Semidefinite programming: Max-Cut & Max_2SAT

Quadratic programs: problem of optimizing a quadratic func. of var's subj to a set of quad. constraints.

Strict Q.P: if each of the const's & the obj. func. consists only of degree zero or degree 2 monomials

Def: let $x \in \mathbb{R}^{n\times n}$ be a non-neg real matrix. We say x is PSD ($x \succeq 0$) iff $\forall a \in \mathbb{R}^n$: $a^T x a \geq 0$ (positive semidefinite)

Theorem: If $x \in \mathbb{R}^{n\times n}$ then the following are equivalent:

1) $x \succeq 0$ (is PSD)
2) x has only non-neg eigen values
3) $\exists\, V \in \mathbb{R}^{m\times n}$, $m \leq n$ s.t. $x = V^T \cdot V$
4) $x = \sum_{i=1}^{n} \lambda_i \omega_i \cdot \omega_i^T$ for $\lambda_i \geq 0$, $\omega_i \in \mathbb{R}^n$ with $\omega_i^T \cdot \omega_i = 1$ & $\omega_i^T \cdot \omega_j = 0$ $_{i \neq j}$

Trace of $A \in \mathbb{R}^{n\times n}$, $Tr(A)$, is the sum of diagonal entries.

Def: Frobenius inner Prod. of A, B, $A \bullet B$ is

$$A \bullet B = tr(A^T \cdot B) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ij}$$

let M_n be the cone of symmetric $\mathbb{R}^{n\times n}$ matrices.

$X \succeq 0$ denotes X is PSD.

let $C, D_1 \dots D_k \in M_n$ and $d_1 \dots d_k \in \mathbb{R}$

SDP formulation

$$\min/\max \sum_{i,j} C_{ij} \cdot x_{ij} \qquad \min/\max\ C \bullet X$$

$$\forall\, 1 \leq \ell \leq k \quad \sum_{i,j} d_{ij\ell} \cdot x_{ij} = d_\ell \qquad D_\ell \bullet X = d_\ell$$

$$X \succeq 0 \qquad\qquad X \succeq 0$$

$$X \in M_n$$

if $C, D_1 \cdots D_k$ are diagonal then this turns into an LP.

- SDP is a convex program. They can be solved in polytime with additive error ε (for any $\varepsilon > 0$). Poly in n & $\log(\frac{1}{\varepsilon})$

- SDP is equivalent to vector program.
- let $V_1 \cdots V_n$ be n-dim vectors $V_i \in \mathbb{R}^n$

$$\min/\max \sum_{i,j} c_{ij} (\vec{v_i} \cdot \vec{v_j})$$

$$\forall\ 1 \le \ell \le k \quad \sum_{i,j} a_{ij\ell} (\vec{v_i} \cdot \vec{v_j}) = b_\ell$$

$$\vec{v_i} \in \mathbb{R}^n$$

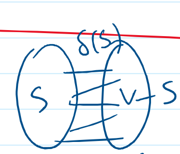
Corresponding SDP is defined as follows; it has n^2 var's y_{ij} replace $\vec{v_i} \cdot \vec{v_j}$ by y_{ij}. Additionally require Y be PSD.

lemma: The Vector Prog & the corresponding SDP are equiv.

proof: We show any feasible sol of V corresponds to a feasible sol of SDP with the same value.

let $W = \begin{bmatrix} \vec{x_1} & \vec{x_2} & \cdots & \vec{x_n} \end{bmatrix}$ where $x_1 \cdots x_n$ is a sol to V. then $X = W^T W$ is feasible to SDP with the same value. (other dir. is similar)

Max_Cut

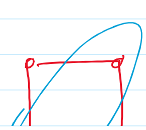


input: undirected $G(V,E)$, $w: E \to \mathbb{R}^+$

Goal: find a subset $S \subseteq V$ that maximizes $w(\delta(S)) = \sum_{e \in \delta(S)} w_e$

The integrality gap for all known LP's is 2.

Random partition gives a $\frac{1}{2}$-approx. for max_cut.



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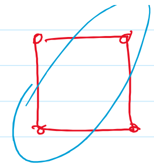


$y_i = \pm 1$ $\quad \max \frac{1}{2} \sum_{1 \le i < j \le n} w_{ij}(1 - y_i \cdot y_j)$ $\qquad$ (IP)

$y_i^2 = 1$

$y_i \in \mathbb{Z}$

This formulates max-cut exactly. We relax this to a vector progr.

