Linear Programming

An LP is the problem of optimizing a linear objective function over n variables $x_1 \dots x_n$ subject to linear equality or inequality constraints

$$\min \quad 2x_1 - 5x_2 + 3x_3$$
$$x_1 + 2x_2 \geq 8$$
$$-3x_1 + 4x_2 + 8x_3 \geq 16$$
$$-x_2 + 12x_3 \geq 24$$

$$\left.\begin{array}{lll} \min & \sum_{j=1}^{n} c_j \cdot x_j & \\ \text{s.t.} & \sum_{j=1}^{n} a_{ij} \cdot x_j \geq b_i & 1 \leq i \leq m \\ & x_j \geq 0 & 1 \leq j \leq n \end{array}\right\} \quad \begin{array}{l} \min \ c^T \cdot x \\ A x \geq \vec{b} \\ x \geq 0 \end{array}$$

Definition: If x satisfies the constraints : feasible solution

- LP is feasible if it has a feasible solution.
- LP is unbounded if for any $\alpha \in \mathbb{R}$ there is a feasible sol x s.t. $c^T \cdot x \geq \alpha$.
- The objective can be maximization

$$\max \ c^T \cdot x$$
$$A \cdot x \leq b$$

$x \geq 0$

- Many problems can be formulated as LP.
- LP has many applications in designing algorithms (both exact & approximate).

Example : A farmer has L hectares of land and can grow three differat products. It has a total of F kg of fertalizer & P kg of pecticide.

Selling prices of products: S_1, S_2, S_3

Amount of fertalizer need per hectare: f_1, f_2, f_3

Amount of pecticide " " " : P_1, P_2, P_3

$$\max \; S_1x_1 + S_2x_2 + S_3x_3$$

$$x_1 + x_2 + x_3 \leq L$$

$$f_1x_1 + f_2x_2 + f_3x_3 \leq F$$

$$P_1x_1 + P_2x_2 + P_3x_3 \leq P$$

$$x_1, x_2, x_3 \geq 0$$

Equivalent forms

- We can translate between min / max:

$$\max \; C^T \cdot x \iff \min \; -C^T \cdot x$$

- Equalities: $a_i^T \cdot x = b \longleftrightarrow \begin{array}{l} a_i^T \cdot x \le b \\ a_i^T \cdot x \ge b \end{array}$

- Using slack variables to express inequalities as equality:

$$a_i^T \cdot x \le b \longleftrightarrow a_i^T \cdot x + s_i = b_i \qquad s_i \ge 0$$

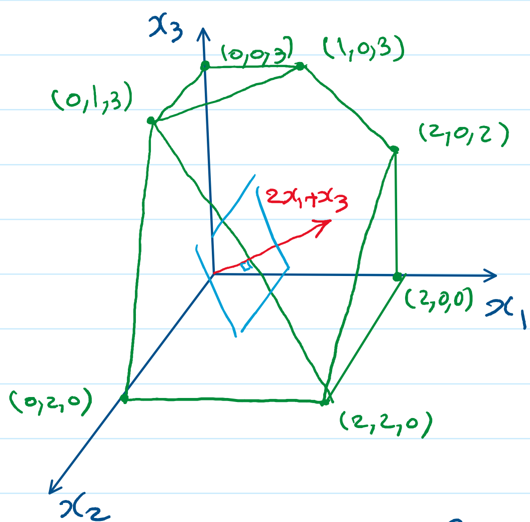
- Canonical & Standard form

$$\text{(canonical)} \quad \begin{array}{l} \min\ c^T \cdot x \\ Ax \ge b \\ x \ge 0 \end{array} \qquad\qquad \begin{array}{l} \min\ c^T \cdot x \\ Ax = b \\ x \ge 0 \end{array} \quad \text{(standard)}$$

Example:

$$\max\ 2x_1 + x_3$$

$$\begin{array}{l} x_1 + x_2 + x_3 \le 4 \\ x_1 \le 2 \\ x_3 \le 3 \\ 3x_2 + x_3 \le 6 \\ x_1, x_2, x_3 \ge 0 \end{array}$$



(2,0,2) is an opt feasible solution

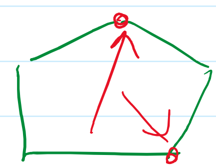
Fact: If LP is feasible there is always an opt "corner" solution (called basic feasible sol)

Assumption: We can assume rank of A is m (i.e. the constraints are linearly independent).

Definition: A basis of $\underline{A}$ is a linearly independent collection of m columns of A, i.e. a non-singular $m \times m$ submatrix $B\,[A_{j_1} \cdots A_{j_m}]$

- A basic solution corresponding to B is $x \in \mathbb{R}^n$ with:
$$\begin{cases} x_j = 0 & \text{if } A_j \notin B \\ x_{j_k} = k\text{'th component of } B^{-1} b & \forall\, 1 \le k \le m \end{cases}$$

- Thus, a basic solution can be found by:
  1) Find a set B of m L.I. columns of A
  2) all x's corresponding to columns <u>NOT</u> in B are 0
  3) Solve the resulting equations to find the other m values.

