Max SAT

input: Boolean formula f over var's $x_1 \dots x_n$ in CNF
plus weight w_j for each clause C_j $1 \le j \le m$

Goal: Find a truth assignment that max the total weight of satisfied clauses.

Special case: $w_j = 1 \longrightarrow$ max # of satisfied clauses.

Max-k-SAT: every clause has up to k literals.
Max-E-k-SAT: " " exactly k " .

Theorem: Max-kSAT is NP-hard for any $k \ge 2$.

(Note that 2SAT is in P).

Simple Rand Alg for Max-SAT

This is perhaps the most obvious Rand Alg: flip a fair coin for each var x_i and set it to T/F with prob $\frac{1}{2}$.

Theorem (Johnson '74) This is a $\frac{1}{2}$-approx for max-SAT

proof: let τ be the rand truth assignment and define

$$Y_j = \begin{cases} 1 & \text{if } C_j \text{ is sat} \\ 0 & \end{cases} \qquad \text{let } W = \sum_{j=1}^{m} w_j \cdot Y_j$$

$$E[W] = \sum_{j=1}^{m} w_j E[Y_j] = \sum_{j=1}^{m} w_j \cdot \Pr[C_j \text{ is sat}] = \sum_j w_j \left(1 - \left(\tfrac{1}{2}\right)^{|C_j|}\right)$$

$$\ge \frac{1}{2} \sum_j w_j \ge \frac{1}{2} OPT$$

Note that if $|C_j| \ge k$ for all j then this is a $\left(1 - \frac{1}{2^k}\right)$-approx. $\longrightarrow$ $\frac{7}{8}$-approx for Max-E3SAT

Thm (Håstad '97): There is no $\left(\frac{7}{8} - \varepsilon\right)$-approx for Max-E3SAT unless P=NP.

Derandomizing using method of cond. expectation

lemma: suppose we have assigned values $a_1 \ldots a_i$ to var's $x_1 \ldots x_i$. Then we can compute the expected value of the sol if the rest of the var's are assigned rand.

proof: let f' be the formula on var's $x_{i+1} \ldots x_n$ obtained from f by substituting the values of $x_1 \ldots x_i$ and simplifying f. Clearly the exp value of sol for any truth assignment to f' can be computed in poly time. Add this to the weight of clauses that are already satisfied by $x_1 \ldots x_i$ & we obtain the exp. value of f given $x_1 \ldots x_i$ are fixed.

– This suggest a simple deterministic alg:

Consider x_1: for each of T/F assignment to x_1 compute the exp value $E[W|x_1=T]$ & $E[W|x_1=F]$ whichever is bigger assign x_1 accordingly. do this for all var's iteratively.

Say this value is v for x_1

$$E[W] = E[W|x_1=T] \cdot Pr[x_1=T] + E[W|x_1=F] \cdot Pr[x_1=F]$$
$$= \frac{1}{2}\left(E[W|x_1=T] + E[W|x_1=F]\right)$$

if we set x_1 as in above then $E[W \mid x_1 = v] \geq E[W] \geq \frac{1}{2}\text{opt}$

Rand. alg with biased coin

First assume we have an instance where all clauses of length 1 have non-negated literals (not $\bar{x}_3 = C_j$)

We set each var $x_i = T$ with prob p ($p > \frac{1}{2}$ TBD).

For any C_j:

- if $|C_j| = 1$ then $\Pr[C_j \text{ is sat}] = p$
- if $|C_j| \geq 2$ then $\Pr[C_j \text{ is sat}] = 1 - \underbrace{p^{\alpha}(1-p)^{\beta}}_{C_j = F} \geq 1 - \underbrace{p^{\alpha+\beta}}_{\substack{\text{because} \\ p > (1-p)}} \geq 1 - p^2$

  α: # of neg. literals

  β: # of non-neg literals

lemma: $\Pr[C_j \text{ is sat}] \geq \min\{p, 1-p^2\}$

Set $p = 1 - p^2 \longrightarrow p = \frac{\sqrt{5}+1}{2} \approx 0.618$

$$E[W] = \sum_j w_j \cdot \Pr[C_j \text{ is sat}] \geq p \sum_j w_j \geq p \cdot \text{opt}$$

Theorem: If all clauses of length 1 have positive literals then this is a p-approx $p = \frac{\sqrt{5} \pm 1}{2}$

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what if there are 1-clauses with neg. literals? e.g. $C_j = \bar{x}_i$

we can easily replace $\bar{x}_i$ with x_i' and x_i with $\bar{x}_i'$.