$v_1, \ldots, v_n \in \mathbb{R}^n$ $\quad \max \frac{1}{2} \sum w_{ij}\underbrace{(1 - \vec{v_i} \cdot \vec{v_j})}_{(1-\cos\theta_{ij})}$ $\qquad \vec{v_i} \cdot \vec{v_j} = \|v_i\| \cdot \|v_j\| \cdot \cos\theta_{ij}$

$\vec{v_i} \cdot \vec{v_i} = 1$ $\qquad$ (VP)

$\vec{v_i} \in \mathbb{R}^n$

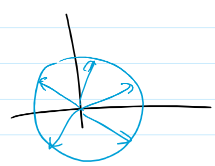
Given a sol y to the I.P. Setting $\vec{v_i} = (y_i, 0, 0, \ldots 0)$ for each $i \in V$ gives a feasible sol of the same obj value for the vector program.

$Z_{VP} \geq OPT_{MC}$

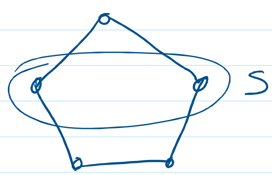
value of opt

Suppose we have an opt sol to the SDP/VP.

- Since $\vec{v_i} \cdot \vec{v_i} = 1$ all vectors lie on the n-dim sphere S_n
- We want to round the sol to an int one.

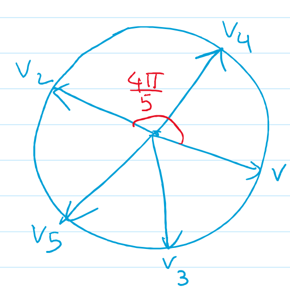


Example: consider C_5 with $w(e) = 1$ $\forall e$



S is opt with value opt=4

The opt VP sol for this graph lies in $\mathbb{R}^2$ in fact and $\widehat{v_i v_{i+1}} = \frac{4\pi}{5}$



$$Z_{VP} = 5\left(\frac{1 - \cos\left(\frac{4\pi}{5}\right)}{2}\right) = \frac{25 + 5\sqrt{5}}{2} = 4.52$$

So $\frac{OPT}{Z_{VP}} \approx 0.884$

The larger θ_{ij} (closer to π) the larger

Max Cut SDP rounding

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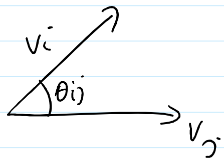
Max_Cut SDP rounding

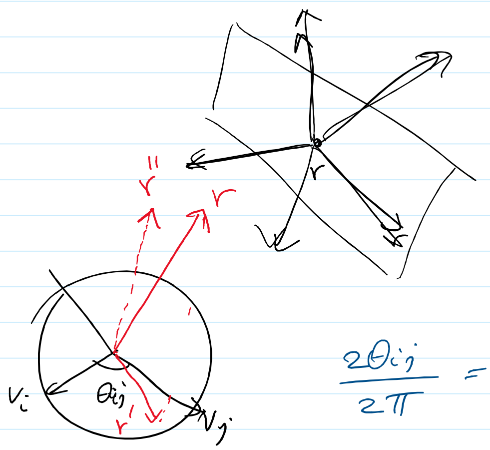
- Solve the VP and get vectors $\vec{v}$
- Choose a random vector r on unit sphere S_n
- let $S = \{ i \mid v_i \cdot r \geq 0 \}$
- return S

Vector r is normal to a hyperplane.
All vectors on the same side as r will be selected (to S)



$\vec{v}_i \cdot \vec{v}_j = \cos(\theta_{ij}) \, |v_i| |v_j|$

$\vec{v}_i \cdot \vec{s}$ and $\vec{v}_j \cdot \vec{s}$ have diff. signs.



$$\frac{2\theta_{ij}}{2\pi} = \frac{\theta_{ij}}{\pi}$$

lemma: $\Pr[v_i \ \& \ v_j \text{ are separated by } r] = \dfrac{\theta_{ij}}{\pi}$

Consider the plane defined by v_i, v_j and project r onto this plane. v_i & v_j are sep <u>iff</u> r' lies between the arcs of θ_{ij}:

$r = r' + r''$

$$\vec{v}_i \cdot r = \vec{v}_i \cdot (r' + r'')$$
$$= \vec{v}_i \cdot r' \quad \text{since } r'' \text{ is ortagonal.}$$
$$\vec{v}_j \cdot r = \vec{v}_j \cdot r'$$

let $x_{ij} = \begin{cases} 1 & \text{if } e = ij \text{ is across the cut} \\ 0 & \text{o.w.} \end{cases}$

