CMPUT 498/501: Advanced Algorithms **Fall 2025**

Lecture 12 (Oct 15, 2025): Integer/Linear Programming, Duality

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12.1 Linear Programming

An LP is the problem of optimizing a linear objective function over n variables $x_1, x_2, \ldots, x_n$ subject to linear equality or inequality constraints. An example of a LP is as follows:

$$\begin{aligned} \text{minimize:} \quad & 2x_1 - 5x_2 + 3x_3 \\ \text{subject to:} \quad & x_1 + 2x_2 \geq 8 \\ & -3x_1 + 4x_2 + 8x_3 \geq 16 \\ & -x_2 + 12x_3 \geq 24 \end{aligned}$$

A general form of the above LP is typically expressed by the following:

$$\begin{aligned} \text{minimize:} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to:} \quad & \sum_{j=1}^{n} a_{ij} x_j \geq b_i \qquad 1 \leq i \leq m \\ & x_j \geq 0 \qquad 1 \leq j \leq n \end{aligned}$$

Its vector form is the following:

$$\begin{aligned} \text{minimize:} \quad & \mathbf{c}^\intercal \mathbf{x} \\ \text{subject to:} \quad & A\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

Definition 1 *If $\mathbf{x}$ satisfies the constraints, then it is a feasible solution*

Some further notes about LPs:

- A LP is feasible if it has a feasible solution.
- A LP is unbounded if for any $\alpha \in \mathbb{R}$ there is a feasible solution $\mathbf{x}$ such that $\mathbf{c}^\intercal \mathbf{x} \geq \alpha$.
- The objective can be maximization, that is:

$$\begin{aligned} \text{maximize:} \quad & \mathbf{c}^\intercal \mathbf{x} \\ \text{subject to:} \quad & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

- Many problems can be formulated as a LP.
- LP has many applications in designing both exact and approximate algorithms.

Example: A farmer has L hectares of land and can grow three different products. It has a total of F kilograms of fertilizer and P kilograms of pesticide. The selling prices of the products are S_1, S_2, S_3 respectively. The amount of fertilizer needed per hectare is f_1, f_2, f_3, and likewise is p_1, p_2, p_3 for pesticide. We can formulate the following LP for this problem:

$$\begin{aligned} \text{maximize:} \quad & S_1x_1 + S_2x_2 + S_3x_3 \\ \text{subject to:} \quad & x_1 + x_2 + x_3 \leq L \\ & f_1x_1 + f_2x_2 + f_3x_3 \leq F \\ & p_1x_1 + p_2x_2 + p_3x_3 \leq P \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

12.2 Equivalent Forms

- We can translate between min / max:

$$\max \mathbf{c}^\intercal \mathbf{x} \leftrightarrow \min -\mathbf{c}^\intercal \mathbf{x}$$

- Equalities can be re-expressed as inequalities:

$$\begin{aligned} \mathbf{a}_i^\intercal \mathbf{x} = b_i \leftrightarrow \mathbf{a}_i^\intercal \mathbf{x} &\leq b_i \\ \mathbf{a}_i^\intercal \mathbf{x} &\geq b_i \end{aligned}$$

- Using slack variables to express inequalities as equality:

$$\mathbf{a}_i^\intercal \mathbf{x} \leq b_i \leftrightarrow a_i^\intercal \mathbf{x} + s_i = b_i \qquad s_i \geq 0$$

- We have both canonical and standard form LPs, respectively:

$$\begin{aligned} \text{minimize:} \quad & \mathbf{c}^\intercal \mathbf{x} \\ \text{subject to:} \quad & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \text{minimize:} \quad & \mathbf{c}^\intercal \mathbf{x} \\ \text{subject to:} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

We consider the following example to show a geometric point of view.

$$
\begin{aligned}
\text{maximize:} \quad & 2x_1 + x_3 \\
\text{subject to:} \quad & x_1 \leq 2 \\
& x_3 \leq 3 \\
& 3x_2 + x_3 \leq 6 \\
& x_1 + x_2 + x_3 \leq 4 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

