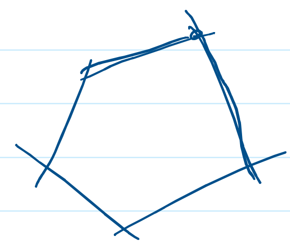
Further notes about LP's

Suppose P is a polyhedron in $\mathbb{R}^n$ defined by set of inequalities $Ax \leq b$. If P is finite and feasible then every bfs (basic feasible solution) corresponds to a set of n linearly independant constraints that all hold with equality, i.e. are "tight."

In other words, if we have an LP with n var's

$$\max c^T \cdot x$$
$$A \cdot x \leq b$$
$$x_i \geq 0$$



then if it is bounded & feasible then for every bfs x^* there is a set of n linearly Independent constraints $a_{ij} \cdot x_i \leq b_j$ $\quad j \in I \subseteq \{1..m\}$ where x^* is the unique solution to the set of equations $a_{ij} \cdot x_i = b_j$, $j \in I$

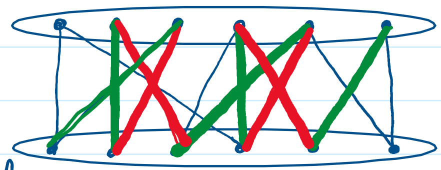
Combinatorial Algorithms for Matching

- We have already seen that for any V.C. C & any matching M: $\quad |C| \geq |M|$

Theorem (König): For bipartite G: $\min_{\text{V.C } C} |C| = \max_{\text{matchg } M} |M|$

- We have already seen a proof via LP.

Definition: Given matching M, an M-alternating path is a path that alternates between M & $E-M$



M-augmenting: if the first & last vertex are not saturated (are exposed).

Lemma: if M is a matching & P is an M-augmenting then $M \Delta P = (M-P) \cup (P-M)$ is a matching of size $|M|+1$.

Proof: Easy.

Theorem: A matching M is maximum <u>iff</u> there is no M-augmenting path.

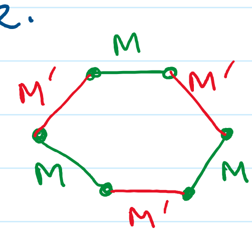
Proof: One direction is easy.

- suppose there is no M-augmenting path but there is a matching M' with $|M'| > |M|$.
- Consider graph $H = (A \cup B, M \cup M')$.

Fact: H has max degree 2.

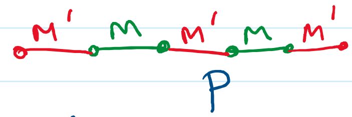
$\longrightarrow$ each component is a path or cycle.

- Each cycle C is even:



- $\exists$ a path (alternating)

P with $|P \cap M'| > |P \cap M|$



$\longrightarrow$ P is an M-augmenting path.

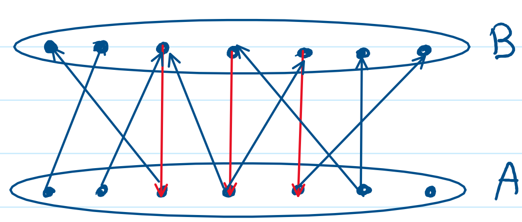
Bipartite Matching & Vertex Cover

- Based on M-augmenting paths we can build an algorithm to find maximum matching

Maximum bipartite Matching $M \leftarrow \phi$ while the is an M-augmenting path P do let $M \leftarrow M \Delta P$ return M

- We provide an $O(m)$ algorithm that finds an M-augmenting path.

- Imagine we make the graph directed $D=(A \cup B, E')$
  - Direct $e \in M$ from B to A
  - Direct $e \notin M$ from A to B



Lemma: There is an M-augmenting path in G iff $\exists$ a dipath from an exposed node in A to an exposed node in B.

Proof: Easy excercise.

Consider an extra node r with di-edges to exposed nodes in A. then run a BFS from r to find a path to exposed nodes in B. $\blacksquare$

Theorem: We can find a maximum matching in time $O(mn)$.