The only prob. is when both x_i & $\overline{x_i}$ appear as a 1-clause. e.g. $C_j = x_i$ & $C_\ell = \overline{x_i}$

WLOG assume $w_j \geq w_\ell$ define $w_i' = w_\ell$ then

$$opt \leq \sum_j w_j - \sum_{\substack{i:\ \text{1-clause with neg} \\ \text{literal \& positive} \\ \text{1-clause}}} w_i'$$

U: set of incides of clauses excluding those with neg. literal

We can still show:

$$E[W] = \sum_j w_j \cdot Pr[C_j \text{ is sat}] \geq \sum_{j \in U} w_j \cdot Pr[C_j \text{ is sat}]$$

$$\geq p \sum_{j \in U} w_j$$

$$\geq p \cdot opt$$

Theorem: Biased coin implies a p-approx for max-SAT.

LP-rounding for Max-SAT

let P_j (N_j) be the set of var's that appear in clause C_j in positive (negative) form. and $\ell_j = |P_j| + |N_j|$. for any x_i we have $y_i \in \{0,1\}$: $y_i = 1 (0)$ iff $x_i = T (F)$.

Also for each C_j we have $z_j = 1$ iff C_j is satisfied.

$$\max \sum_j w_j \cdot z_j$$

$$\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j \quad \forall C_j \in C$$

Rand. Round for max-SAT

- Solve the LP. let (y^*, z^*) be the opt sol

[illegible]

$$\sum_{i\in P_j} y_i + \sum_{i\in N_j}(1-y_i) \geq z_j \quad \forall C_j \in C$$

$$0 \leq z_i, y_i \leq 1$$

be the opt sol

- for each x_i set it to T with prob y_i^*
- let $\hat{x}$ be the int sol obtained.

Thm (Goemans/Williamson '94):

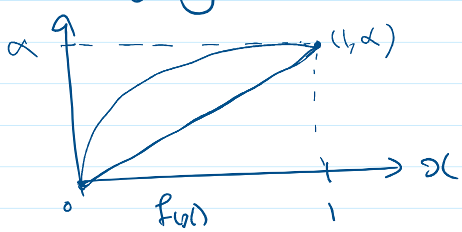
R.R. is a $\underbrace{\left(1-\frac{1}{e}\right)}_{\approx 0.632}$-approx for max-SAT

Proof: We use the following two facts

Fact 1: Arithm/Geom mean inequality: non neg #s $a_1 \dots a_k$:

$$\frac{a_1 + \dots + a_k}{k} \geq \sqrt[k]{a_1 a_2 \dots a_k} \qquad \left(\frac{\sum_i a_i}{k}\right)^k \geq \prod_i a_i$$

Fact 2: if $f(x)$ is concave over $[0,1]$ (i.e. $f''(x) \leq 0$) and $f(0)=0$ and $f(1)=\alpha$ then the func. f is lower bounded by the line going thru $(0,0)$ and $(1,\alpha)$



let W_j be the weight contributed by C_j to the total

Lemma: $E[W_j] \geq \underbrace{\left[1-\left(1-\frac{1}{k}\right)^k\right]}_{\geq 1-\frac{1}{e}} w_j \cdot z_j^*$ for every clause C_j with $l_j = k$.

