# LP and duality

In general:

$$\min \sum_{i=1}^{n} c_i \cdot x_i \qquad \xleftrightarrow{\text{dual}} \qquad \max \sum_{i=1}^{m} b_i \cdot y_i$$

$$\forall\, 1 \le i \le m \quad \sum_{j=1}^{n} a_{ij} \cdot x_j \ge b_i \qquad\qquad 1 \le j \le n \quad \sum_{i=1}^{m} a_{ij} \cdot b_i \le c_j$$

$$x_j \ge 0 \qquad\qquad y_i \ge 0$$

$$\underline{P} \quad \boxed{\begin{array}{l} \min\ c^T \cdot x \\ A \cdot x \ge b \\ x \ge 0 \end{array}} \qquad\qquad \underline{D} \quad \boxed{\begin{array}{l} \max\ b^T \cdot y \\ A^T \cdot y \le c \\ y \ge 0 \end{array}}$$

**Weak duality:** if $\vec{x}$ & $\vec{y}$ are <u>any</u> feasible solutions to $\underline{P}$ & $\underline{D}$ (respectively) then

$$c^T \cdot x \ge b^T \cdot y$$

**Proof:**

$$\sum_{j=1}^{n} c_j \cdot x_j \ge \sum_{j=1}^{n} \left( \overbrace{\sum_{i=1}^{m} a_{ij} \cdot y_i}^{\le c_j} \right) x_j$$

$$\ge \sum_{i=1}^{m} \left( \sum_{j=1}^{n} a_{ij} \cdot x_j \right) y_i$$

$$\ge \sum_{i=1}^{m} b_i \cdot y_i$$

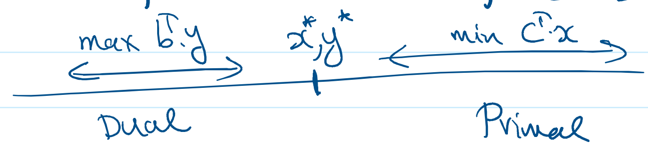
**Corollary:** if $\underline{D}$ is unbounded (opt value $\to +\infty$)

then $\underline{P}$ is not feasible.

if $\underline{P}$ is unbounded (goes to $\longrightarrow -\infty$) then $\underline{D}$ is not feasible.

**Strong duality:** P has a finite feasible solution <u>iff</u> D has a finite feasible solution.

Also if $x^*$ and $y^*$ are optimum for P & D then $c^T.x^* = b^T.y^*$




**Theorem (complementary slackness)** let $\vec{x}$ & $\vec{y}$ be two feasible solutions to $\underline{P}$ and $\underline{D}$. Then both $\vec{x}$ and $\vec{y}$ are optimum <u>iff</u> the following hold:

- Primal complementary slackness:

$$\forall\ 1 \leq j \leq n: \text{ either } x_j = 0 \text{ or } \sum_{i=1}^{m} a_{ij}\, y_i = c_j$$

- Dual Complementary Slackness:

$$\forall\ 1 \leq i \leq m: \text{ either } y_i = 0 \text{ or } \sum_{j=1}^{n} a_{ij} \cdot x_j = b_i$$

| D \ P | unbounded | infeasible | Feasible |
|---|---|---|---|
| unbounded | impossible | possible | impossible |
| infeasible | possible | Possible | impossible |
| feasible | impossible | impossible | Possible & equal |

| | | | | |
|---|---|---|---|---|
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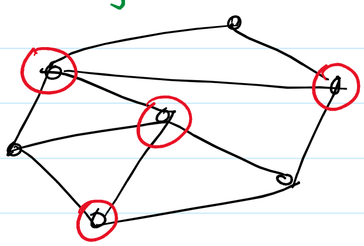
## Examples of writing problems as IP/LP