Proof: size of matching is $O(n)$; finding an M-augmenting path takes $O(m)$.

improved time: If one uses BFS to find "shortest" M-augmenting paths then it can be shown that one can find vertex disjoint M-augmenting paths
– So can increase $|M|$ by more than 1.

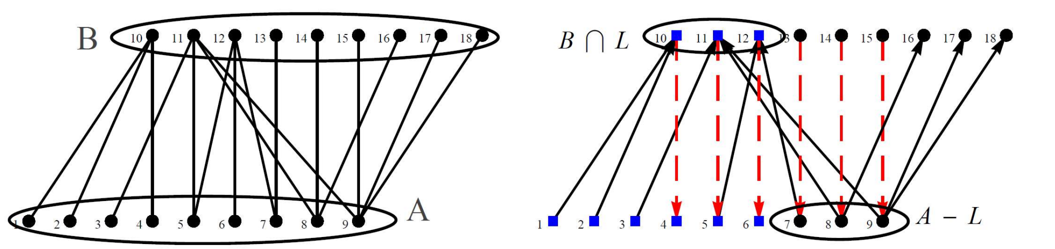
Theorem: Using BFS one can find maximum matching in time $O((m+n)\sqrt{n})$.

Fastest: A breakthrough result in FOCS22 (112 pages!) show bipartite matching in time $O(m^{1+o(1)})$. (Chen et al. FOCS 22).

What about min V.C. for bipartite G?

Consider the digraph $D=(A\cup B, E')$ where edges of M are directed from B to A (and $e\in E-M$ from A to B).

- let L be the set of vertices that can be reached from an exposed node of A



(a) graph G (b) graph D

Lemma: When the matching algorithm terminates $C^* = (A-L)\cup(B\cap L)$ is a vertex cover and $|C^*| = |M|$.

proof: Suppose C^* is not a V.C. $\longrightarrow$ $\exists$ an edge $ab\in E$ s.t. $a\in A\cap L$ & $b\in B-L$

Claim: $ab\notin M$

proof: Suppose not, so $ab\in M$ and is directed $b\to a$; since $a\in L$, it can be reached only by ba (only edges in M) from an exposed node, (note that a itself is not exposed as $ab\in M$) so b is also reached from the exposed node

$\longrightarrow b \in L$ but $b \in B - L$ ⚡

So $ab \in E - M \longrightarrow$ directed from a to b

Since $a \in L$ and ab is an edge so b is reachable from an exposed node $\longrightarrow b \in L$ ⚡

So $e \notin M \wedge e \notin E - M$, So C^* is V.C.

Prove $\underline{|C^*| \leq |M|}$:

- Note no vertex in $A - L$ is exposed (since exposed vertices of A are in L).

- No $b \in B \cap L$ is exposed (or else $\exists$ an M-augmenting path).

- Also we showed $\nexists\, ab \in E$ with $a \in A - L$, $b \in B \cap L$

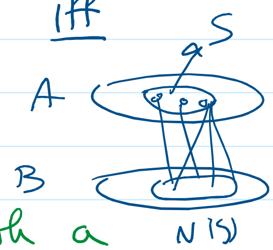
$\longrightarrow |M| \geq |(A-L) \cup (B \cap L)| = |C^*|$ ∎

- This gives another proof of König's theorem.

- We can prove the following using König's theorem

Theorem (Hall 1935): A bipartite graph $G = (A \cup B, E)$ has a matching that saturates all of A <u>iff</u>

$\forall S \subseteq A: |N(S)| \geq |S|$



$N(S)$ = vertices in B with a

neighbor in S

Weighted Bipartite Graph Matching

Problem: $G(A \cup B, E)$, $w: E \longrightarrow \mathbb{R}^+$, find max-weight matching.

Assumptions:

- G is complete bipartite (add 0-weight edges)
- $|A| = |B|$ (add dummy vertices & 0-edges)
- $w_e \geq 0$ (if not add $W = \max_e |w_e|$ to all)

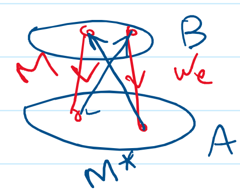
minimization: Can define $c_e = W - w_e$

- We use $w(M)$: weight of matching M.

