Lecture 34: NP-Completeness

Agenda:

- Example proofs of NP-completeness
  - CNF-SAT $\leq_p$ 3-CNF-SAT
  - 3-CNF-SAT $\leq_p$ 3-Coloring
  - 3-Coloring $\leq_p$ 4-Coloring

Reading:

- Textbook pages 995 – 1021

CNF-SAT $\leq_p$ 3-CNF-SAT:

- We have already shown that

  SAT $\leq_p$ 3-CNF-SAT.

  Therefore, by 3-CNF-SAT $\in NP$, 3-CNF-SAT is NP-complete.

- We next show: CNF-SAT $\leq_p$ 3-CNF-SAT

  For every CNF formula f, in polynomial time we can construct a 3-CNF formula g such that

  f is satisfiable if and only if g is satisfiable

- Construction details:
  - for 1 literal clause (x_i), construct a 3-CNF formula: $(x_i \vee y_1 \vee y_2) \wedge (x_i \vee y_1 \vee \neg y_2) \wedge (x_i \vee \neg y_1 \vee y_2) \wedge (x_i \vee \neg y_1 \vee \neg y_2)$
  - for 2 literal clause $(x_i \vee x_j)$, construct a 3-CNF formula: $(x_i \vee x_j \vee y_1) \wedge (x_i \vee x_j \vee \neg y_1)$
  - for 3 literal clause $(x_i \vee x_j \vee x_k)$: unchanged
  - for more than 3 literal clause $(x_{i_1} \vee x_{i_2} \vee \ldots \vee x_{i_k})$, construct a 3-CNF formula: $(x_{i_1} \vee x_{i_2} \vee y_1) \wedge (\neg y_1 \vee x_{i_3} \vee y_2) \wedge \ldots \wedge (\neg y_{k-3} \vee x_{i_{k-1}} \vee x_{i_k})$

  Note: for different clauses, the y literals are distinct !!

  g is the conjunctive form of these 3-CNF formulae.

  Construction time is linear in the number of Boolean operations in formula f.

- Example: $f = (x_1) \wedge (x_2 \vee \neg x_3) \wedge (\neg x_4 \vee x_5 \vee x_6 \vee \neg x_1 \vee x_3)$

- Proof of "if and only if"

Proof of "if and only if":

- Construction details:
  - for 1 literal clause (x_i), construct a 3-CNF formula:
    $(x_i \vee y_1 \vee y_2) \wedge (x_i \vee y_1 \vee \neg y_2) \wedge (x_i \vee \neg y_1 \vee y_2) \wedge (x_i \vee \neg y_1 \vee \neg y_2)$
  - for 2 literal clause $(x_i \vee x_j)$, construct a 3-CNF formula:
    $(x_i \vee x_j \vee y_1) \wedge (x_i \vee x_j \vee \neg y_1)$
  - for 3 literal clause $(x_i \vee x_j \vee x_k)$: unchanged
  - for more than 3 literal clause $(x_{i_1} \vee x_{i_2} \vee \ldots \vee x_{i_k})$, construct a 3-CNF formula:
    $(x_{i_1} \vee x_{i_2} \vee y_1) \wedge (\neg y_1 \vee x_{i_3} \vee y_2) \wedge \ldots \wedge (\neg y_{k-3} \vee x_{i_{k-1}} \vee x_{i_k})$

only if:
  - 1 literal clauses
  - 2 literal clauses
  - 3 literal clauses
  - more then 3 literal clauses

if:
  - for 1 literal clause (x_i),
    $(x_i \vee y_1 \vee y_2) \wedge (x_i \vee y_1 \vee \neg y_2) \wedge (x_i \vee \neg y_1 \vee y_2) \wedge (x_i \vee \neg y_1 \vee \neg y_2)$
  - for 2 literal clause $(x_i \vee x_j)$,
    $(x_i \vee x_j \vee y_1) \wedge (x_i \vee x_j \vee \neg y_1)$
  - for 3 literal clause $(x_i \vee x_j \vee x_k)$,
  - for more than 3 literal clause $(x_{i_1} \vee x_{i_2} \vee \ldots \vee x_{i_k})$,
    $(x_{i_1} \vee x_{i_2} \vee y_1) \wedge (\neg y_1 \vee x_{i_3} \vee y_2) \wedge \ldots \wedge (\neg y_{k-3} \vee x_{i_{k-1}} \vee x_{i_k})$

- Conclusion: CNF-SAT $\leq_p$ 3-CNF-SAT.

