Flexible Approximators for Approximating Fixpoint Theory

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AFT and Motivation for Flexible approximators

- AFT framework to study operators on bi-lattices.
  - study various nonmonotonic semantics uniformly (..).
  - prove general results
    - avoid syntactical details + "prove for one" → "obtain several"
    - no need to prove for slightly diff. circumstances.
- Stable Revision op (with two params). → uses default negation.
  - Problem: combine various formalisms → missing entailed negation
  - Solution: Flexible approximators → ternary op.
    - third param. aids to access entailed negation.
  - Question: is the extended AFT correct?

Background

- Poset:Elements of a set are partially ordered.
- Lattice:a poset $\langle \mathcal{L}, \leq \rangle$ ... has a LUB (Join) and a GLB(Meet)
- Chain:a linearly ordered subset of $\mathcal{L}$
- ChainComplete:a poset $\langle \mathcal{L}, \leq \rangle$, has least element $\perp$ and $C \subseteq \mathcal{L}$ has a LUB.
- CompleteLattice:if every $S \subseteq \mathcal{L}$ has LUB and GLB.
- BiLattice:$\langle \mathcal{L}^2, \leq, \leq_p \rangle$, precision order ($\leq_p$) and truth order $\leq$
  - for all $x, y, x', y' \in \mathcal{L}$, $(x, y) \leq_p (x', y')$ if $x \leq x'$ and $y' \leq y$
- Consistent Pair:For $(x, y) \in \mathcal{L}^2$, if $x \leq y$ and exact: if $x = y$.
- $\mathcal{L}^2$:set of all consistent pair
- Interval:$(x, y) \in \mathcal{L}^2$ defines $[x, y] = \{z | x \leq z \leq y\}$,
- Operator:$\mathcal{O}$ is monotone $\rightarrow (x, y \in \mathcal{L},\ x \leq y \rightarrow \mathcal{O}(x) \leq \mathcal{O}(y))$.
  - possesses fixpoints and a least fixpoint.

Stable Revision Operator

- a Semantic in NM formalism $\rightarrow$ an approximator (monotone in bi-lattice)
- kripkee-klenee fixpoint $\mathcal{A}$
  - leave as "unknown", what cannot be proven (3-valued)
- "Unfounded atoms"- key features of ASP
  - unfounded atoms $\rightarrow$ those can be assigned to false.
  - a pair (u, v), u: *true*, v:*possibly true*, $\bar{u}$: *false*
  - computation of unfounded : Stable Revision Op.
    - Given a (u, v), maps (u, v) to $(lfp(\mathcal{A}^1(\cdot, v)), lfp(\mathcal{A}^2(u, \cdot)))$
      $\mathcal{A}^1(\cdot, v)$: computes what must be true with a fixed v.
      $\mathcal{A}^2(u, \cdot)$: computes what is possible true with a fixed u.

Issues with entailed negation

- Given a (u, v), *false* atoms are in both u and v
  - (i) $a \notin v$ (entailed) and
  - (ii) $a \notin u$, but $a \in v$ (default)
- $\mathcal{A}^2(u, \cdot)$: no access v (no access in the entailed negation).
- Lack of access to v is problematic (while computing $\mathcal{A}^2(u, \cdot)$)
  - (i) Underestimation
  - (ii) Overestimation

Overestimation Example

- FOL-program $KB = (L, \Pi)$; $L = \{P(a)\}$;
  $\Pi = \{P(a) \leftarrow P(a);\ P(b) \leftarrow F\}$
- Given $(\bot, \top)$ (i.e. (u, v)) as least atom, compute $(lfp(\mathcal{A}^1(\cdot, \top)), lfp(\mathcal{A}^2(\bot, \cdot)))$
  - $\mathcal{A}^2(\bot, \bot)$: first step to compute $lfp(\mathcal{A}^2(\bot, \cdot))$
  - $P(a)$ is false in $(\bot, \bot)$ - contradiction with L.
  - Overestimation, as $\mathcal{A}^2$ does not have access to v.

Underestimation Example ..

-$KB = (L, \Pi)$

$$L = \{\forall x\, C(x) \supset (A(x) \vee D(x))\}$$

and

$$\Pi = \{A(a) \leftarrow A(a).\ B(a) \leftarrow not\ A(a).$$

$$D(a) \leftarrow not\ B(a).\ C(a) \leftarrow not\ C'(a).\ C'(a) \leftarrow not\ C(a).\}$$

$$(\bot, \top) \Rightarrow (\bot, \{c, c', b, d\}) \Rightarrow (\{b\}, \{c, c', b, d\}) \Rightarrow (\{b\}, \{c, c', b\}) \Rightarrow$$

$$(\{b\}, \{c', b\}) \Rightarrow ...$$

-$C(a), C'(a), B(a), D(a)$ in second pair are possibly true.
- $C(a)$ is not possibly true because $\neg C(a)$ is entailed by L
- (if)infer $\neg C(a)$, block derivation of $C(a)$
- but, without access to v of (u, v), $\neg C(a)$ is not derivable.