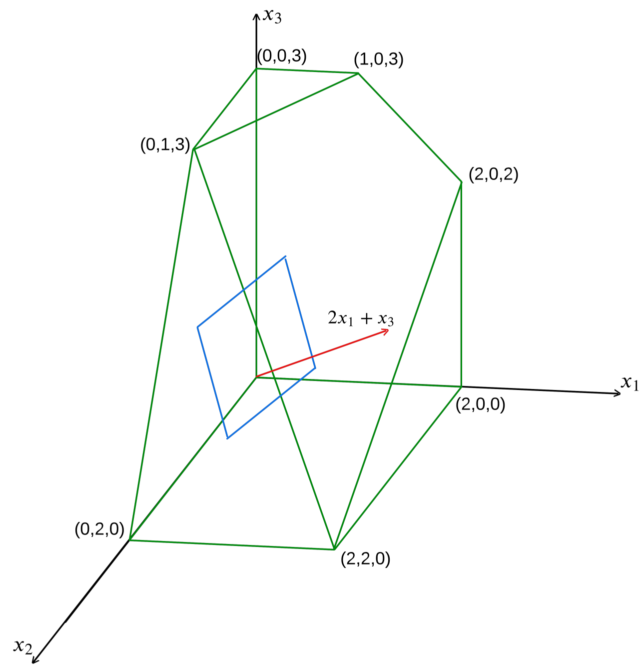

Figure 12.1: (2,0,2) is an optimal feasible solution.

Definition 2 *A hyperplane in* $\mathbb{R}^n$ *is a set of points* $\{\mathbf{x} \in \mathbb{R}^n : a_1x_1 + a_2x_2 + \cdots + a_nx_n = b\}$ *for some given* a_i*'s and* b *(with not all* a_i*'s* $= 0$*).*

A hyperplane defines two half spaces $\mathbf{ax} \leq \mathbf{b}$ and $\mathbf{ax} \geq \mathbf{b}$.

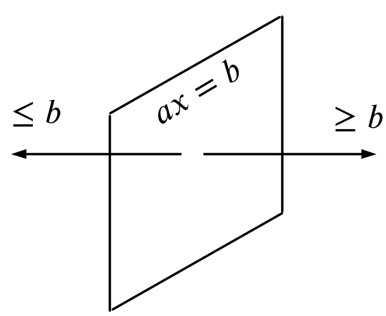

Figure 12.2: The two half spaces defined by the hyperplane above.

Definition 3 *A polyhedron is the convex body defined by the intersection of a finite number of half spaces.*

Definition 4 *A polytope is a bounded polyhedron.*

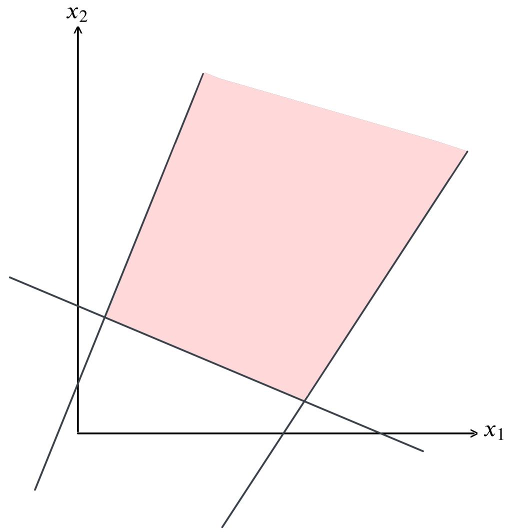

Figure 12.3: The shaded region is a polyhedron.

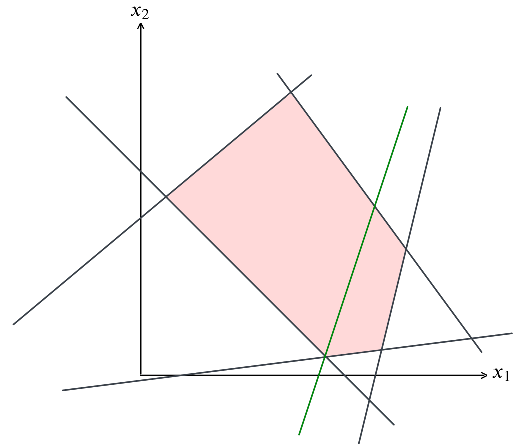

Figure 12.4: The shaded region is a polytope. Note the green line is a face of the polytope.