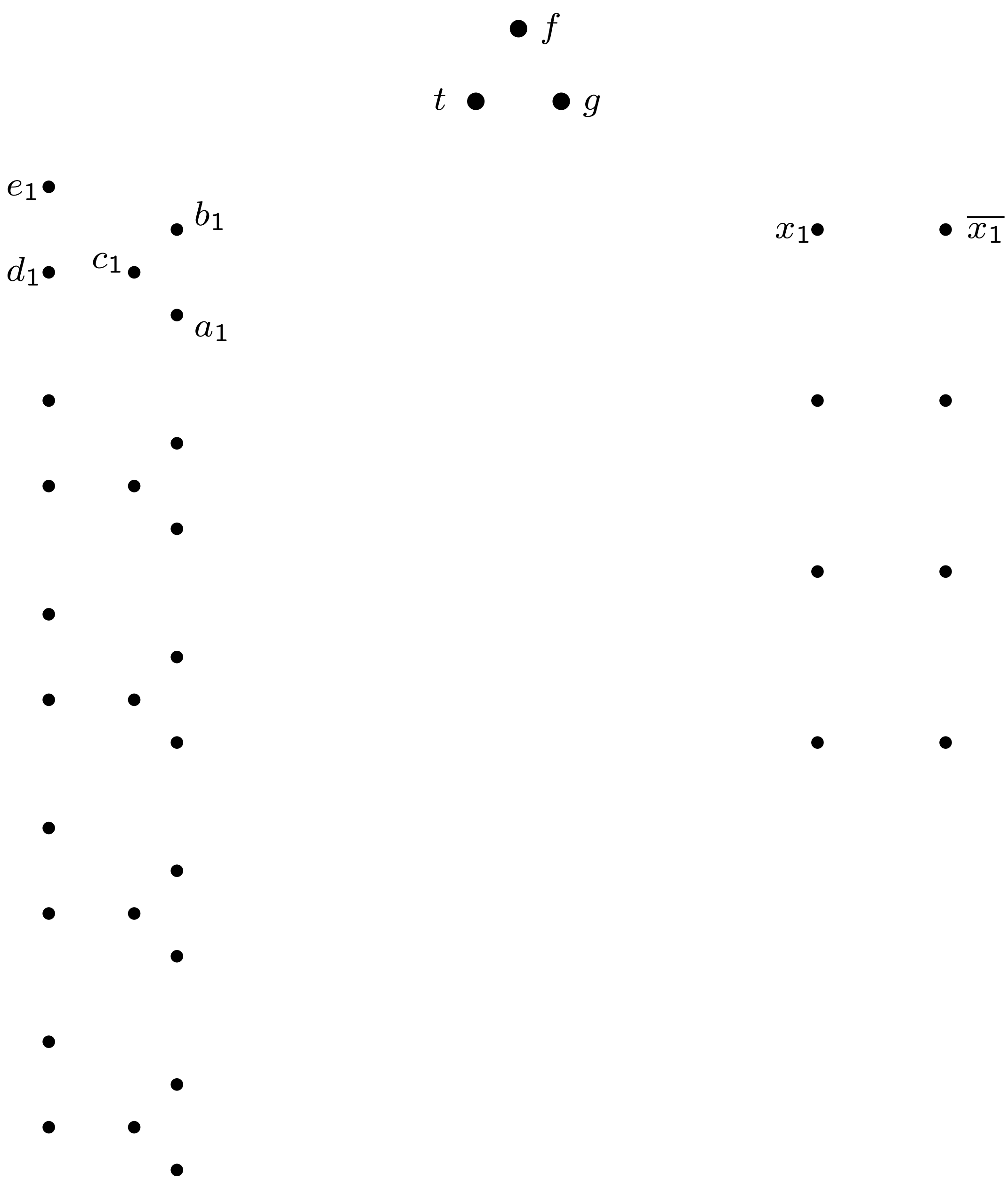
3-CNF-SAT $\leq_p$ 3-Coloring:

