# Lecture 9: Recurrences

Agenda:

- Master theorem (full version from textbook)
- The limit of master theorems
- Recurrence conclusions

Reading:

- Textbook pages 62 – 75

## Master Theorem (simple version):

Let $a \geq 1$ and $b > 1$ be constants, let $f(n) \in \Theta(n^c)$ for some positive $c$, and let $T(n)$ be defined by the recurrence

$$T(n) = aT(\frac{n}{b}) + f(n).$$

Then $T(n)$ can be bounded asymptotically as follows:

1. if $c = \log_b a$, then $T(n) \in \Theta(n^c \log n)$,
2. if $c < \log_b a$, then $T(n) \in \Theta(n^{\log_b a})$,
3. if $c > \log_b a$, then $T(n) \in \Theta(n^c)$.

## Some examples:

1. $T(n) = \begin{cases} 2T(\frac{n}{2}) + n & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$

2. $T(n) = \begin{cases} 4T(\frac{n}{2}) + n^3 & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$

3. $T(n) = \begin{cases} 7T(\frac{n}{2}) + n^2 & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$

4. $T(n) = \begin{cases} 7T(\frac{n}{2}) + \dfrac{n^2}{\lg n} & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$

## Some examples:

1. $T(n) = \begin{cases} 2T(\frac{n}{2}) + n & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$

   $a = 2, b = 2, c = 1 \Longrightarrow c = \log_b a$, so $T(n) \in \Theta(n \log n)$

2. $T(n) = \begin{cases} 4T(\frac{n}{2}) + n^3 & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$

   $a = 4, b = 2, c = 3 \Longrightarrow c > \log_b a = 2$, so $T(n) \in \Theta(n^3)$

3. $T(n) = \begin{cases} 7T(\frac{n}{2}) + n^2 & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$

   $a = 7, b = 2, c = 2 \Longrightarrow c < \log_b a$, so $T(n) \in \Theta(n^{\log_2 7})$

4. $T(n) = \begin{cases} 7T(\frac{n}{2}) + \dfrac{n^2}{\lg n} & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$

   $a = 7, b = 2, c = ??? \Longrightarrow$, so simple Master Theorem does NOT apply

## Exercise:

In the last example, show that $T(n) \in O(n^{\log_2 7})$.

## Master Theorem (full version):

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined by the recurrence

$$T(n) = aT(\frac{n}{b}) + f(n).$$

Then $T(n)$ can be bounded asymptotically as follows:

1. If $f(n) \in \Theta(n^{\log_b a})$, then $T(n) \in \Theta(n^{\log_b a} \log n)$,

2. if $f(n) \in O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) \in \Theta(n^{\log_b a})$,

3. if $f(n) \in \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(\frac{n}{b}) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large $n$, then $T(n) \in \Theta(f(n))$.

## The last example:

$$T(n) = \begin{cases} 7T(\frac{n}{2}) + \dfrac{n^2}{\lg n} & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$a = 7, b = 2, \log_b a > 2$

$f(n) = \frac{n^2}{\lg n} \in O(n^{\log_b a - \epsilon})$ for any constant $0 < \epsilon < \log_b a - 2$.

Therefore, $T(n) \in \Theta(n^{\log_b a}) = \Theta(n^{\lg 7})$.

Limit of Master Theorem:

$$T(n) = \begin{cases} 4T(\frac{n}{2}) + \dfrac{n^2}{\lg n} & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$a = 4, b = 2, \log_b a = 2$

$f(n) = \frac{n^2}{\lg n} \notin \Theta(n^2)$;

$f(n) = \frac{n^2}{\lg n} \in O(n^2)$ but $f(n) = \frac{n^2}{\lg n} \notin O(n^{2-\epsilon})$ for any positive constant $\epsilon$.

What we can do to get the closed form?

— iterated substitution!!!

$$\begin{aligned}
& T(2^k) \\
= \; & 4 \times T(2^{k-1}) + \tfrac{2^{2k}}{k} \\
= \; & 4^2 \times T(2^{k-2}) + 4 \times \tfrac{2^{2(k-1)}}{k-1} + \tfrac{2^{2k}}{k} \\
= \; & 4^2 \times T(2^{k-2}) + \tfrac{2^{2k}}{k-1} + \tfrac{2^{2k}}{k} \\
\\
= \; & 4^3 \times T(2^{k-3}) + 4^2 \times \tfrac{2^{2(k-2)}}{k-2} + \tfrac{2^{2k}}{k-1} + \tfrac{2^{2k}}{k} \\
= \; & 4^3 \times T(2^{k-3}) + \tfrac{2^{2k}}{k-2} + \tfrac{2^{2k}}{k-1} + \tfrac{2^{2k}}{k} \\
\\
= \; & 4^k \times T(1) + \tfrac{2^{2k}}{k-(k-1)} + \ldots + \tfrac{2^{2k}}{k-1} + \tfrac{2^{2k}}{k} \\
= \; & 4^k \times T(1) + 2^{2k} \left( \tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \tfrac{1}{k} \right) \\
= \; & 4^k \times T(1) + 4^k \times H(k)
\end{aligned}$$