Definition 5 *Consider a polytope $P \subseteq \mathbb{R}^n$ and a half-space S define by a hyperplane H. If $P \cap S$ is non-empty, then the intersection of P and H is a face of P (equivalently, a face is the set of points satisfying a valid inequality of P with equality).*

A *facet* is a face of dimension $n-1$. A vertex a face of dimension 0, and a line/edge is a face of dimension 1. Note that a hyperplane defining a facet corresponds to a defining half-space of P but the converse may not be true.

Fact: Any interior point of P can be written as a convex combination of its vertices. For points x_1, x_2 and $\lambda \in [0,1]$, $x = \lambda x_1 + (1-\lambda)x_2$ is called a convex combination of x_1, x_2.

We have the following fact: If a LP is feasible, there is always an optimal "corner" solution called a basic feasible solution (BFS).

Definition 6 *A a point $x \in \mathbb{R}^n$ is called an extreme point of a polyhedron P if there do not exist two other points x_1, x_2 inside P and $\lambda \in [0,1]$ such that $x^* = \lambda x_1 + (1-\lambda)x_2$.*

Definition 7 *Given a polyhedron P, a point x^* inside P (i.e. a feasible solution) is called a basic feasible solution if there are n linearly independent constraints of P which x^* satisfies with equality.*

For a bfs x^*, there is a set I of size n where the constraints $a_i \cdot x^* = b_i$ for $i \in I$ are satisfied and for indices $j \notin I$ constraints $a_j \cdot x^* \leq b_j$ hold. A bfs correspond to extrem points of the polyhedron (and are also called vertex solutions).

Thus, a basic solution can be found by: find a set B of n linearly independent rows of A and change the inequality to equality; solve the resulting equations to find x

BFS's are very important. If a LP is feasible, it has a BFS (they cannot be written as a convex combination of other feasible solutions).

Lemma 1 *Let $\mathbf{x}$ be a BFS to $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq 0$. Then there is a unique $\mathbf{c}$ such that $\mathbf{x}$ is the unique optimal feasible solution to*

$$\begin{aligned} \textit{minimize:} \quad & \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ \textit{subject to:} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

Theorem: Let $\mathbf{x}^*$ be a vertex of $P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$. Then there exists $I \subseteq \{1, \ldots, n\}$ such that $\mathbf{x}^*$ is the unique solution to

$$A_I\mathbf{x} = \mathbf{b}_I, \quad A_I \text{ linearly independent.}$$

Proof: Exercise.

Hence, vertices of a polytope correspond to basic feasible solutions of an LP. Each BFS is defined by a set of n linearly independent equations that hold with equality.

Solving LPs

Simplex: The method used in practice moves from one vertex to another (pivoting). Worst-case time is exponential, but it works well in practice.

Ellipsoid: The first polynomial-time LP solver [K79]. Not practical, but it had big consequences in combinatorial optimization and designing other algorithms.

LP Feasibility vs. Optimum Solutions

The main problem is to find one feasible solution, as optimization can be reduced to feasibility by adding the objective as a constraint and performing binary search on α:

$$\begin{aligned} \text{maximize:} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to:} \quad & A\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

can be reduced to

$$\begin{aligned} A\mathbf{x} &\geq \mathbf{b} \\ \mathbf{c}^T\mathbf{x} &\leq \alpha \\ \mathbf{x} &\geq 0. \end{aligned}$$

Important Feature of Ellipsoid: Separation Oracle

Given a proposed solution $\mathbf{x}$ to LP, the oracle returns “Yes” (feasible) or “No” along with a constraint that is violated by $\mathbf{x}$.

The Ellipsoid algorithm finds a feasible solution by making polynomially many calls to a separation oracle.

Why is this powerful? As long as we have a polynomial time separation oracle, we can solve the LP in polynomial time, even if the number of constraints (potentially exponentially) large.

Duality

Consider the following LP:

$$\begin{aligned} \min \quad & 10x_1 + 6x_2 + 4x_3 && (12.1)\\ \text{s.t.} \quad & 2x_1 + x_2 - x_3 \geq 2 && (12.2)\\ & x_1 + x_2 + x_3 \geq 3 && (12.3)\\ & x_1, x_2, x_3 \geq 0. && (12.4) \end{aligned}$$

Question: is the optimal value $Z^* < 50$?

To answer this, we can check whether there exists a feasible solution with value ≤ 50, say $\mathbf{x} = (1, 1, 1)$.

Question: Is $Z^* > 10$?

To answer, we need to find lower bounds.