- Note:
  3-CNF-SAT is on Boolean formulae; 3-Coloring is on graphs
- Reduction:
  should construct a graph out of the Boolean formula !
- Construction details:
  - assuming in the given 3-CNF formula f
    1. n variables: $x_1, x_2, x_3, \ldots, x_n$
    2. m clauses: $C_1, C_2, C_3, \ldots, C_m$
    3. $C_j = y_1^j \vee y_2^j \vee y_3^j$ (y_k^j is a literal)
  - vertices in the graph G:
    1. t (TRUE), f (FALSE), g
    2. x_i and $\overline{x_i}$, $i = 1, 2, 3, \ldots, n$
    3. a_j, b_j, c_j, d_j, e_j, $j = 1, 2, 3, \ldots, m$
  - edges in the graph G:
    1. (t, f), (f, g), (g, t)
    2. (g, x_i) and $(g, \overline{x_i})$, $i = 1, 2, 3, \ldots, n$
    3. $(x_i, \overline{x_i})$, $i = 1, 2, 3, \ldots, n$
    4. (a_j, b_j), (b_j, c_j), (c_j, a_j), and $(c_j, d_j), (d_j, e_j)$, $j = 1, 2, 3, \ldots, m$
    5. (t, d_j), (t, e_j), $j = 1, 2, 3, \ldots, m$
    6. (e_j, y_1^j), (b_j, y_2^j), (a_j, y_3^j), $j = 1, 2, 3, \ldots, m$

3-CNF-SAT $\leq_p$ 3-Coloring (cont'd):

- An example reduction:

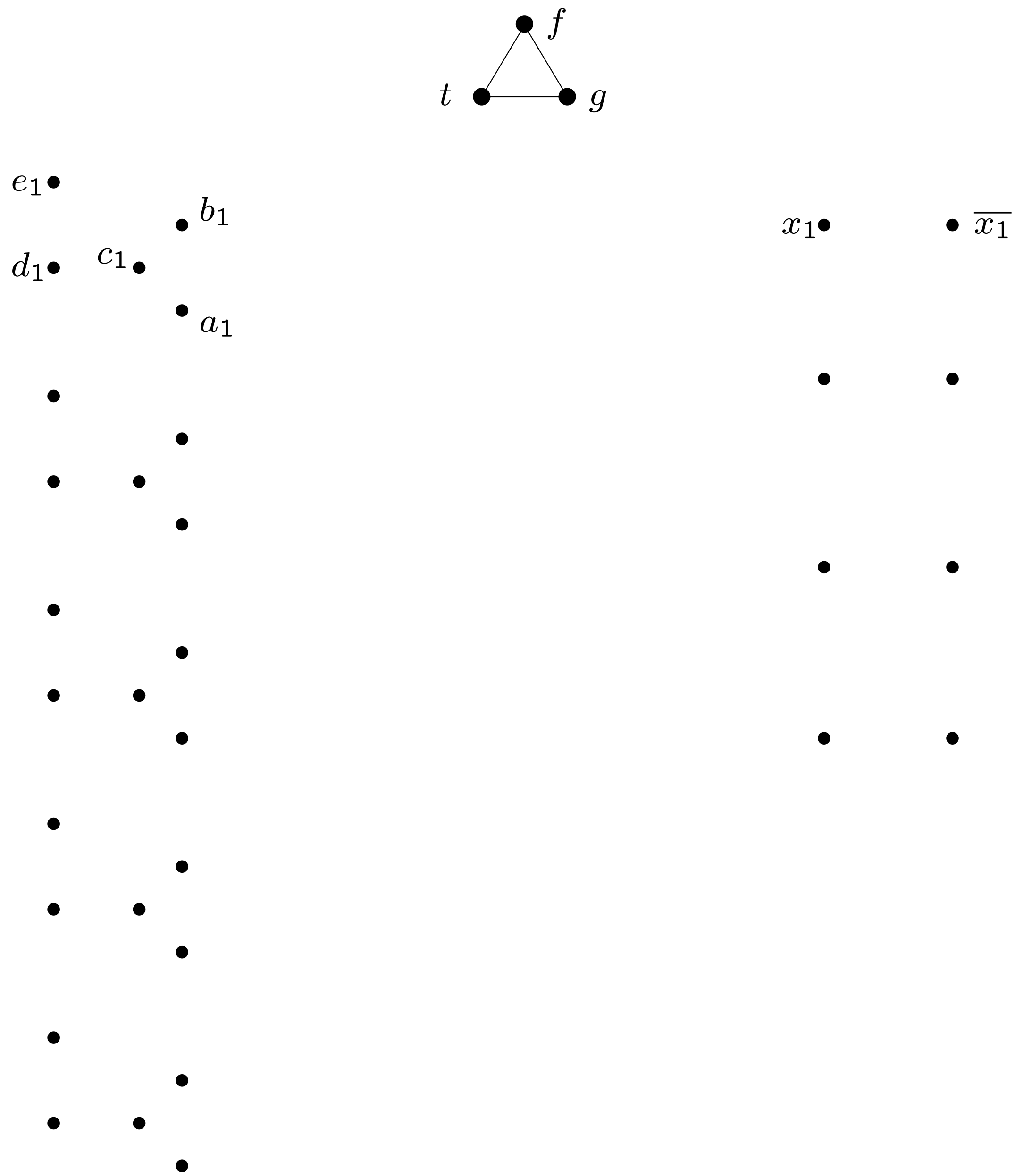
$$f = (x_1 + \overline{x_2} + x_3)(x_1 + x_2 + \overline{x_4})(\overline{x_1} + \overline{x_3} + x_4)(x_2 + x_3 + \overline{x_4})(\overline{x_2} + \overline{x_3} + x_4)$$



