The Discrete Fourier Transform

C306
Fall 2001
Martin Jagersand

Last time: Continuous Fourier transform

Forward transform

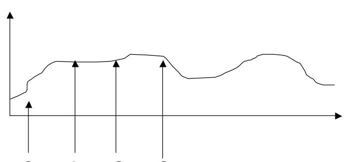
$$F(\mathbf{u}) = \int_{-\infty}^{\infty} f(\mathbf{x}) e^{-j2\pi \mathbf{u}\mathbf{x}} \, \mathbf{dx}$$

Inverse

$$f(\mathbf{x}) = \int_{-\infty}^{\infty} F(\mathbf{u}) e^{j2\pi \mathbf{u}\mathbf{x}} \, \mathbf{du}$$

Discrete functions

- Continuous function: f(x)



- Discretized at t = 0, 1, 2, 3,…
- (f_0, f_1, f_2, f_3, …)

Discrete Fourier Transform (DFT)

Forward transform:

$$F(\mathbf{u}) = \frac{1}{\sqrt{\mathbf{N}}} \sum_{x=0}^{\mathbf{N}-1} f(x) e^{-j\frac{2\pi ux}{\mathbf{N}}}$$

Reverse transform:

$$f(\mathbf{x}) = \frac{1}{\sqrt{\mathbf{N}}} \sum_{u=0}^{\mathbf{N}-1} F(u) e^{j\frac{2\pi ux}{\mathbf{N}}}$$

DFT in matrix form

$$\begin{bmatrix}F(0)\\F(2)\\\vdots\\F(N-1)\end{bmatrix} = \frac{1}{\sqrt{N}}\begin{bmatrix}1 & 1 & \cdots & 1\\ 1 & e^{-\frac{j2\pi}{N}} & & e^{-\frac{j2\pi(N-1)}{N}}\\ \vdots & & \ddots & \vdots\\ 1 & e^{-\frac{j2\pi(N-1)}{N}} & & e^{-\frac{j2\pi(N-1)^2}{N}}\end{bmatrix}\begin{bmatrix}f(0)\\f(2)\\\vdots\\f(N-1)\end{bmatrix}$$

$$F(u) = \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1} f(x)e^{-\frac{j2\pi ux}{N}}$$

- Note: u=Ax form! Inverse: A^{T*}

Compactly written

Let: $\mathbf{b} = \begin{bmatrix}0\\1\\\vdots\\N-1\end{bmatrix}$

A can be written: $$A = \frac{1}{\sqrt{N}}\exp\left(-j\frac{2\pi \mathbf{b}^T\mathbf{b}}{N}\right)$$

Note: pointwise

Gonzalez 3.2-2: F=Af

3.2-3: f=A^{T*}F

For images: 2-Dim DFT

- Gonzales 3.2-9:

$$F(u,v) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)e^{-j\frac{2\pi}{N}(ux+vy)}$$

- Written with the A-matrix:

Forward $F = AfA^T$

Inverse $f = A^{T*}fA^{*}$

2-Dim DFT separates

- Notice: We can separate sums over x and y:

$$F(u,v) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)e^{-j\frac{2\pi}{N}(ux+vy)}$$

$$= \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1} e^{-j\frac{2\pi ux}{N}}\left(\frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} e^{-j\frac{2\pi vy}{N}} f(x)\right)$$

- With respect to: x y

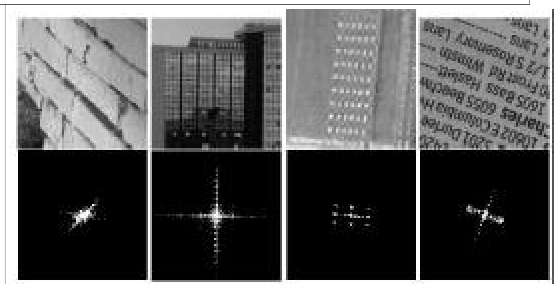
Same as $F = AfA^T$ if f 2D image

Displaying the Fourier Transform

- Practically the Fourier transform coefficients have a large range
- Usually the magnitude $|F(u,v)|$ is what we want to see
- Compress range variation to see as an image:

$$D(i,j) = c*\log(1+|F(u,v)|)$$

Manmade object images:

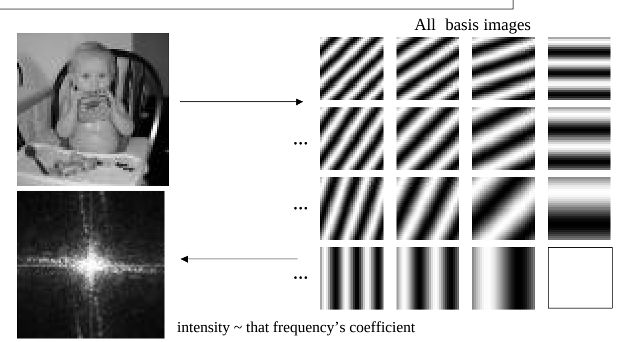


- What does the F-t express?

Periodicity

- The fourier transform highlights/identifies periodic structure in images.
- This is seen as peaks (usually several) along the same axis as the repetition in the original image

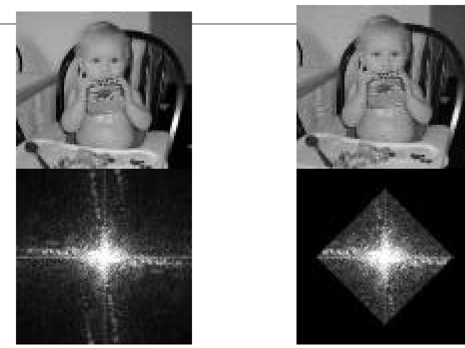
What does the F-t express?



Frequency spectrum

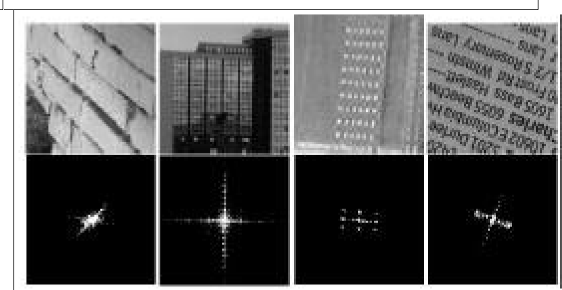
- By convention the lowest frequency components are shifted to the center of the image. (use fftshift in matlab)
- Fourier transform coefficients tells us the relative composition of low and high spatial frequencies, I.e. the frequency spectrum
- Usually more lows than highs

Cropping in Fourier space



Data Reduction: only use *some* of the existing frequencies

Properties: Rotation



- Notice direction of F-t:

Properties: Rotation

- Rotating f(x,y) by an angle θ_0 rotates F(u,v) by the same angle
- expressing the variables in polar coordinates

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ u = w\cos\phi,\ \ v = w\sin\phi$$

- substitution leads to

$$f(r, \theta+\theta_0) \leftrightarrow F(w, \phi+\theta_0)$$

Properties: Translation

$$f(x,y)e^{j2\pi(u_0x+v_0y)/N} \leftrightarrow F(u-u_0, v-v_0)$$

$$f(x-x_0, y-y_0) \leftrightarrow F(u,v)e^{-j2\pi(ux_0+vy_0)/N}$$

- the magnitude is unaffected because

$$\left|F(u,v)e^{-j2\pi(ux_0+vy_0)/N}\right| = \left|F(u,v)\right|$$

Properties: Distributivity

- the Fourier transform and its inverse are distributive over addition but not over multiplication:

$$\mathrm{F}\{f_1(x,y)+f_2(x,y)\} = \mathrm{F}\{f_1(x,y)\} + \mathrm{F}\{f_2(x,y)\}$$

- and

$$\mathrm{F}\{f_1(x,y) * f_2(x,y)\} \neq \mathrm{F}\{f_1(x,y)\} * \mathrm{F}\{f_2(x,y)\}$$

Properties: Periodicity

- discrete Fourier transform and inverse are periodic with period N

$$F(u,v) = F(u+N,v) = F(u,v+N) = F(u+N,v+N)$$