Lecture 33: NP-Completeness

Agenda:

- SAT and Cook's theorem
- Class NP-c
- NP-completeness proof steps
- Example proof of NP-completeness:
  SAT $\leq_p$ 3-CNF-SAT

Reading:

- Textbook pages 995 – 1021

SAT — formula satisfiability and Cook's Theorem:

- Boolean formula (recursive definition):
  - Boolean variable x_i (takes value TRUE or FALSE)
  - suppose f and g are Boolean formulae
  - $\neg f$ is a Boolean formula
  - $f \vee g$ is a Boolean formula
  - $f \wedge g$ is a Boolean formula
  - $f \rightarrow g$ is a Boolean formula
  - $f \leftrightarrow g$ is a Boolean formula
- A Boolean formula is <u>satisfiable</u> if there is an assignment on the Boolean variables such that the formula evaluates TRUE
- Example: $((x_1 \rightarrow x_2) \vee \neg((\neg x_1 \leftrightarrow x_3) \vee x_4)) \wedge \neg x_2$

  It is satisfiable: there is a truth assignment $x_1 = F$, $x_2 = F$, $x_3 = T$, $x_4 = T$
- SAT problem:

  Instance: a Boolean formula

  Query: is the formula satisfiable?

  <u>Cook's Theorem</u> SAT is NP-complete.

  `Proof not required.`

SAT variants:

- Conjunctive normal form satisfiability (CNF-SAT):
  - literal: x_i (a Boolean variable) or $\neg x_i$ (negation of x_i)
  - Boolean CNF formula: $C_1 \wedge C_2 \wedge \ldots \wedge C_m$, where
    C_j is called a clause
    C_j is the OR of one or more literals
  - example:
    $(\neg x_3 \vee x_7) \wedge (x_1) \wedge (\neg x_2 \vee x_3 \vee \neg x_5 \vee x_6) \wedge (x_2 \vee x_3 \vee x_4)$
  - CNF-SAT:
    Instance: a Boolean CNF formula
    Query: is the formula satisfiable?

  Note: CNF-SAT is a subproblem of SAT

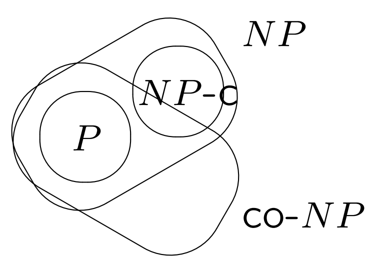
- 3-CNF-SAT:
  - CNF formula
  - every clause contains exactly 3 distinct literals
  - example:
    $(\neg x_3 \vee x_4 \vee x_7) \wedge (x_1 \vee x_4 \vee x_6) \wedge (\neg x_2 \vee x_3 \vee x_6) \wedge (x_2 \vee x_3 \vee x_5)$
  - 3-CNF-SAT:
    Instance: a Boolean 3-CNF formula
    Query: is the formula satisfiable?

  Note: 3-CNF-SAT is a subproblem of CNF-SAT

Class NP-c:

- Class NP-c: all NP-complete problems
  - a subclass of NP
  - NP-complete problem:
    1. is in NP
    2. every problem in NP is reducible to it
  - Cook's theorem implies that NP-c is NOT empty

- The relationships among 4 classes (most likely):



- Notes:
  1. if any NP-complete problem is in P, then $P = NP$ — unlikely
  2. least likely: $P = NP = \text{co}-NP$
  3. most likely: P, NP, co-NP all different
  4. big open CS problem: is it $P = NP$ or not?

     Don't try to answer at this moment!

Karp's consequence of Cook's theorem:

- Cook showed that SAT is in NP-c
- Soon after, Karp showed that 21 other problems in NP-c
- How did Karp do this so quickly?

  Using the transitivity of reduction and SAT as the base NP-complete problem:

  1. SAT reduces to 3-CNF-SAT
  2. 3-CNF-SAT reduces to 3-Coloring
  3. 3-Coloring reduces to k-Coloring (fixed $k \geq 3$)
  4. SAT reduces to k-Clique
  5. k-Clique reduces to k-Independent Set
  6. ...
- Now, there are thousands of problems in NP-c
- How to prove the NP-completeness (recall):
  1. <u>Prove</u> that $\Pi \in NP$
  2. <u>Look for</u> a NP-complete problem Π'
  3. <u>Prove</u> that Π' reduces to Π
- We will show:

  SAT $\leq_p$ 3-CNF-SAT (today)