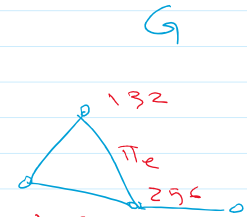
$W = \sum w_{ij} \cdot x_{ij}$

$$W = \sum_{i,j} w_{ij} \cdot x_{ij}$$

$$E[W] = \sum_{i,j} w_{ij} \cdot \Pr[v_i \ \&\ v_j \text{ are sep}] = \sum_{i,j} w_{ij} \cdot \frac{\theta_{ij}}{\pi}$$

let $\alpha_{GW} = \frac{2}{\pi} \min_{0 \le \theta \le \pi} \frac{\theta}{1 - \cos\theta}$ $\longrightarrow$ for any θ: $\frac{\theta}{\pi} \geq \alpha\left(\frac{1-\cos\theta}{2}\right)$

$$\longrightarrow E[W] \geq \alpha \sum_{i,j} \frac{1}{2} w_{ij} (1 - \cos\theta_{ij}) = \alpha \cdot Z_{SDP}$$



labels: $\{1 \ldots k\}$

$\pi_e : [k] \longleftrightarrow [k]$

Yes: $1-\varepsilon$

NO: $\delta(\varepsilon)$

lemma: $\alpha_{GW} \geq 0.8785$

Theorem (Khot et al.): Assuming UGC, there is no $(\alpha_{GW} - \varepsilon)$-approx for Max Cut for any $\varepsilon > 0$.

Max-2SAT using SDP

We saw that LP gives a $\frac{3}{4}$-approx for Max 2SAT and the gap was $\frac{3}{4}$ (tight).

We show how we can do better using SDP.

$y_i = \pm 1$ for x_i

$y_0 = y_i$ iff x_i is True

Value of a clause C: $v(C) = 1$ iff C is Sat.

$$v(x_i) = \frac{1 + y_i y_0}{2} \qquad v(\bar{x}_i) = \frac{1 - y_i y_0}{2}$$

$$v(x_i \vee x_j) = 1 - v(\bar{x}_i) v(\bar{x}_j) = 1 - \frac{1 - y_i y_0}{2} \cdot \frac{1 - y_j y_0}{2}$$

$$= \frac{1}{4}\left(3 + y_i y_0 + y_j y_0 - y_i y_j \cancel{y_0^2}\right)$$

$$= \frac{1 + y_i y_0}{4} + \frac{1 + y_j y_0}{4} - \frac{1 - y_i y_j}{4}$$

So each clause is a liner comb. of $1 + y_i y_j$ and $1 - y_i y_j$

$$\max \sum^{w_i \cdot v(C_i)} \left[a_{ij} (1 + y_i y_j) + b_{ij} (1 - y_i y_j) \right]$$

$$y_i^2 = 1$$
$$y_i = \pm 1$$

$$\max \sum a_{ij} (1 + v_i \cdot v_j) + b_{ij} (1 - v_i \cdot v_j)$$

$$v_i \cdot v_i = 1$$
$$v_i \in \mathbb{R}^{n+1}$$

like Max-Cut : take a random $r \in S_{n+1}$ round each y_i to 1 iff $r \cdot v_i \geq 0$

lemma: $E[W] \geq \alpha \cdot Z_{SDP}$

$$E[W] = 2 \sum a_{ij} \cdot \Pr[y_i = y_j] + b_{ij} \cdot \Pr[y_i \neq y_j]$$

$$\widehat{v_i v_j} = \theta_{ij}$$

$$\Pr[y_i \neq y_j] = \frac{\theta_{ij}}{\pi} \geq \frac{\alpha}{2}(1 - \cos\theta_{ij})$$

$$\Pr[y_i = y_j] = 1 - \frac{\theta_{ij}}{\pi} \geq \frac{\alpha}{2}(1 + \cos\theta_{ij})$$

$$\rightarrow E[W] \geq \alpha \sum a_{ij}(1 + \cos\theta_{ij}) + b_{ij}(1 - \cos\theta_{ij}) = \alpha \cdot Z_{SDP}$$

Theom (LLZ/JN2):

Max-2SAT has an SDP roundig alg with ratio 0.94016 and this is best possible under UGC.

General CSP: range of values $\{1 .. k\}$

$$(x_1, x_5, x_6) \in \begin{cases} 2, 1, 6 \\ 5, 5, 3 \\ 4, 8, 11 \end{cases}$$