1) First Algorithm: Negative cycles.

- Build digraph $D = (A \cup B, E')$ as before: $\begin{cases} e \in M & B \text{ to } A \\ e \notin M & A \text{ to } B \\ & -w_e \end{cases}$

- let M be the current matching (M is a perfect matching always).



- if $w(M^*) > w(M)$ then $H = (A \cup B, M \cup M^*)$ is union of cycles $\longrightarrow$ $\exists$ a negative cycle!

$$w(H) = \sum_{C \in \mathcal{C}} w(C) = w(M) - w(M^*)$$

$\mathcal{C}$ $\rightarrow$ set of cycles in H

– We can use Bellman-Ford Shortest path to find one in $O(mn)$ time; swap edges.

Maximum Weight bipart Matching: Algorithm 1

$M \leftarrow$ any perfect matching

Build D from M

while there is a negative cycle C do

$M \leftarrow M \Delta C$

update M

return M

Theorem: A perfect weighted matching M is optimum <u>iff</u> there is no negative cycle C.

Proof: One direction is easy.

Suppose M is not maximum & $\nexists$ negative cycle C

– let M' be a max-weight matching & build $D' = M \cup M'$

– D' is union of disjoint edges & cycles

$$C \subseteq (M \cup M') - (M \cap M')$$

– Note: $W(M' - M) > W(M - M')$

$\rightarrow$ there is a negative cycle!

Time Complexity: $O(mn \times (W_{max} - W_{min}))$; inefficient

Goldberg & Tarjan '89: if one finds negative cycles with min average weight $\frac{w(c)}{|C|} \longrightarrow$ time complexity will be strongly-polynomial.

2) Second Algorithm: Hungarian Method

Developed by Kuhn (1955) using Egerváry's theorem.

later improved by Munkres'57, Iri'60, Edmond & Karp'70

- At each step $\underline{k}$ it has a max-weight matching with exactly $\underline{k}$ edges.
- Start with $M = \emptyset$
- In each iteration, build D (as before):

$e \in M$ directed B to A

$e \notin M$ " A to B with $-w_e$

- If there is an M-augmenting path P then

$M' = M \Delta P$: $\quad w(M') = w(M) - w(P \cap M) + w(P - M)$

$= w(M) - \ell(P)$

$\ell(P) = w(P \cap M) - w(P - M)$

$|M'| = |M| + 1$; so $k+1$

Theorem: Augmenting a max-matching of size k by a shortest augmenting path produces a max matching of size k+1.

Proof: Induction on k: $k \geq 0$

- suppose M is a max-matching of size k & P a shortest M-augmenting path
- Suppose $M' = M \Delta P$ is not maximum; let N be a maximum size $k+1$ matching: $w(N) > w(M')$
- let $H = (A \cup B, M \cup N) \subseteq D$; Since $|N| > |M|$, $\exists$ an M-augmenting path P' in H.
- Since P is the shortest M-augmenting in D and $H \subseteq D$:

$$\ell(P') \geq \ell(P) \qquad (1)$$

- consider $N' = N \Delta P'$ obtained by applying P' in reverse to N ($|N'| = k$).

$$w(N') \leq w(M) \qquad (2)$$

this is because N' is a matching of size k and M is the max. matching of size k.

- From (1) & (2): $\underbrace{w(N') - \ell(P')}_{w(N)} \leq \underbrace{w(M) - \ell(P)}_{w(M')}$

$$w(N) \leq w(M').$$

Max-weight bipartite Matching: Hungarian Method

$M \leftarrow \underset{w_e}{\arg\max}\{e\}$

Build D

while $|M| < \frac{n}{2}$ do

  find shortest path P in D

  $M \leftarrow M \Delta P$

  update D

return M

- Using Bellman-Ford takes $O(mn)$ for shortest Path

$\longrightarrow$ Total time complexity: $O(mn^2)$