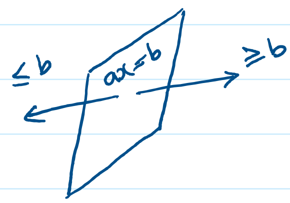


- BFS's are very important. If an LP is feasible it has BFS (they cannot be written as convex combination of other feasible solutions).

Lemma: let x be a bfs to $Ax = b$, $x \ge 0$. Then there is a unique $\vec{c}$ s.t. x is the unique opt feasible sol to
$$\min\ c^T x \quad Ax = b \quad x \ge 0$$

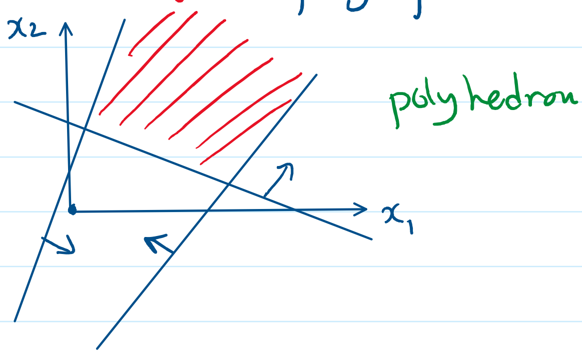
Definition: A hyperplane in $\mathbb{R}^n$ is a set of points $\{x \in \mathbb{R}^n : a_1x_1 + a_2x_2 + \cdots + a_nx_n = b\}$ for some given a_i's & b (with not all a_i's $= 0$).

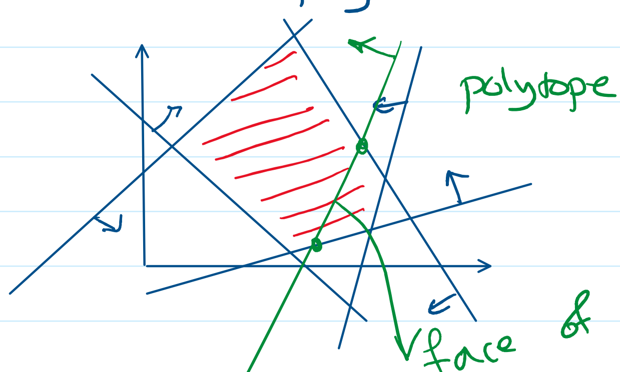
– A hyperplane defines two half spaces $a \cdot x \geq b$, $ax \leq b$



Definition: A polyhedron is the convex body defined by intersection of a finite # of half-spaces

Definition: A polytope is a bounded poly-hedron





Definition: Consider a polytope $P \subseteq \mathbb{R}^n$ and a half-space S defined by a hyperplane H. If $P \cap S$ is non-empty then the intersection of P and H is a face of P. (equivalently, face is the set of points satisfying a valid inequality of P with equality)

Facet: A face of dimension $n-1$ is a facet.

Vertex : A face of dimension 0 is a vertex.

line / edge: A face of dimension 1 is an edge.

Note : A hyperplane defining a facet corresponds to a defining half-space of P but the converse may not be true.

- Fact : Any interior point of P can be written as convex combination of its vertices.

Definition (equivalent): x is a vertex in P if

$\nexists\, y \neq 0$ s.t. $x+y \wedge x-y \in P$

Theorem: let x^* be a vertex of $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$

Then there is $I \subseteq \{1 \dots m\}$ s.t. x^* is the unique solution to: $\forall\, i \in I \quad \sum_j a_{ij} \cdot x_j = b_j$

Proof : Exercise.

- So vertices of polytope correspond to bfs's of an LP. Each bfs is defined by a set of n L.I. equations that hold with equality.

Solving LP's

Simplex: The method used in practice, moves from one vertex to another (pivots).

- Worst case time is exponential but works well in practice.

Ellipsoid: The first polytime LP solver (Khachyan '79)

- Not practical but has big consequences in Combinatorial optimization & designing other algorithms.

LP feasibility vs. optimum solution:

the main problem is to find one feasible solution as optimization can be reduced to feasibility by adding the objective as a constraint & do binary search:

$$\begin{array}{l} \min\ c^T x \\ Ax \geq b \\ x \geq 0 \end{array} \longrightarrow \begin{array}{l} \text{feasibility} \\ Ax \geq b \\ c^T x \leq R \\ x \geq 0 \end{array} \nearrow \text{binary search for } R$$

Important feature of Ellipsoid:

Separation oracle: Given a proposed solution x for LP, either returns "yes" (i.e. x is feasible) or "No" along with a constraint that is violated by x.

- Ellipsoid algorithm finds a feasible solution by making polynomially many calls to a separation oracle.

- Why is this important/useful?

As long as we have a polynomial time separation oracle we can solve the LP in polytime even if the # of constraints is (potentially exponentially) large!

