Fourier Transforms

C306
Fall 2001
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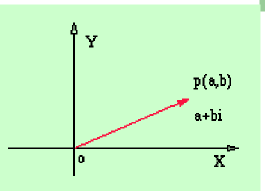
Today's lecture

- Introduction to complex numbers
- Continuous Fourier Transform

Complex Numbers- Definition

Imaginary unit: $i^2 = -1$

Rectangular form:



$$x = \mathrm{Re}(z) - \textit{real part}$$
$$y = \mathrm{Im}(z) - \textit{imaginary part}$$

Complex Numbers - Properties

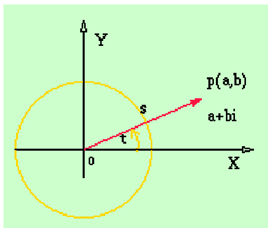
Conjugate: $\bar{z} = x - iy$

Modulus: $|z| = \sqrt{x^2 + y^2}$

Addition
$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$$
$$= (x_1 + x_2) + i(y_1 + y_2)$$

Multiplication
$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$= (x_1 x_2 + y_1 y_2) + i(x_1 y_2 + x_2 y_1$$

Polar Form



$$\begin{cases} x = r\cos q \\ y = r\sin q \end{cases} \qquad \begin{cases} r = \sqrt{x^2 + y^2} \\ \tan q = \dfrac{y}{x} \end{cases}$$

r – modulus

θ - argument

$$z = x + iy = r(\cos q + i\sin q) = re^{iq}$$

Polar form - Properties

Conjugate: $|z| = re^{-iq}$

Multiplication: $z_1 z_2 = r_1 e^{iq_1} r_2 e^{iq_2} = r_1 r_2 e^{i(q_1+q_2)}$

Power: $z^n = \left(re^{iq}\right)^n = r^n e^{inq}$

Polar form - Formulas

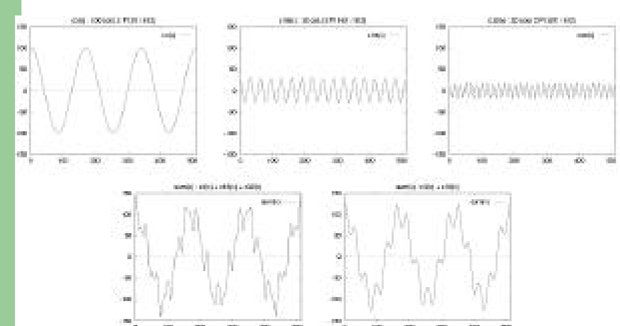
- De Moivre's Formula

$$(\cos q + i\sin q)^n = \cos nq + i\sin nq$$

- Euler Formula

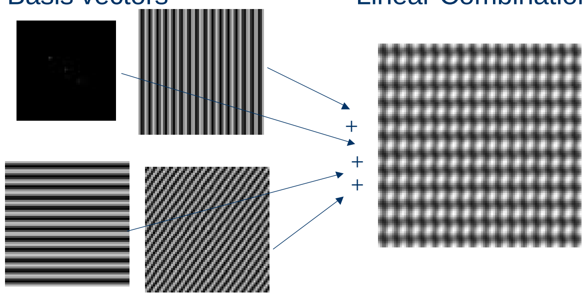
$$\cos q = \frac{e^{iq} + e^{-iq}}{2}, \quad \sin q = \frac{e^{iq} - e^{-iq}}{2}$$

Combination of sines



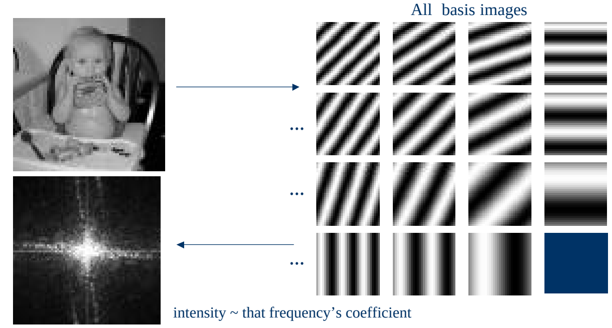
Constructing an image

Basis vectors | Linear Combination



Analyzing an image

All basis images



intensity ~ that frequency's coefficient

Till now …

Im = a_1 I_1 + a_2 I_2 + … + a_n I_n

Real Basis

Im can be recovered from **a** if **I** invertible

Fourier transform = real and imaginary part

Fourier Transform - definition

Direct:

$$\Im\{f(x)\} = F(u) = \int_{-\infty}^{\infty} f(x) e^{-i2pux} dx =$$

$$= \int_{-\infty}^{\infty} f(x)(\cos 2pux - i \sin 2pux) dx =$$

$$= \int_{-\infty}^{\infty} f(x) \cos 2pux \, dx - i \int_{-\infty}^{\infty} f(x) \sin 2pux \, dx$$

even | odd

Oddness and Evenness

Any function … $f(x) = O(x) + E(x)$

Where $O(x)$ is an odd function and $E(x)$ is an even function

$$E(x) = \frac{1}{2}\left(f(x) + f(-x)\right)$$

$$O(x) = \frac{1}{2}\left(f(x) - f(-x)\right)$$

The Fourier transform …

$$\Im\{f(x)\} = 2\int^{\infty} E(x)\cos(2pux)dx - 2i\int^{\infty} O(x)\sin(2pux)dx$$

FT of a **real even** function is **real** (and even)

real odd function is **imaginary** (and odd)

FT- spectrum and phase

F(u) = R(u) + i I(u)

- Spectrum $|F(u)| = \sqrt{R^2(u) + I^2(u)}$
- Phase $\Phi(u) = \mathrm{atan}\frac{I(u)}{R(u)}$

$$F(u) = R(u)e^{i\Phi(u)}$$

Inverse FT

$$\Im^{-1}\{F(u)\} = f(x) = \int_{-\infty}^{\infty} F(u)e^{i2pux}du$$

Convolution

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du$$

Convolution theorem

$$\Im\{f_1(x) * f_2(x)\} = \Im\{f_1(x)\}\Im\{f_2(x)\} = F_1(u)F_2(u)$$

Product theorem

$$\Im\{f_1(x) f_2(x)\} = \Im\{f_1(x)\} * \Im\{f_2(x)\} = F_1(u) * F_2(u)$$

Obs: Discrete case – FFT + convolution theorem = fast convolution

Properties

	f(x)	F(u)
Linearity	$af_1(x)+bf_2(x)$	$aF_1(u)+bF_2(u)$
Time shifting	$f(x-x_0)$	$F(u)e^{-2piux_0}$
Frequency shifting	$f(x)e^{-2piux}$	$F(u-u_0)$
Scaling	$f(ax)$	$\lvert a \rvert^{-1} F(u/a)$