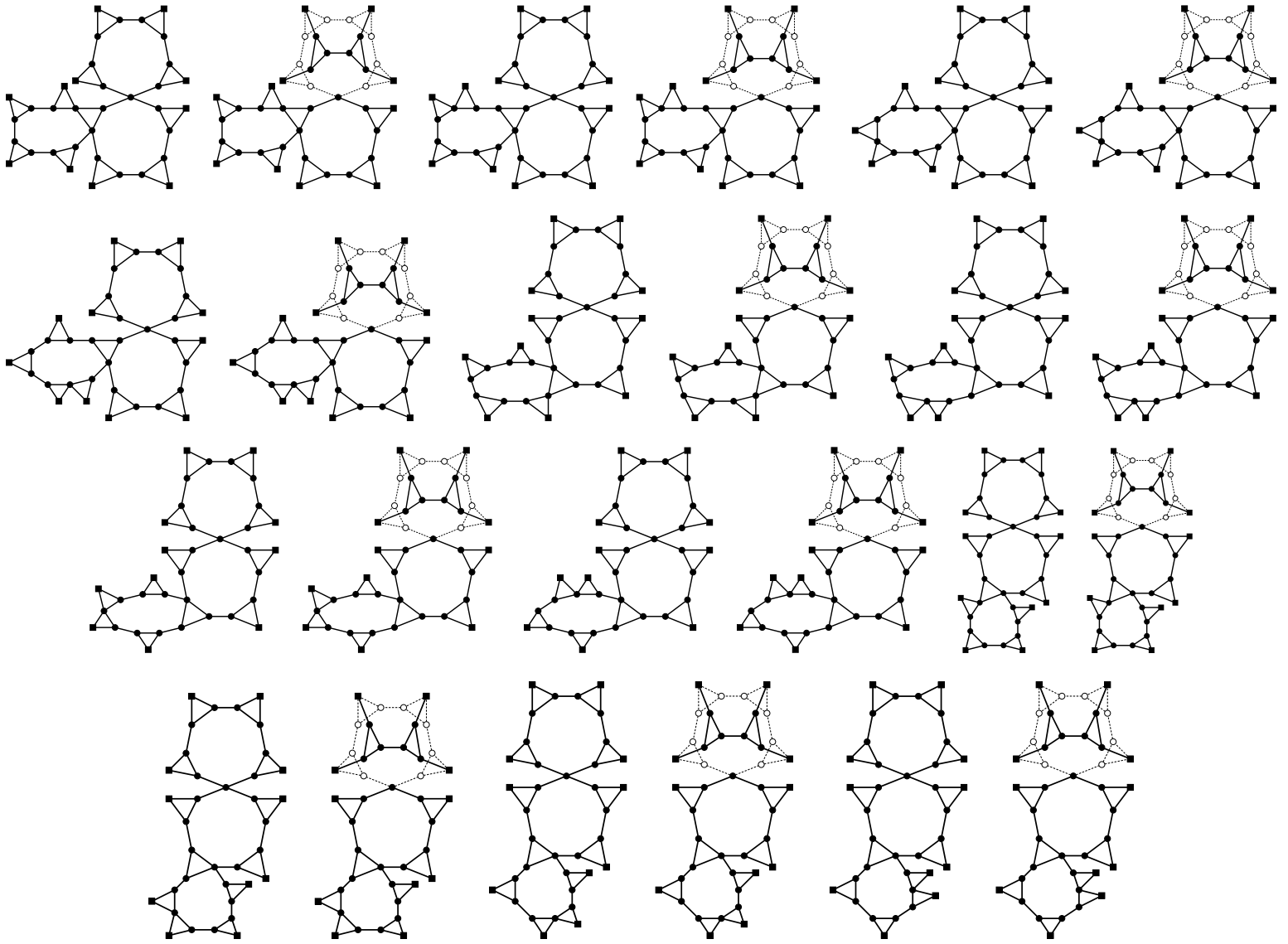
A semi-simple face sharing its type 1 vertex with a type 1 face (Lemma 3.10)



Triple structure of kind 1 (Lemma 3.12)







Triple structure of kind 2 (Lemma 3.13)