Lecture 23: Disjoint Sets

Agenda:

- 2^{nd} implementation —— forest of rooted trees (review)
- Improvements:
  - Union by rank `rUnion`
  - Compressed find `cFind`

Reading:

- Textbook pages 505 – 522

2^{nd} implementation — forest of rooted trees

- Forest of rooted trees:
  - elements of a set $\longleftrightarrow$ nodes in the rooted trees
  - representative of a set $\longleftrightarrow$ root of the tree
  - each node needs only 'parent' $\longrightarrow$ implement via an array
  - $P(x)$ — parent of x, for $x = 1, 2, \ldots, n$

- **procedure** MakeSet(x) **initialize parent for x

```
P(x) ← x
```

- **procedure** Find(x) **return root of the tree containing x

```
while P(x) ≠ x do
    x ← P(x)
return x
```

- **procedure** Union(x, y) **make root of x's tree
  **a child of root of y's tree

```
rx ← Find(x)
ry ← Find(y)
P(rx) ← ry
```

- Running time per operation — $\Theta(n)$

Union by rank — `rUnion`:

- Observation: running time affected by the depth of the element(s) in both `find` and `union`
- Goal: to reduce the height of the tree
- Tree height determined by how we do the `union`
- Improvement — union by rank (denoted as `rUnion`)
  - idea: when union two sets, root of shorter tree becomes a child of the root of higher tree
  - rank of an element x — height of subtree rooted at x
  - pseudocode:

```
procedure rUnion(x, y)    **make smaller rank root
                          **child of the other root
   rx ← Find(x)
   ry ← Find(y)
   if rank(rx) > rank(ry) then
      P(ry) ← rx
   else
      P(rx) ← ry
      if rank(rx) = rank(ry) then
         rank(ry) ← rank(ry) +1
```

- Need to initialize the rank for every element:

```
procedure MakeSet(x) **initialize parent for x

   P(x) ← x
   rank(x) ← 0
```

Finding connected components of a graph:

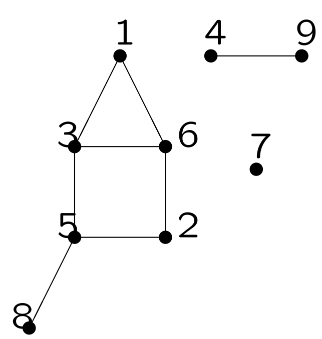
- procedure ConnectedComponents(G)

```
    for each vertex v ∈ V(G) do
        MakeSet(v)
    for each edge (x, y) ∈ E(G) do
        if not SameComponent(x, y, G) then
            rUnion(x, y)            ** previously, Union(x, y)
```

- procedure SameComponent(x, y, G)

```
    return Find(x) = Find(y)
```

- An example:



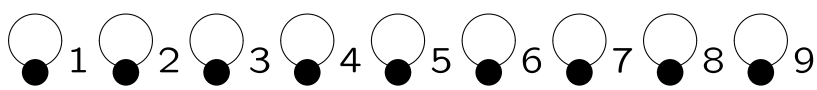
$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$

Finding connected components of a graph:

- Graph $G = (V, E)$:
  $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
  $E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$

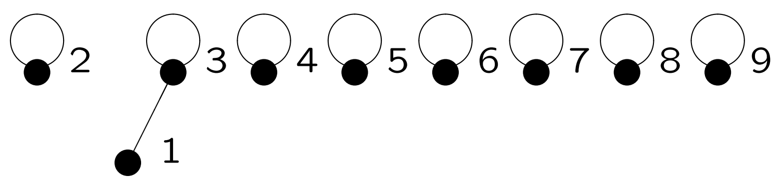
- After `MakeSets`:



Finding connected components of a graph:

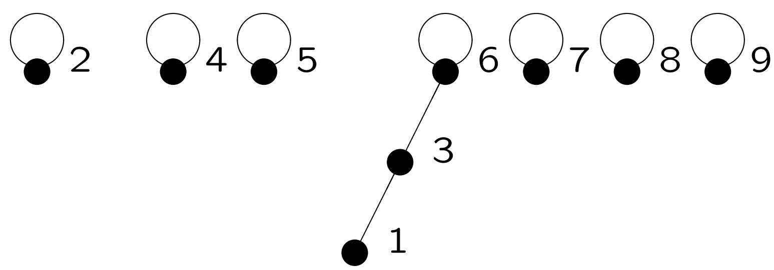
- Graph $G = (V, E)$:
  $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
  $E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$

- After considering edge $(1, 3)$:

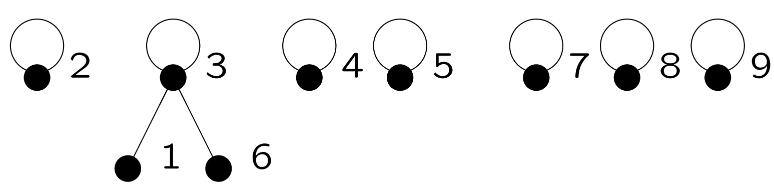


Finding connected components of a graph:

- Graph $G = (V, E)$:
  $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
  $E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$
- After considering edge $(1, 6)$:
  by original `union`:



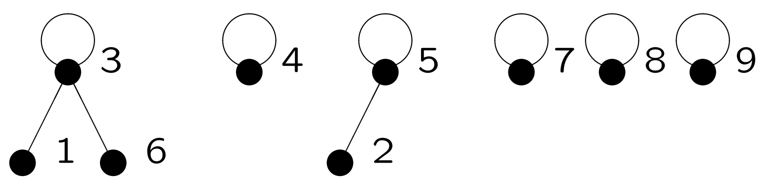
by `rUnion`:



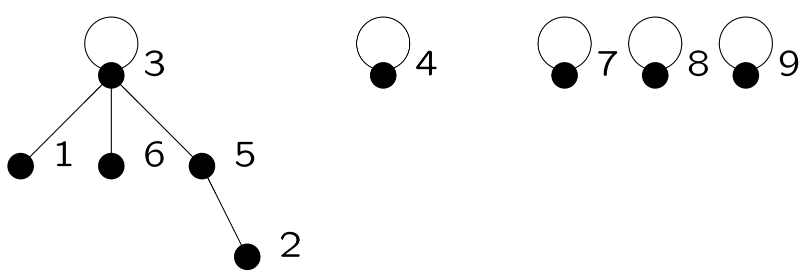
Finding connected components of a graph:

- Graph $G = (V, E)$:
  $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
  $E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$

- After considering edge $(2, 5)$:



- After considering edge $(2, 6)$:



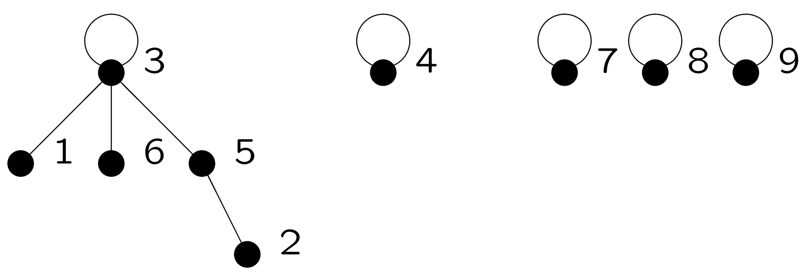
Finding connected components of a graph:

- Graph $G = (V, E)$:

  $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

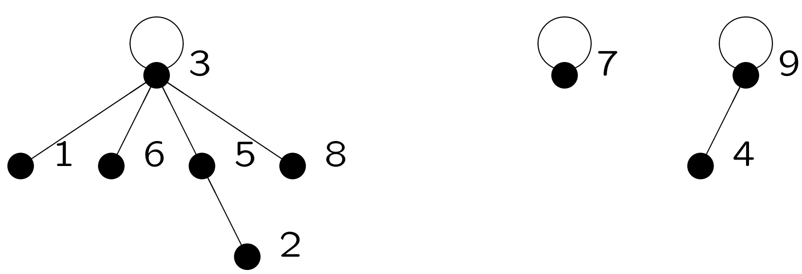
  $E = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 9\}, \{5, 8\}\}$

- After considering edges $(3, 5)$ and $(3, 6)$



  (no change since $3, 5, 6$ are already in a same component):

- After considering edge $(4, 9)$ and $(5, 8)$:



- Therefore, there are 3 connected components

Analysis of (`rUnion + Find`):

- n `MakeSet` and $(m-n)$ `Union/Find`
- Each `MakeSet` — $\Theta(1)$ time
- Each `Find` — $\Theta(\text{depth}(x))$ time
  - T_x — the tree containing x
  - #elements in $T_x \geq 2^{\text{height}(T_x)}$ — proof? by induction
  - so, $\text{depth}(x) \leq \text{height}(T_x) \leq \lg(\#\text{elements in } T_x) \leq \lg n$
- Each `rUnion` — $\Theta(\text{depth}(x) + \text{depth}(y))$ time
- Worst case:
  $\Theta(n + (m-n)\lg n) = \Theta(m \lg n)$ time (assuming $m >> n$)
- On average, $\Theta(\lg n)$ per operation

Compressed find — `cFind`:

- Another observation — to make the trees as short as possible
- Idea:
  - during the time we `Find` the root of the tree containing x
  - we pass all the elements on the x-to-root path
  - re-examine them and make their parents the root
  - `Find` TWICE to make the tree shorter
- pseudocode (non-recursive):

```
procedure cFind(x)

    t ← x
    while P(t) ≠ t do          **find the root
        t ← P(t)
    root ← t
    t ← x
    while P(t) ≠ t do          **change the parent to root
        x ← t
        t ← P(t)
        P(x) ← root
    return root
```

- pseudocode (recursive):

```
procedure cFind(x)

    if P(x) ≠ x do             **x isn't the root
        P(x) ← cFind(P(x))
    return P(x)
```

Analysis of (`rUnion + cFind`):

- n `MakeSet` and $(m-n)$ `Union/Find`
- Each `MakeSet` — $\Theta(1)$ time
- Each `cFind` — $\Theta(\text{depth}(x))$ time
  - $\lg^* n$ — the smallest t such that $2^{2^{2^{\cdot^{\cdot^{\cdot^{2^2}}}}}} \geq n$ where there are t 2's

n	2 ...	4 ...	16 ...	65536 ...	2^{65536}	...
$\lg^* n$	1	2	3	4	5	...

  - depth(x) grows more slowly than $\lg^* n$

    proof not required ... just memorize it ...

- Each `rUnion` — $\Theta(\text{depth}(x) + \text{depth}(y))$ time
- Worst case:
  $O(n + (m-n)\lg^* n) = O(m \lg^* n)$ time (assuming $m >> n$)
- On average, $O(\lg^* n)$ per operation — almost constant

Have you understood the lecture contents?

well	ok	not-at-all	topic
☐	☐	☐	disjoint sets?
☐	☐	☐	3 operations
☐	☐	☐	forest of rooted trees
☐	☐	☐	finding connected components
☐	☐	☐	union by rank
☐	☐	☐	running time analysis
☐	☐	☐	compressed find
☐	☐	☐	running time