3-CNF-SAT $\leq_p$ 3-Coloring (cont'd):

- An example reduction:

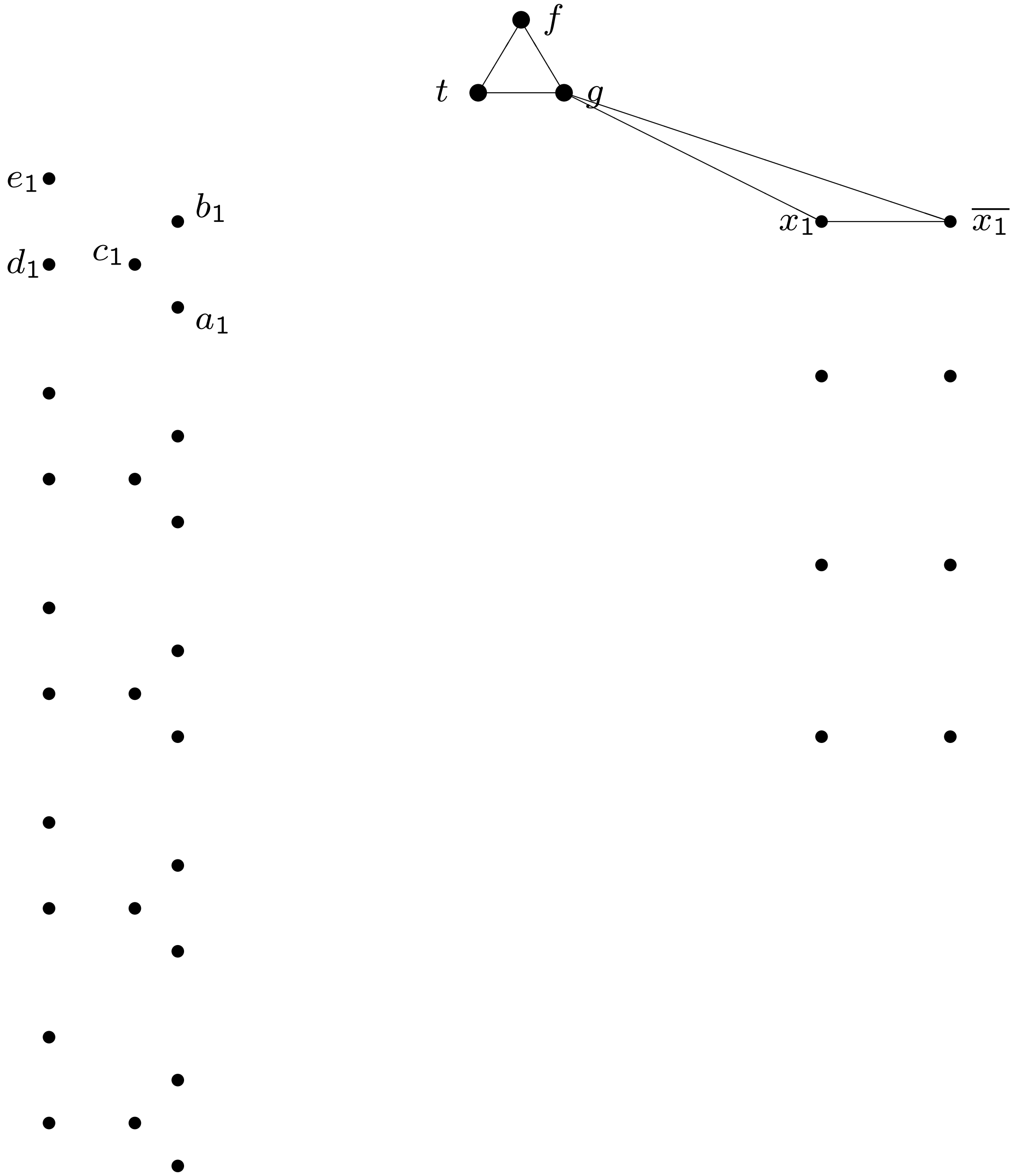
$$f = (x_1 + \overline{x_2} + x_3)(x_1 + x_2 + \overline{x_4})(\overline{x_1} + \overline{x_3} + x_4)(x_2 + x_3 + \overline{x_4})(\overline{x_2} + \overline{x_3} + x_4)$$