**min vertex cover:** Given a graph $G(V,E)$, find a smallest set of vertices $C \subseteq V$ s.t. each edge $e \in E$ has at least one end-point in $C$: $\quad x_v \in \{0,1\} \longleftrightarrow v \in C$



$$\min \sum_{v} x_v$$

$$\forall e = uv: \quad x_u + x_v \geq 1$$

$$x_v \in \{0,1\} \xrightarrow{LP} 0 \leq x_v \leq 1$$

**max-flow:** Consider the max-s-t-flow problem:

**Input:** Given (di)graph $G(V,E)$, nodes $s,t \in V$, and edge capacities: $c: E \to \mathbb{Z}^{\geq 0}$

**Goal:** Find a flow $f$ going from $s$ to $t$ with max size:

1) $f(e) \leq c(e) \quad \forall e$
2) flow conservation at all nodes $\in V - \{s,t\}$
3) total flow going out of $s$ = total flow going into $t$

$$\max \sum_{e \ni s} f_e$$

$$\forall v \in V - \{s,t\} \quad \sum_{e \in \delta^{in}(v)} f_e = \sum_{e \in \delta^{out}(v)} f_e$$

$$\forall v \in V - \{s,t\} \quad \sum_{e \in \delta^+(v)} f_e = \sum_{e \in \delta^-(v)} f_e$$

$$\sum_{e \in \delta^-(s)} f(e) = \sum_{e \in \delta^+(t)} f_e$$

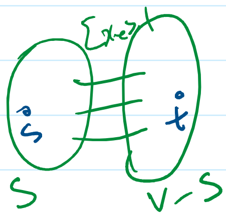
$$\forall e \quad 0 \le f(e) \le c(e)$$

Consider the MST problem & formulate as an IP

$x_e \in \{0,1\}$ indicates $e$ is in MST

$$\min \sum x_e \cdot c_e$$

$$\forall S \subset V, S \neq \phi : \quad \sum_{e \in \delta(S)} x_e \ge 1$$




$$\sum_e x_e = n-1$$

$$0 \le x_e \le 1 \leftarrow x_e \in \{0,1\}$$

$$LP \le IP$$

$$\min \sum x_e \cdot c_e$$

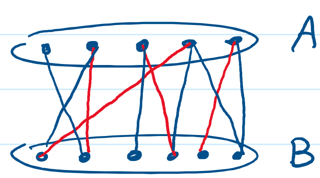
$$\forall S \subseteq V : \quad \sum_{e \in S} x_e \le |S| - 1$$

$$\sum_{e \in E} x_e = n-1$$

$$0 \le x_e \le 1$$

## Bipartite Matching

- Given a bipartite graph $G = (A \cup B, E)$ with two parts $A$ & $B$

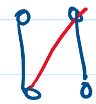



- A matching $M \subseteq E$ is a subset of edges with no common end-points

Question: Find a maximum size matching.

Perfect Matching: Matching $M$ is perfect if each node of $V$ has degree 1; participates in $M$ (Saturated)

Maximal matching: $M$ is maximal if no edges can be added to $M$.



M: maximal
M: maximum

- We can formulate an Integer Program (IP)

$\forall e \in E: \quad x_e \in \{0,1\} \qquad x_e = \begin{cases} 1 & e \in M \\ 0 & \text{o.w.} \end{cases}$

$$\max \sum_{e \in E} x_e$$

$$\sum_{e \ni u} x_e \leq 1 \quad \forall u \in A$$

$$\sum_{e \ni v} x_e \leq 1 \quad \forall v \in B$$

$$x_e \in \{0,1\} \longrightarrow LP \quad 0 \leq x_e \leq 1$$

- In general, solving IP is NP-Complete.

- We can relax the last constraint to obtain an LP: $\quad x_e \in \{0,1\} \xrightarrow{\text{relax}} 0 \leq x_e \leq 1$

Fact: Consider any IP, $\underline{I}$ and let $\underline{P}$ be its LP relaxation (w.o.L.G assume both are maximization). Let $Z$ be the optimum value of $I$ and $Z^*$ be the optimum value of $\underline{P}$. $\quad Z^* \geq Z$

(any feasible Solution to $\underline{I}$ is also feasible for $\underline{P}$)

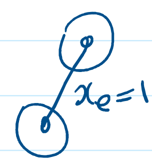
Bipartite Matching LP:

$$\max \sum_{e \in E} x_e$$

$$\sum_{e \ni u} x_e \leq 1 \quad \forall u \in V \qquad LP(1)$$

$$(1 \geq) \; x_e \geq 0$$

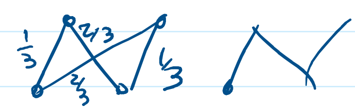
Theorem: If $G$ is bipartite then any Solution of LP(M) can be rounded to an integer solution of $\geq$ value.