  CNF-SAT $\leq_p$ 3-CNF-SAT

  3-CNF-SAT $\leq_p$ 3-Coloring

  3-Coloring $\leq_p$ 4-Coloring $\leq_p$ k-Coloring ($k \geq 5$)

Proof of "SAT $\leq_p$ 3-CNF-SAT":

- An instance I of SAT: a Boolean formula f
- Construct an instance J of 3-CNF-SAT out of I: a Boolean 3-CNF formula g
  1. construction takes polynomial time in the length of f, which can be measured by the number of Boolean operations in f
  2. f is satisfiable iff g is satisfiable
- Construction details:
  - create a distinct variable for every Boolean operation in f, and then re-write the formula

    for example: $f(x_1, x_2, x_3) = \neg x_1 \vee (x_2 \leftrightarrow x_3)$

    create y_1 for $\neg$; y_2 for $\vee$; and y_3 for $\leftrightarrow$. Then we will have an equivalent formula

$$\begin{aligned} f'(x_1, x_2, x_3, y_1, y_2, y_3) \ = \quad & y_2 \\ & \wedge(y_2 \leftrightarrow (y_1 \vee y_3)) \\ & \wedge(y_1 \leftrightarrow (\neg x_1)) \\ & \wedge(y_3 \leftrightarrow (x_2 \leftrightarrow x_3)) \end{aligned}$$

    The new formula f' is in conjunctive form, in which each term involves at most 3 literals.

    construction time so far:

    Θ(# of Boolean operations in f)

Construction details (cont'd):

- – draw a truth table for every term and write down the CNF formula

  for example: for term $y_3 \leftrightarrow (x_2 \leftrightarrow x_3)$

y_3	x_2	x_3	$y_3 \leftrightarrow (x_2 \leftrightarrow x_3)$
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	F

  construction time constant (since only 3 literals)

  – write down an equivalent disjunctive normal form (DNF) formula for negation of the term

  for example: for term $y_3 \leftrightarrow (x_2 \leftrightarrow x_3)$

  $$\begin{aligned}\neg(y_3 \leftrightarrow (x_2 \leftrightarrow x_3)) \quad = \quad & (y_3 \wedge x_2 \wedge \neg x_3) \\ & \vee(y_3 \wedge \neg x_2 \wedge x_3) \\ & \vee(\neg y_3 \wedge x_2 \wedge x_3) \\ & \vee(\neg y_3 \wedge \neg x_2 \wedge \neg x_3)\end{aligned}$$

  construction time constant

  – the term itself is the negation of the negation (DeMorgan Laws)

  for example: for term $y_3 \leftrightarrow (x_2 \leftrightarrow x_3)$

  $$\begin{aligned}y_3 \leftrightarrow (x_2 \leftrightarrow x_3) \quad & = \quad \neg(\neg(y_3 \leftrightarrow (x_2 \leftrightarrow x_3))) \\ & = \quad (\neg y_3 \vee \neg x_2 \vee x_3) \\ & \qquad \wedge(\neg y_3 \vee x_2 \vee \neg x_3) \\ & \qquad \wedge(y_3 \vee \neg x_2 \vee \neg x_3) \\ & \qquad \wedge(y_3 \vee x_2 \vee x_3)\end{aligned}$$

  construction time constant

Proof of "SAT $\leq_p$ 3-CNF-SAT" — conclusion:

- An instance I of SAT: a Boolean formula f
- Construct an instance J of 3-CNF-SAT out of I: a Boolean 3-CNF formula g
  1. construction takes linear time in the number of Boolean operations in formula f, and thus polynomial time
  2. f is satisfiable iff g is satisfiable

     `Proof.`

     `notice that` g `is equivalent to` f `in the sense that those` y `literals can be assigned values according to the values of` x `literals.`

     `therefore, if` f `is satisfiable, then` g `is satisfiable;`

     `on the other hand, if` g `is satisfiable, then the` x `literals inherit the values in the truth assignment is an assignment on which` f `evaluates to TRUE.`

     `in the other words,` f `is satisfiable iff` g `is satisfiable.`
- Conclusion:
  - we just showed SAT $\leq_p$ 3-CNF-SAT
  - 3-CNF-SAT $\in NP$
  - therefore, 3-CNF-SAT is NP-complete

Have you understood the lecture contents?

well	ok	not-at-all	topic
□	□	□	SAT and Cook's theorem
□	□	□	Class NP-c, properties
□	□	□	How to prove the NP-completeness
□	□	□	SAT $\leq_p$ 3-CNF-SAT