3-CNF-SAT $\leq_p$ 3-Coloring (cont'd):

- An example reduction:

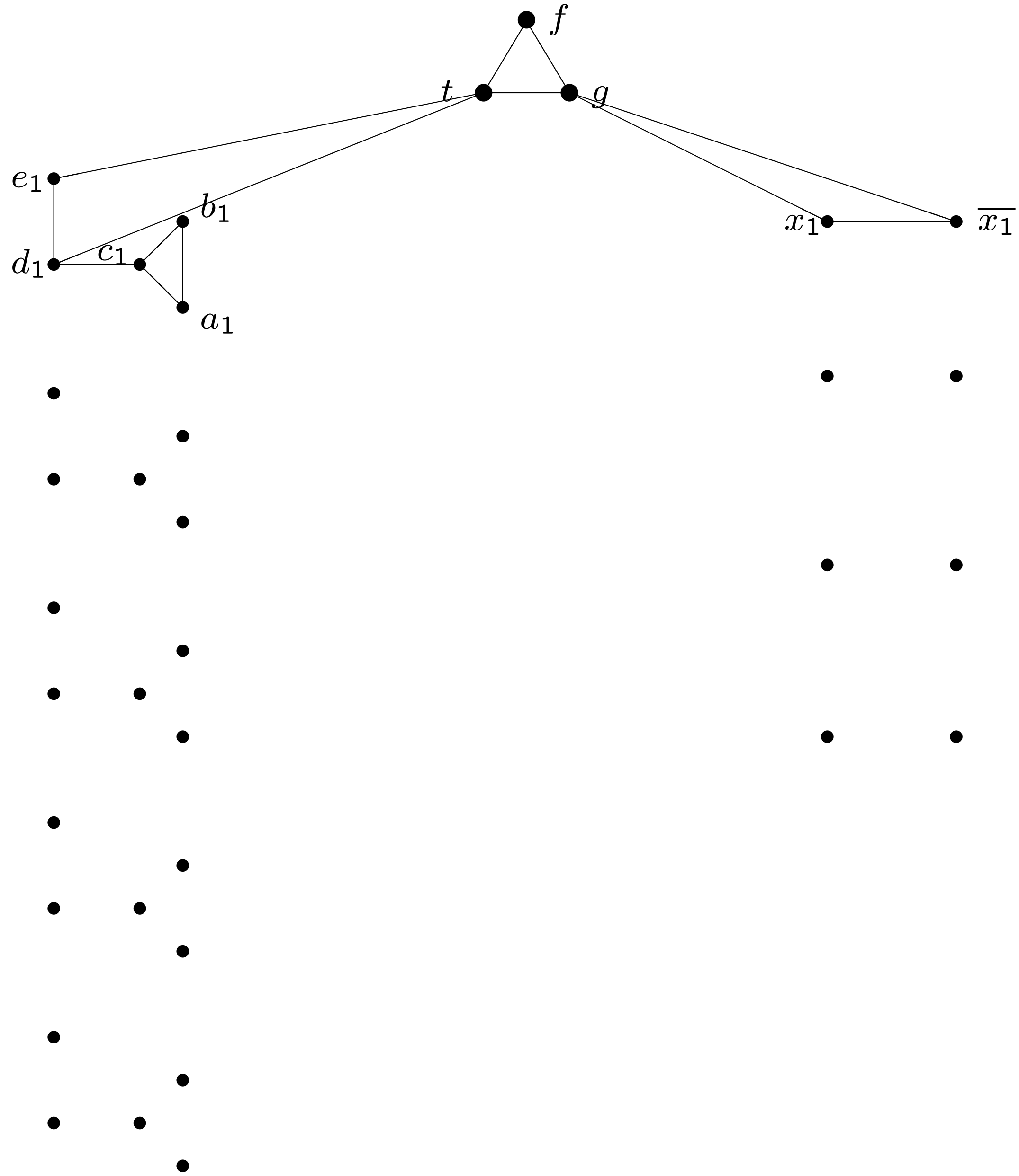
$$f = (x_1 + \overline{x_2} + x_3)(x_1 + x_2 + \overline{x_4})(\overline{x_1} + \overline{x_3} + x_4)(x_2 + x_3 + \overline{x_4})(\overline{x_2} + \overline{x_3} + x_4)$$