Proof: Consider any (fractional) solution $x^*$.

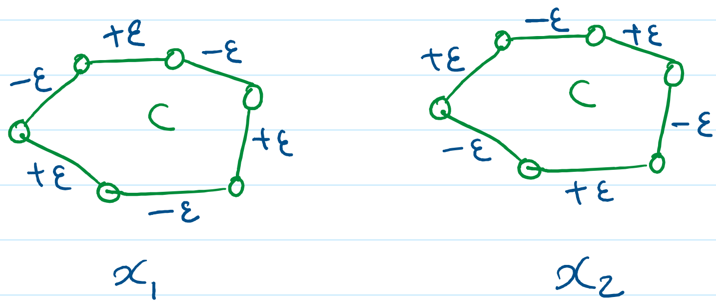
– let support of $x^*$ be $\text{supp}(x^*) = \{e \in E : x^*_e > 0\}$.

– W.O.L.G we assume $x^*$ is total fractional (i.e. $0 < x^*_e < 1 \quad \forall e \in E$), why?




– Eliminate cycles: If there is any cycle $C$ in $\text{supp}(x^*)$ we can eliminate it in the following way:

– let $\varepsilon$ be the smallest value s.t. $0 \leq x_e \pm \varepsilon \leq 1$ $\quad e \in C$



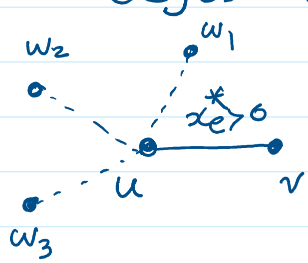
- Note: $\frac{1}{2}(x_1(C) + x_2(C)) = x^*(C)$

so $x^*$ is a convex combination of $x_1$ & $x_2$

- Value of LP solution does not change, at least one edge becomes 0/1 $\longrightarrow$ support size decreases.

- Round forest: Now $\text{supp}(x^*)$ is a forest. Pick any leaf node $v$ with $x^*_e > 0$, $e = uv$.

- round $x^*_e$ to 1, eliminate $x^*_e$ and round all other edges incident to $u$ to zero




- Total value of $x^*$ incident to $u$ was $\leq 1$ before and now is $= 1$; so value of LP has not gone down.

- We still have a feasible solution.

Perfect matching: Given (bipartite) $G$, find a perfect matching.

Weighted: Suppose $w: E \longrightarrow \mathbb{R}^+$. find a perfect

Weighted: Suppose $w: E \to \mathbb{R}^+$, find a perfect matching of max (min) weight.

$$\max \sum w_e \cdot x_e$$

$$\sum_{e \ni u} x_e = 1 \quad u \in V \qquad LP(2)$$

$$x_e \geq 0$$

Theorem: If G is bipartite any bfs of LP(2) is integral.

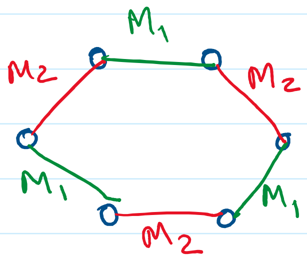
This means this LP is integral: all corners correspond to integer solutions.

Proof: We show any fractional solution is <u>not</u> a bfs (can be written as convex combinations of two other feasible solutions).

- let $x^*$ be any (totally) fractional solution.
- Since $\sum_{e \ni u} x_e^* = 1 \ \forall u$, each node $u$ has $\geq 2$ edges in $\mathrm{supp}(x^*)$.

$\longrightarrow$ $\mathrm{supp}(x^*)$ has a cycle $C$.

- We can partition $C$ into two matchings $M_1$ & $M_2$




$x^1$ : $M_1 + \varepsilon$, $M_2 - \varepsilon$

$x^2$ : $M_1 - \varepsilon$, $M_2 + \varepsilon$

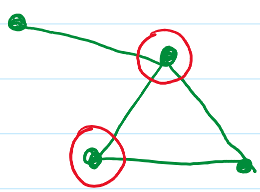
pick $\varepsilon$ as before

$x^*(C) = \frac{1}{2}(x^1(C) + x^2(C))$

$\rightarrow$ $x^*$ is not a bfs.