Contributions ...

1) Original AFT on $\mathcal{L}^c$, Our AFT on $\mathcal{L}^2$
   - changes notion of approximator + enrich algebric manipulation
2) Enable $\mathcal{A}^2$ to have access to v
   - fix Underestimation and Overestimation.
3) Prove correctness.

Extend to $\mathcal{L}^2$

- Approximator $\mathcal{A}$ is $\leq_p$-monotone .... $\mathcal{A}(z,z) = (\mathcal{O}(z), \mathcal{O}(z))$
- (What if) $\mathcal{A}(z,z)$ inconsistent?
- So, we define $\mathcal{A}(z,z)$ on $\mathcal{L}^2$.

Definition

Let $\mathcal{O}$ be an op. on $\mathcal{L}$; $\mathcal{A} : \mathcal{L}^2 \to \mathcal{L}^2$ is an *approximator* of $\mathcal{O}$ iff.

- For all $x \in \mathcal{L}$, if $\mathcal{A}(x,x)$ is consistent then $\mathcal{A}(x,x) = (\mathcal{O}(x), \mathcal{O}(x))$.
- $\mathcal{A}$ is $\leq_p$-monotone.

-Stable Revision Op. $(St_{\mathcal{A}})$: persistently reachable / non-reachable

- has fixpoints (stable models) and a least fix point(well founded fixpoints)
- $St_{\mathcal{A}}(u,v) = (lfp(\mathcal{A}^1(\cdot, v)), lfp(\mathcal{A}^2(u, \cdot))$.

Issues with $\mathcal{A}^2$ ...

- $lfp(\mathcal{A}^1(\cdot, v))$ is well defined as $\mathcal{A}^1$ is monotone on $\mathcal{L}$
- But, two issues with $lfp(\mathcal{A}^2(u, \cdot))$
  - $\mathcal{A}^2(u, \cdot)$ no access to v.
  - $\mathcal{A}^2(u, \cdot) \notin [u, \top]$ ($lfp(\mathcal{A}^2(u, \cdot))$ is ill defined)

Adding third param. to $St_{\mathcal{A}}$ ...

- $St_{\mathcal{A}}(u, v, v) = (lfp(\mathcal{A}_v^1(\cdot, v)), lfp(\mathcal{A}_v^2(u, \cdot)).$
-lfp construction, Given a pair (u, v):

$$x_0 = \bot, x_1 = \mathcal{A}_v^1(x_0, v), ..., x_{\alpha+1} = \mathcal{A}_v^1(x_\alpha, v), ... \quad (1)$$

$$y_0 = u, y_1 = \mathcal{A}_v^2(u, y_0), ..., y_{\alpha+1} = \mathcal{A}_v^2(u, y_\alpha), ... \quad (2)$$

- So, given an operator $\mathcal{O}$ on $\mathcal{L}$,
  - $\mathcal{A}_v : \mathcal{L}^3 \to \mathcal{L}^2$ is the approximator.

-$\mathcal{A}_v$ is $\leq_p$ monotone.

Lemma

Let $\mathcal{A} : \mathcal{L}^3 \to \mathcal{L}^2$ be an approximator, and $v, v' \in \mathcal{L}$ s.t. $v \leq v'$. For all $(x, y), (x', y') \in \mathcal{L}^2$, if $(x, y) \leq_p (x', y')$, then $\mathcal{A}_{v'}(x, y) \leq_p \mathcal{A}_v(x', y')$.

Tackle Inconsistency ...

- Reliability : $\mathcal{A}$, $(u, v) \in \mathcal{L}^2$ is called $\mathcal{A}$-*reliable* if $(u, v) \leq_p \mathcal{A}_v(u, v)$.
- But, in $\mathcal{L}^2$, may be $\mathcal{A}_v^2(u, \cdot) \notin [u, \top]$
- Sufficient Condition: $\mathcal{A}_v^2(u, u) \geq u$ holds.

Lemma

Let $\mathcal{A} : \mathcal{L}^3 \to \mathcal{L}^2$ be an approximating operator. For any $(u, v) \in \mathcal{L}^2$, if $A_v^2(u, u) \geq u$, then for every $z \in [u, \top]$, $\mathcal{A}_v^2(u, z) \in [u, \top]$.

- Stable Revision operator for the extended AFT:

$$St_{\mathcal{A}}(u, v) = \begin{cases} (lfp(\mathcal{A}_v^1(\cdot, v)), lfp(\mathcal{A}_v^2(u, \cdot))) \text{ where } \mathcal{A}_v^2(u, \cdot) \text{ is on } [u, \top] \\ (lfp(\mathcal{A}_v^1(\cdot, v)), lfp(\mathcal{A}_v^2(u, \cdot))) \text{ where } \mathcal{A}_v^2(u, \cdot) \text{ is on } [\bot, \top] \end{cases}$$

- Solves Inconsistency!
- $\mathcal{A}$-prudent: $(u, v) \in \mathcal{L}^2$ is called $\mathcal{A}$-*prudent* if $u \leq lfp(\mathcal{A}_v^1(\cdot, v))$ (improves u)
- $\mathcal{L}^{rp}$ the set of all $\mathcal{A}$-reliable and $\mathcal{A}$-prudent pairs in $\mathcal{L}^2$

Properties

- chain property:

Lemma

$(u, v) \in \mathcal{L}^{rp}$, let $St(u,v) = (u', v')$. Then, $(u, v) \leq_p (u', v')$, and (u', v') is $\mathcal{A}$-reliable and $\mathcal{A}$-prudent.