3-CNF-SAT $\leq_p$ 3-Coloring (cont'd):

- An example reduction:

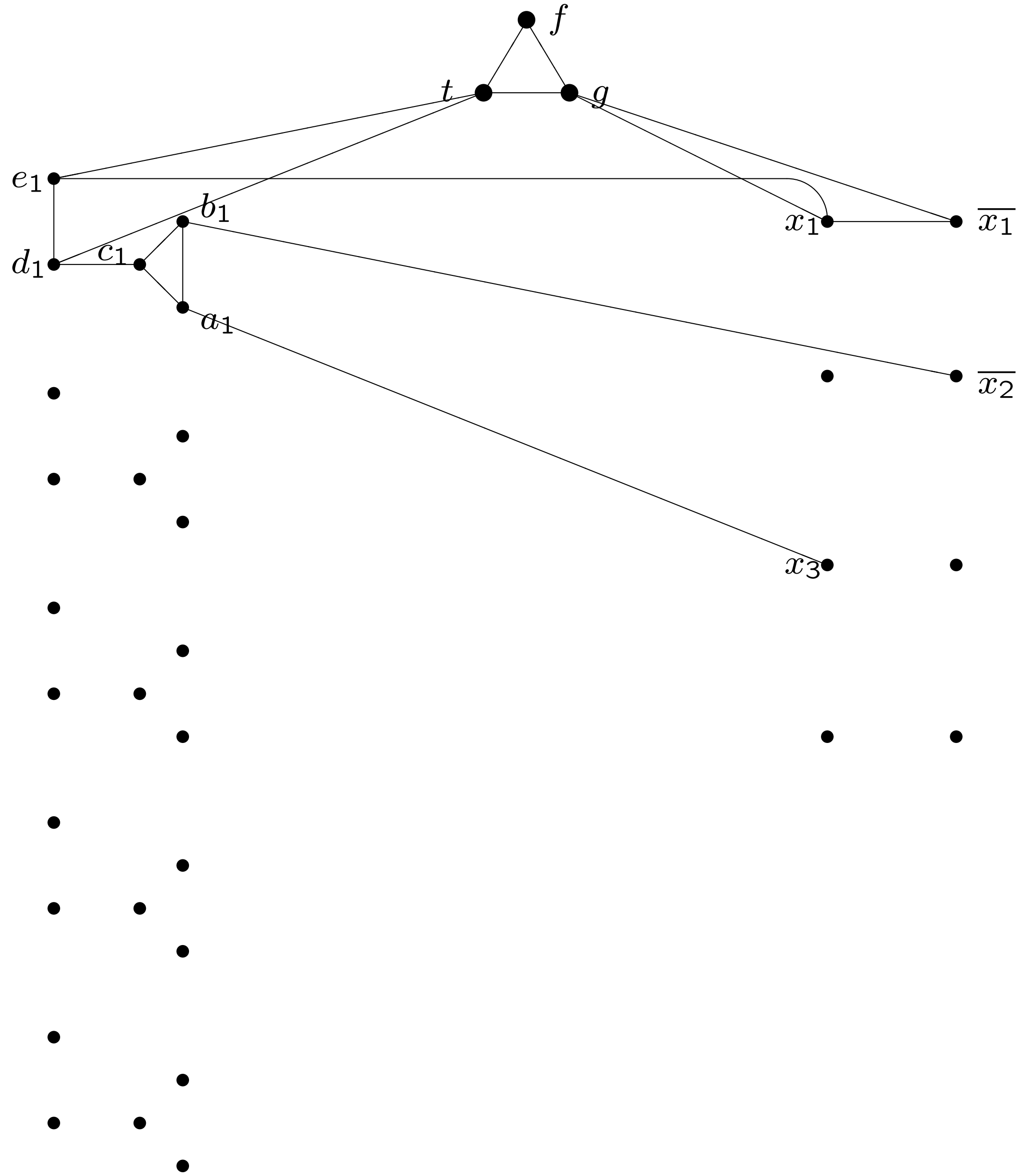
$$f = (x_1 + \overline{x_2} + x_3)(x_1 + x_2 + \overline{x_4})(\overline{x_1} + \overline{x_3} + x_4)(x_2 + x_3 + \overline{x_4})(\overline{x_2} + \overline{x_3} + x_4)$$