Therefore, $T(n) = n^2 \times T(1) + n^2 \times H(\lg n) \in \Theta(n^2 H(\lg n))$.

Further we have $H(k) \in \Theta(\log k)$ — you should able to prove it, thus $T(n) \in \Theta(n^2 H(\lg n)) = \Theta(n^2 \log \log n)$.

Limit of Master Theorem:

$$T(n) = \begin{cases} 4T(\frac{n}{2}) + n^2 \lg n & \text{if } n \geq 2 \\ T(1) & \text{if } n = 1 \end{cases}$$

$a = 4, b = 2, \log_b a = 2$

$f(n) = n^2 \lg n \notin \Theta(n^2)$;

$f(n) = n^2 \lg n \in \Omega(n^2)$ but $f(n) = n^2 \lg n \notin \Omega(n^{2+\epsilon})$ for any positive constant $\epsilon$.

What we can do to get the closed form?

— iterated substitution!!!

$$\begin{aligned}
& T(2^k) \\
= \; & 4 \times T(2^{k-1}) + 2^{2k}k \\
= \; & 4^2 \times T(2^{k-2}) + 4 \times 2^{2(k-1)}(k-1) + 2^{2k}k \\
= \; & 4^2 \times T(2^{k-2}) + 2^{2k}(k-1) + 2^{2k}k \\
\\
= \; & 4^3 \times T(2^{k-3}) + 4^2 \times 2^{2(k-2)}(k-2) + 2^{2k}(k-1) + 2^{2k}k \\
= \; & 4^3 \times T(2^{k-3}) + 2^{2k}(k-2) + 2^{2k}(k-1) + 2^{2k}k \\
\\
= \; & 4^k \times T(1) + 2^{2k}(k-(k-1)) + \ldots + 2^{2k}(k-1) + 2^{2k}k \\
= \; & 4^k \times T(1) + 2^{2k}\,(1 + 2 + 3 + \ldots + k) \\
= \; & 4^k \times T(1) + 4^k \times \tfrac{k(k+1)}{2}
\end{aligned}$$

Therefore, $T(n) = n^2 \times T(1) + n^2 \times \dfrac{\lg n(\lg n + 1)}{2} \in \Theta(n^2 \lg^2 n))$.

## An exercise — dealing with floor & ceiling:

Prove that $T(n)$ defined by the following recurrence is in $O(\log n)$:

$$T(n) = \begin{cases} 1, & \text{if } n = 1, \\ T(\lceil \frac{n}{2} \rceil) + 1, & \text{if } n \geq 2 \end{cases}$$

- Examine some small cases:

  $T(1) = 1$

  $T(2) = 2$

  $T(3) = T(4) = 3$

  $T(5) = T(6) = T(7) = T(8) = 4$

  ...

  Guess: $T(n) = k + 1$, for any $2^{k-1} < n \leq 2^k$

- Prove the above guessed (by induction).

- Now you only need to get the closed form for $n$ being a power of 2 ...

- By iterated substitution, $T(2^k) = k + 1$ (again, prove by induction)

- So, $T(n = 2^k) = \lg n + 1$

- So, guess for any $n$, $T(n) \leq \lceil \lg n \rceil + 1$

  Prove it by induction

- Conclusion: since $T(n) \leq \lceil \lg n \rceil + 1 \leq \lg n + 2 \leq 2 \lg n$, for $n \geq 4$,

  $T(n) \in O(\lg n) = O(\log n)$

## Have you understood the lecture contents?

| well | ok | not-at-all | topic |
|---|---|---|---|
| ☐ | ☐ | ☐ | Master Theorem: simple version |
| ☐ | ☐ | ☐ | Master Theorem: full version |
| ☐ | ☐ | ☐ | applying Master Theorem |
| ☐ | ☐ | ☐ | limit of Master Theorem |
| ☐ | ☐ | ☐ | iterated substitution method |
| ☐ | ☐ | ☐ | closed form guessing |
| ☐ | ☐ | ☐ | prove by math induction |