Duality

Consider the following LP:

$$\begin{array}{llr} \min & 10x_1 + 6x_2 + 4x_3 & (1) \\ \text{s.t.} & 2x_1 + x_2 - x_3 \geq 2 & (2) \\ & x_1 + x_2 + x_3 \geq 3 & (3) \\ & x_1, x_2, x_3 \geq 0 & \end{array}$$

- let z^* be the opt value

- Question: is $z^* \leq 50$?

- To answer it is enough to find a feasible solution with value ≤ 50, e.g. $x = (1,1,1)$

Question: Is $Z^* \geq 10$? To answer need to find lower bounds.

- Compare (1) & (3): Since coefficients of x_1, x_2, x_3 are all $\geq$ in (1) v.s. (3) we can say:

  $Z^* \geq 3$. In fact 4 times (3) implies

$$4x_1 + 4x_2 + 4x_3 \geq 12 \longrightarrow Z^* \geq 12.$$

- We can take any convex combinations of (2) & (3) as long as coefficients are $\leq$ those in (1):

min $10x_1 + 6x_2 + 4x_3$ (1)

s.t. $y_1(2x_1 + x_2 - x_3) \geq 2y_1$ (2)

$y_2(x_1 + x_2 + x_3) \geq 3y_2$ (3)

need:

$$2y_1 + y_2 \leq 10$$
$$y_1 + y_2 \leq 6$$
$$-y_1 + y_2 \leq 4$$

maximize $2y_1 + 3y_2$

Example 2:

max $3x_1 + x_2 + 2x_3$

$\times y_1$ $x_1 + x_2 + 3x_3 \leq 30$

$\times y_2$ $2x_1 + 2x_2 + 5x_3 \leq 24$

$\times y_3$ $4x_1 + x_2 + 2x_3 \leq 36$

$\xrightarrow{\text{dual}}$

min $30y_1 + 24y_2 + 36y_3$

$$y_1 + 2y_2 + 4y_3 \geq 3$$
$$y_1 + 2y_2 + y_3 \geq 1$$
$$3y_1 + 5y_2 + 2y_3 \geq 2$$

In general:

$$\min \sum_{i=1}^{n} c_i \cdot x_i \quad \xleftrightarrow{\text{dual}} \quad \max \sum_{i=1}^{m} b_i \cdot y_i$$

$$\forall_{1 \le i \le m} \sum_{j=1}^{n} a_{ij} \cdot x_j \geq b_i \qquad\qquad 1 \le j \le n \sum_{i=1}^{m} a_{ij} \cdot b_i \leq c_j$$

$$x_j \geq 0 \qquad\qquad y_i \geq 0$$

$$\underline{P} \quad \boxed{\begin{array}{l} \min c^T \cdot x \\ A \cdot x \geq b \\ x \geq 0 \end{array}} \qquad\qquad \underline{D} \quad \boxed{\begin{array}{l} \max b^T \cdot y \\ A^T \cdot y \leq c \\ y \geq 0 \end{array}}$$

Weak duality: if $\vec{x}$ & $\vec{y}$ are <u>any</u> feasible solutions to $\underline{P}$ & $\underline{D}$ (respectively) then

$$c^T \cdot x \geq b^T \cdot y$$

Proof:

$$\sum_{j=1}^{n} c_j \cdot x_j \geq \sum_{j=1}^{n} \Big(\overbrace{\sum_{i=1}^{m} a_{ij} \cdot y_i}^{\leq c_j} \Big) x_j$$

$$\geq \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} a_{ij} \cdot x_j \Big) y_i$$

$$\geq \sum_{i=1}^{m} b_i \cdot y_i$$

Corollary: if $\underline{D}$ is unbounded (opt value $\longrightarrow +\infty$) then $\underline{P}$ is not feasible.

if $\underline{P}$ is unbounded (goes to $\longrightarrow -\infty$) then $\underline{D}$

is not feasible.

Strong duality: P has a finite feasible solution <u>iff</u> D has a finite feasible solution.

Also if x^* and y^* are optimum for P & D then $c^T.x^* = b^T.y^*$

Theorem (complementary slackness) let $\vec{x}$ & $\vec{y}$ be two feasible solutions to <u>P</u> and <u>D</u>. Then both $\vec{x}$ and $\vec{y}$ are optimum <u>iff</u> the following hold:

- Primal complementary slackness:

$$\forall\ 1 \le j \le n:\ \text{either}\ x_j = 0\ \text{or}\ \sum_{i=1}^{m} a_{ij} y_i = c_j$$

- Dual Complementary Slackness:

$$\forall\ 1 \le i \le m:\ \text{either}\ y_i = 0\ \text{or}\ \sum_{j=1}^{n} a_{ij} . x_j = b_i$$

D \ P	unbounded	infeasible	Feasible
unbounded	impossible	possible	impossible
infeasible	possible	Possible	impossible
feasible	impossible	impossible	Possible & equal

Examples of writing problems as IP/LP

Min vertex cover: Given a graph $G(V,E)$, find a smallest