# Linear Subspace Transforms

PCA, Karhunen-Loeve, Hotelling

c306, 2001

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## Image dependent basis

- Other transforms, (Fourier and Cosine) uses a fixed basis.
- Better compression can be achieved by tailoring the basis to the image (pixel) statistics.

## Sample data

- Let **x** $=[x_1, x_2, \ldots, x_w]$ be a sequence of random variables (vectors)

Example 1: x = Column flattening of an image, so **x** is a sample over several images.

Example 2: x = Column flattening of an image block. **x** could be sampled from only one image.

Example 3: x = optic flow field.

## Natural images = stochastic variables

- The samplings **x** $=[x_1, x_2, \ldots, x_w]$ of natural image intensities or optic flows are far from uniform!
- Draw $x_1, x_2, \ldots, x_w$: Localized clouds of points.
- Particularly localized for e.g.
  - Groups of faces or similar objects
  - Lighting
  - Small motions represented as intensity changes or optic flow. (Note coupling through optic flow equation)

## Dimensionality reduction

- Find an ON basis B which "compacts" x
- Can write:

  $x = y_1B_1 + y_2B_2 + \ldots + y_wB_w$
- But can drop some and only use a subset, i.e.

  $x = y_1B_1 + y_2B_2$

Result: Fewer y values encode almost all the data in x

## Estimated Covariance Matrix

$$C_{k,l} = \frac{1}{W}\left(\sum_{j}^{W-1} \vec{x}^{k\,T}_{j} \vec{x}^{l}_{j}\right) - \vec{m}^T\vec{m} \quad (eqn1)$$

$$\vec{m} = \frac{1}{W}\left(\sum_{i}^{W-1} \vec{x}_i\right) \quad (eqn2)$$

Note 1: Can more compactly write:

C = X'*X and m = sum(x)/w

Note 2: Covariance matrix is both symmetric and real, therefore the matrix can be diagonalized and its eigenvalues will be real

## Coordinate Transform

- Diagonalize C, ie find B s.t.

  $B'CB = D = \mathrm{diag}(\ l_j\ )$

Note:

1. C symmetric => B orthogonal
2. B columns = C eigenvectors

## Definition KL, PCA, Hotelling transform

- The forward coordinate transform

  $y = B'*(x-m_x)$
- Inverse transform

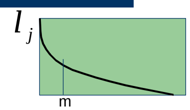
  $x = B*y + m_x$

Note: $m_y = E[y] = 0$ zero mean

$C = E[yy'] = D$ diagonal cov matrix

## Study eigenvalues

- Typically quickly decreasing:




- Write:

  $x = y_1 B_1 + y_2 B_2 + \ldots + y_m B_m + \ldots + y_w B_w$

- But can drop some and only use a subset, i.e.

  $x_m = y_1 B_1 + y_2 B_2 + \ldots + y_m B_m$

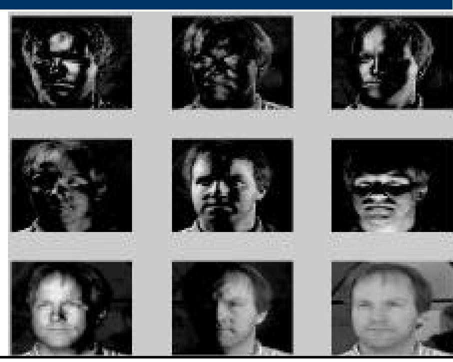
Result: Fewer y values encode almost all the data in x

## Remaining error

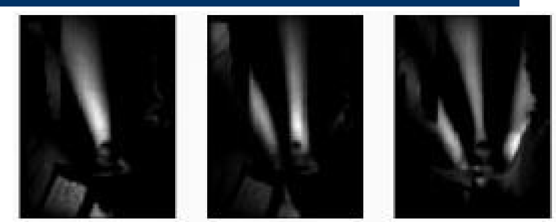
- Mean square error of the approximation is

$$MSE = \sum_{j=K}^{N-1} l_j$$

## Example eigen-images 1: Lighting variation



## Example eigen-images 2 Motions



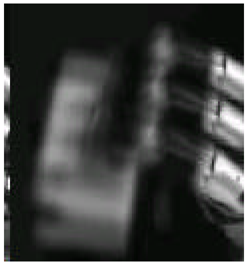
Motions with my arm sampled densely

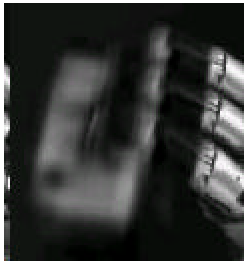
## Human Arm Animation

Training sequence

Simulated Animation

## Modulating Basis Vectors





Vectors: 5 and 6 1-6

## Practically:

C=XX' quite large, eig(C) impractical.

Instead:

1. C=X'X, [d,V]= eig(C), eigenvectors U=XV
2. [U,s,V] = svd(X)
   (since C = XX' = Us^2U')

## Summary Hotelling,PCA,KLT

- Advantages:
1. Optimal coding in the least square sense
2. Theoretcially well founded
3. Image dependent basis: We can interpret variation in our particular set of images.

- Disadvantages:
1. Computationally intensive
2. Image dependent basis (we have to compute it)