Proof:

$\underbrace{\qquad\qquad}_{C_j \text{ not sat}}$

$$\Pr[C_j \text{ is sat}] = 1 - \prod_{i \in P_j} (1-y_i^*) \prod_{i \in N_j} y_i^*$$

$$\geq 1 - \left(\frac{\sum_{i \in P_j} (1-y_i^*) + \sum_{i \in N_j} y_i^*}{k} \right)^k \quad \text{use. A.G.M}$$

$$\geq 1 - \left(1 - \frac{\sum_{i \in P_j} y_i^* + \sum_{i \in N_j} (1-y_i^*)}{k} \right)^k$$

by const. for C_j $\quad \geq 1 - \left(1 - \frac{z_j^*}{k}\right)^k$

Consider func. $g(z) = 1 - \left(1 - \frac{z}{k}\right)^k$ $\qquad g(0) = 0$

$g(1) = 1 - \left(1 - \frac{1}{k}\right)^k$

and $g(\cdot)$ is concave in $[0,1]$.

$\xrightarrow{\text{by fact 2}}$ $g(z) \geq \left(1 - \left(1 - \frac{1}{k}\right)^k\right) \cdot z$

So $\Pr[C_j \text{ is sat}] \geq \left[1 - \left(1 - \frac{1}{k}\right)^k\right] z_j^*$ ∎

$$E[W] = \sum_j w_j \cdot \Pr[C_j \text{ is sat}]$$

$$\geq \left[1 - \left(1 - \frac{1}{k}\right)^k\right] \sum w_j \cdot z_j^* \geq \left(1 - \frac{1}{e}\right) opt$$

Since $k \to \infty$ $\left(1 - \frac{1}{k}\right)^k \to \frac{1}{e}$

Remark: This alg performs better than $1 - \frac{1}{e}$ if C_j's are small

Remark: we can also Derand. this.

A $\frac{3}{4}$-approx for max-SAT

Take the better of:

- Rand. Alg. $\leftarrow$ $a=0$

- Rand Rounding alg. (LP) $\leftarrow$ if $a=1$

Suppose we set $a=0/1$ with prob $\frac{1}{2}$ and

Let Z^* be the opt value of LP and W_j be the cont. of clause C_j in our alg.

Lemma: For each C_j : $E[W_j] \geq \frac{3}{4} \underbrace{w_j \cdot Z_j^*}$

Proof: Suppose C_j has k literals. and define $\alpha_k = 1 - \frac{1}{2^k}$ and $\beta_k = 1-(1-\frac{1}{k})^k$

$$E[W_j \mid a=0] \geq (1-\frac{1}{2^k})w_j \geq \alpha_k w_j \cdot Z_j^* \quad (\text{Since } Z_j^* \leq 1)$$

$$E[W_j \mid a=1] \geq \left[1-(1-\frac{1}{k})^k\right] w_j \cdot Z_j^* = \beta_k \cdot w_j \cdot Z_j^*$$

$$E[W_j] = E[W_j \mid a=0] \cdot \underbrace{Pr[a=0]}_{1/2} + E[W_j \mid a=1] \cdot \underbrace{Pr[a=1]}_{\frac{1}{2}}$$

$$\geq \frac{1}{2}[\alpha_k + \beta_k]\, \underbrace{w_j \cdot Z_j^*}$$

$\alpha_1 + \beta_1 = \alpha_2 + \beta_2 = \frac{3}{2}$ for $k \geq 3$ $\alpha_k + \beta_k \geq \frac{7}{8} + (1-\frac{1}{e}) \geq \frac{3}{2}$

So $E[W] = \sum_j E[W_j] \geq \frac{3}{4} \sum w_j \cdot Z_j^* \geq \frac{3}{4} OPT$ ∎

We can also derandomize this alg:

Derand $\frac{3}{4}$-approx for max-SAT

- Use the deterministic of rand alg to find Sol τ_1
- " " " " " LP_rand " " " τ_2
- output the better of τ_1 & τ_2

Thm: This is a deterministic $\frac{3}{4}$-approx for max-sat.

Tight example for LP (integrality gap):

$(x_1 \vee x_2) \wedge (\bar{x}_1 \vee x_2) \wedge (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_2)$ all $w_j = 1$

opt = 3 but if we set $y_i = \frac{1}{2}$ and $z_j = 1$ we have a feasible LP with value 4.

$\longrightarrow$ integrality gap = $\frac{4}{3}$

Best known ratio for Max-SAT ≈ 0.7846

based on a plausible conj $\xrightarrow[\text{be}]{\text{might}}$ 0.8331-approx