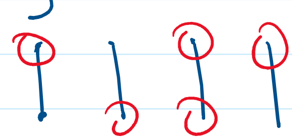
Corollary: We can use LP to find perfect matching and also maximum weight perfect matching.

Vertex Cover: Given a graph $G=(V,E)$ a set $C \subseteq V$ is a vertex cover if every edge $e \in E$ has an end-point in $C$.



Fact: if $C$ is a V.C. then there is no edges in $V-C$. (so $V-C$ is an independent set)

Corollary: For any matching $M$ and any V.C. $C$ : $|C| \geq |M|$.



Proof: Each edge of $M$ must have one end in $C$.

$\longrightarrow$ minimum V.C. $\geq$ maximum matching

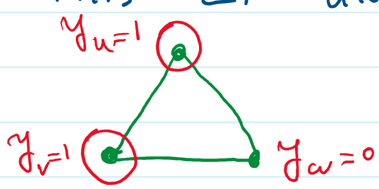
- let's consider an LP relaxation for V.C.

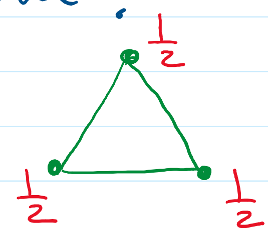
$$\min \sum_{v \in V} y_v$$

$$y_u + y_v \geq 1 \quad \forall\, uv \in E$$

$$y_u \geq 0 \quad \forall\, u \in V$$

- Is this LP always integral?

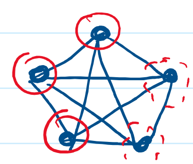



integer V.C. $= 2$ LP V.C. $= \frac{3}{2}$

- if we consider complete graph $K_n$:

integer $= n-1$ LP$= \frac{n}{2}$



**Integrality gap:** Given a (min) Integer program $I$ and the corresponding LP relaxation $P$ we define Integrality gap of $P$ as:

$$\sup_{\pi} \frac{\text{value of } I \text{ on instance } \pi}{\text{value of } P \text{ on instance } \pi}$$

- For eg. we see that for V.C. the gap is at least $\frac{n-1}{\ } = 2 - \frac{1}{\ }$

at least $\frac{n-1}{n/2} = 2 - \frac{1}{n}$

## Duality of matching & vertex cover:

$$\begin{array}{lll|lll} & \max \sum x_e & & & \min \sum y_v & \\ \forall u \in V: & \sum_{e \ni u} x_e \leq 1 & & \forall uv \in E: & y_u + y_v \geq 1 & \\ LP(M) & x_e \geq 0 & & DP(C) & y_u \geq 0 & \end{array}$$

- For bipartite graphs we saw LP(M) has always an integer optimum solution.
- We can show the same for dual LP DP(C)

**Theorem:** Any feasible solution $y^*$ to DP(C) can be rounded to an integer solution of value $\leq \sum_v y_v^*$ if $G = (A \cup B, E)$ is bipartite.

**Proof:** Pick a random value $0 \leq \alpha \leq 1$ and define $A^* = \{u \in A \mid y_u \geq \alpha\}$ $B^* = \{v \in B \mid y_v \geq 1-\alpha\}$

- We claim $A^* \cup B^*$ is a V.C.:

$\forall uv \in E: \quad y_u + y_v \geq 1 \longrightarrow$ if $(u \notin A^* \wedge v \notin B^*)$

then $y_u + y_v < \alpha + (1-\alpha)$

- As for $|A^* \cup B^*|$:

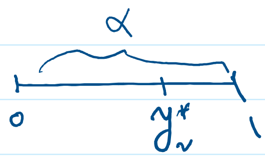
– As for $|A^* \cup B^*|$:

$$\mathbb{E}\left[|A^* \cup B^*|\right] = \sum_{u \in A} \Pr[y_u^* \geq \alpha] + \sum_{v \in B} \Pr[y_v^* \geq 1-\alpha]$$

$$\overset{(*)}{=} \sum_{u \in A} y_u^* + \sum_{v \in B} y_v^*$$

$$= \sum_{v \in V} y_v^*$$

(*) $\alpha$ is chosen randomly




– Since the "expected" value of the integer solution $A^* \cup B^*$ is at most $\sum_v y_v^*$ so there is an integer solution of the same value.

Corollary: For bipartite graphs both the maximum matching & minimum V.C. have the same value.