- $\leq_p$-monotonicity :

Lemma

$(u, v), (u', v') \in \mathcal{L}^{rp}$, $(u, v) \leq_p (u', v')$, then $St(u, v) \leq_p St(u', v')$.

Theorem

structure $\langle \mathcal{L}^{rp}, \leq_p \rangle$ is a chain-complete poset, has least element $(\bot, \top)$, and St well-defined, increasing, and $\leq_p$-monotone op.

Definition

well-founded fixpoint of $\mathcal{A}$: least fixpoint
stable fixpoints : the exact fixpoints of $St_{\mathcal{A}}$

Application to FOL programs

- FOL-program $KB = (L, \Pi)$
  - L is a FO theory and Π a *rule base* $H \leftarrow A_1, \ldots, A_m, not\ B_1, \ldots, not\ B_n$, (Π)
  - HB_Π: Herbrand Base, $P(t_1, \ldots, t_n)$: (P is predicate, $t_1, \ldots, t_n$: constants)
  - $I \subseteq HB_\Pi$ (Interpretation) $\|$ $\bar{I} = HB_\Pi - I$ $\|$ $\neg.I = \{\neg A | A \in I\}$

-consistency:(I, J) is consistent with L if $L \cup I \cup \neg.\bar{J}$ is consistent.

-consistent extension:(I', J') is a *consistent extension of* (I, J) if $I \subseteq I' \subseteq J' \subseteq J$

Definition

$(I, J) \in (2^{HB_\Pi})^2$, and ϕ a literal.

- $(I, J) \models_L \phi$ iff, if ϕ is an atom A then $A \in I$, if ϕ is a negative literal *not* A then $A \notin I$, and if ϕ is an FOL-formula then $L \cup I \cup \neg.\bar{J} \models \phi$. (Entailment)
- $(I, J) \Vdash_L \phi$ iff for all (I', J') of (I, J), $(I', J') \models_L \phi$. (Consistency)

Approximation in FOL

- Operator to be approximated:

$$\mathcal{K}_{KB}(I) = \{hd(r) \mid r \in \Pi, (I, I) \models_L body(r)\} \cup \{A \in HB_\Pi \mid (I, I) \models_L A\}$$

- Approximator of $\mathcal{K}_{KB}(I)$:

Definition

(Operator $\Phi_{KB,v}$: Standard Semantics) For all $H \in HB_\Pi$,

- $H \in \Phi^1_{KB,v}(I, J)$ iff one of the following holds (true atoms)
  (a) $(I, v) \models_L H$.
  (b) $\exists r \in \Pi$ with $hd(r) = H$, s.t. $\forall \phi \in body(r)$, $(I, J) \Vdash_L \phi$.
- $H \in \Phi^2_{KB,v}(I, J)$ iff $(I, v) \not\models_L \neg H$ and one of the following holds (possibly true atoms)
  (a) $\exists I', J'(I \subseteq I' \subseteq J' \subseteq J)$, $(I', J' \cup \bar{J}) \models_L H$.
  (b) $\exists r \in \Pi$ with $hd(r) = H$, s.t. $\forall \phi \in body(r)$,
  $\exists I', J'(I \subseteq I' \subseteq J' \subseteq J)$, $(I', J' \cup \bar{J}) \models_L \phi$.

Fix of Underestimation

-Example:

Example

$KB = (L, \Pi)$,

$$L = \{\forall x\, C(x) \supset (A(x) \vee D(x))\}$$

and

$$\Pi = \{A(a) \leftarrow A(a).\ B(a) \leftarrow not\ A(a).$$

$$D(a) \leftarrow not\ B(a).\ C(a) \leftarrow not\ C'(a).\ C'(a) \leftarrow not\ C(a).\}$$

We use $St_{\Phi_{KB}}$ in this case. Generates:

$$(\emptyset, HB_\Pi) \Rightarrow (\emptyset, \{c, c', b, d\}) \Rightarrow (\{b\}, \{c, c', b, d\})$$
$$\Rightarrow (\{b\}, \{c, c', b\}) \Rightarrow (\{b\}, \{c', b\}) \Rightarrow (\{c', b\}, \{c', b\})$$

- $A(a)$ is false $\rightarrow$ $B(a)$ (Second pair) $\rightarrow$ $D(a)$ is false (third pair)
- $\neg C(a)$ entailed by L, $C(a)$ is no longer possible true.

Future work and Question

- Multi context system (use of entailed negation?).
- Grounded Fix point (does not handle inconsistency. Use this result?)
- Question??