Compare Eq. (12.1) and Eq. (12.3): Since coefficients are nonnegative, we can say

$$Z^* \geq 4\,(x_1 + x_2 + x_3) \geq 12.$$

In fact, we can take any (non-negative) linear combination of Eq. (12.2) and Eq. (12.3) with weights such that the coefficients in the combination are less than or equal to those in Eq. (12.1). That is, consider

$$\begin{aligned} \min \quad & 10x_1 + 6x_2 + 4x_3 & \\ \text{s.t.} \quad & 2x_1 + x_2 - x_3 \geq 2, & \longleftarrow \text{Multiply } y_1 \geq 0\\ & x_1 + x_2 + x_3 \geq 3, & \longleftarrow \text{Multiply } y_2 \geq 0\\ & x_1, x_2, x_3 \geq 0, & \end{aligned}$$

and we have Z^* is greater than or equal to

$$\begin{aligned} \max \quad & 2y_1 + 3y_2\\ \text{s.t.} \quad & 2y_1 + y_2 \leq 10,\\ & y_1 + y_2 \leq 6,\\ & -y_1 + y_2 \leq 4,\\ & y_1, y_2 \geq 0. \end{aligned}$$

Example:

Primal	Dual
min $3x_1 + x_2 + 2x_3$	max $30y_1 + 24y_2 + 36y_3$
subject to:	subject to:
$x_1 + x_2 + 3x_3 \geq 30$	$y_1 + 2y_2 + 4y_3 \leq 3$
$2x_1 + 2x_2 + 5x_3 \geq 24$	$y_1 + 2y_2 + y_3 \leq 1$
$4x_1 + x_2 + 2x_3 \geq 36$	$3y_1 + 5y_2 + 2y_3 \leq 2$

In general, we have

Primal	Dual
min $\mathbf{c}^T\mathbf{x}$	max $\mathbf{b}^T\mathbf{y}$
subject to:	subject to:
$A\mathbf{x} \geq \mathbf{b}$	$A^T\mathbf{y} \leq \mathbf{c}$
$\mathbf{x} \geq 0$	$\mathbf{y} \geq 0$

Weak Duality

If $\mathbf{x}$ and $\mathbf{y}$ are any feasible solutions to the primal (P) and dual (D) respectively, then

$$\mathbf{c}^T\mathbf{x} \geq \mathbf{b}^T\mathbf{y}.$$

Proof: From $A\mathbf{x} \geq \mathbf{b}$ and $A^T\mathbf{y} \leq \mathbf{c}$ with $\mathbf{x}, \mathbf{y} \geq 0$, we have

$$\mathbf{b}^T\mathbf{y} \leq \mathbf{x}^T A^T \mathbf{y} = \mathbf{y}^T A\mathbf{x} \leq \mathbf{c}^T\mathbf{x}.$$

Corollary: If the dual (D) is unbounded (i.e., the objective can go to $+\infty$), then the primal (P) is infeasible.

Likewise, if the primal is unbounded (objective $\to -\infty$), then the dual is infeasible.

Strong Duality

The primal (P) has a finite feasible solution, if and only if the dual (D) also has a finite feasible solution. Let $\mathbf{x}^*$ and $\mathbf{y}^*$ be the optimum of P and D, respectively, and then we have

$$\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*.$$

Theorem(Complementary Slackness Theorem): Let $\mathbf{x}$ and $\mathbf{y}$ be feasible solutions to the primal and dual respectively. Then $\mathbf{x}$ and $\mathbf{y}$ are optimal if and only if the following conditions hold:

- Primal Complementary Slackness:

$$x_j \left(c_j - (A^T\mathbf{y})_j\right) = 0 \quad \forall 1 \leq j \leq n,$$

- Dual Complementary Slackness:

$$y_i \left((A\mathbf{x})_i - b_i\right) = 0 \quad \forall 1 \leq i \leq m.$$

That is, for each constraint or variable, either the inequality is tight or the corresponding multiplier is zero.

P \ D	unbounded	infeasible	feasible
unbounded impossible	impossible	possible	impossible
infeasible	possible	possible	impossible
feasible	impossible	impossible	possible & equal

12.3 References

[K79] Khachiyan, Leonid Genrikhovich. "A polynomial algorithm in linear programming." In *Doklady Akademii Nauk*, vol. 244, no. 5, pp. 1093-1096. Russian Academy of Sciences, 1979.