3-CNF-SAT $\leq_p$ 3-Coloring (cont'd):

- An example reduction:

$$f = (x_1 + \overline{x_2} + x_3)(x_1 + x_2 + \overline{x_4})(\overline{x_1} + \overline{x_3} + x_4)(x_2 + x_3 + \overline{x_4})(\overline{x_2} + \overline{x_3} + x_4)$$



3-CNF-SAT $\leq_p$ 3-Coloring (cont'd):

- "f satisfiable $\Longrightarrow$ G is 3-colorable":
  - a truth assignment
  - t colored T, f colored F, g colored G
  - $y_1^j = T \Longrightarrow$
    e_j colored F or G
    d_j colored G or F
    c_j colored T or F, or T or G
  - $y_1^j = F \Longrightarrow$
    e_j colored G
    d_j colored F
    c_j colored T or G

- "G is 3-colorable $\Longrightarrow$ f satisfiable":
  - a 3-coloring scheme
  - assume t colored T, f colored F, g colored G
  - e_j colored $F \Longrightarrow$
    y_1^j colored T, thus C_j is satisfied
  - e_j colored $G \Longrightarrow$
    d_j colored F
    c_j colored T or G
    one of a_j and b_j must be colored F
    one of y_2^j and y_3^j must be colored T, thus C_j is satisfied

3-CNF-SAT $\leq_p$ 3-Coloring — conclusion:

- We just proved 3-CNF-SAT $\leq_p$ 3-Coloring
- We have proved 3-CNF-SAT is NP-complete
- We know 3-Coloring is in NP
- Therefore, 3-Coloring is NP-complete.

3-Coloring $\leq_p$ 4-Coloring:

- An instance of 3-Coloring: a (simple, undirected, connected) graph $G = (V, E)$
- A new graph $G' = (V', E')$ is constructed as follows:
  1. $V' = V + h$, h is a new vertex
  2. $E' = E \cup \{(v, h) \mid v \in V\}$
- Given a 3-coloring scheme for G, in G', color the vertices in V the same and color h using an additional color.
- Given a 4-coloring scheme for G', in G, color the vertices the same. This is a valid coloring scheme since their color in G' must be distinct from the color for h, and thus there are at most 3 colors for them.
- Conclusion:
  1. 3-Coloring has been proven NP-complete
  2. 3-Coloring $\leq_p$ 4-Coloring
  3. 4-Coloring is in NP
  4. therefore, 4-Coloring is NP-complete
- Extension:

  For any $k \geq 3$, k-Coloring is NP-complete.

  Exercise.

Have you understood the lecture contents?

well	ok	not-at-all	topic
☐	☐	☐	How to prove the NP-completeness
☐	☐	☐	CNF-SAT $\leq_p$ 3-CNF-SAT
☐	☐	☐	3-CNF-SAT $\leq_p$ 3-Coloring
☐	☐	☐	3-Coloring $\leq_p$ 4-Coloring
☐	☐	☐	k-Coloring is NP-complete