Intro to image transforms

C306
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Image point operations

- So far we have considered operations on one image point, ie contrast stretching, histogram eq. Etc

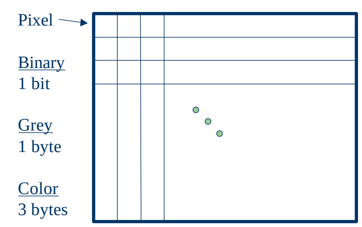
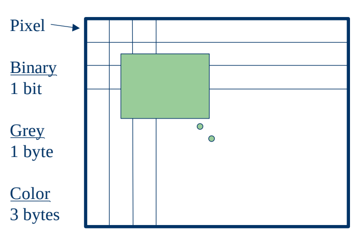


Image transforms

- By contrast, image transforms are defined over regions of an image or the whole image



Examples: Image transforms

- Fourier transform
  - For analysis of frequency spectrum in a image
- Cosine, sine transform
  - Compression, e.g. jpeg, mpeg
- Hotelling, KL, PCA
  - Statistically optimal basis. Used in compression, recognition and image variability representation

Image transforms

- Image transforms are some of the most important methods in image processing
- Will be used later in:
  1. Filtering
  2. Restoration
  3. Enhancement
  4. Compression
  5. Image analysis

Mathematically

- Learning the math can be daunting…

$$\begin{bmatrix} F(0) \\ F(2) \\ \vdots \\ F(N-1) \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-\frac{j2\pi}{N}} & & e^{-\frac{j2\pi(N-1)}{N}} \\ \vdots & & \ddots & \vdots \\ 1 & e^{-\frac{j2\pi(N-1)}{N}} & & e^{-\frac{j2\pi(N-1)^2}{N}} \end{bmatrix} \begin{bmatrix} f(0) \\ f(2) \\ \vdots \\ f(N-1) \end{bmatrix}$$

$$F(\mathbf{u}) = \frac{1}{\sqrt{\mathbf{N}}} \sum_{x=0}^{\mathbf{N}-1} f(x) e^{-\frac{j2\pi ux}{\mathbf{N}}}$$

- Note: u=Ax form! A^{T*}

Vector spaces

- We will start with the concepts of
  1. vector spaces and
  2. coordinate changes,

just as used in the camera models:

Examples:

– Euclidean world space $\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix} \in \mathbb{R}^3$

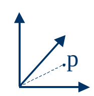
– Image space, (q by q image): $\begin{bmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} & \cdots & \mathbf{a}_{1q} \\ \mathbf{a}_{21} & & & \\ \vdots & & \ddots & \\ \mathbf{a}_{q1} & & & \mathbf{a}_{qq} \end{bmatrix} \Leftrightarrow \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_{q^2} \end{bmatrix} \in \mathbb{R}^{q^2}$

Basis (of a vector space)

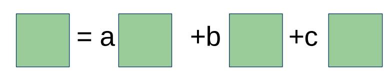
- A member can be described as a linear comb

$$\mathbf{x} = \mathbf{a}\mathbf{e}_1 + \mathbf{a}\mathbf{e}_2 + \ldots + \mathbf{a}\mathbf{e}_m$$

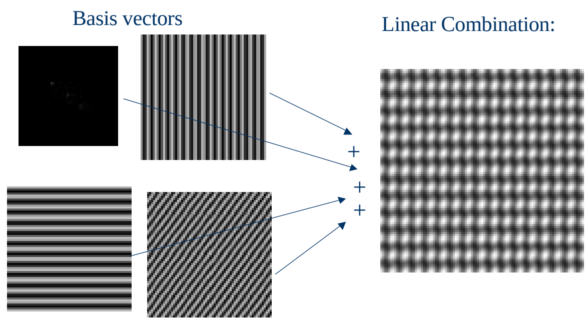
- Example: 3D vectors



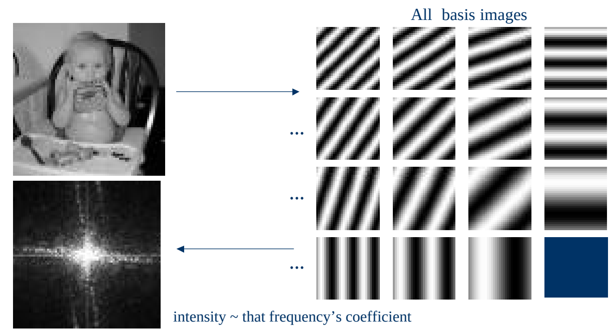
- Example: Image



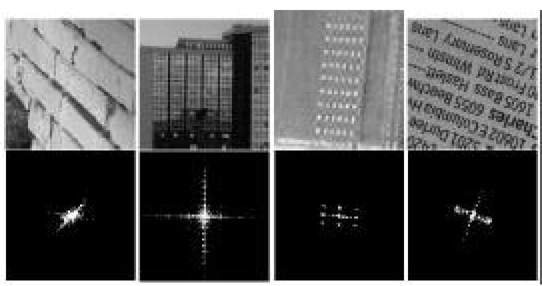
Constructing an image



Analyzing an image

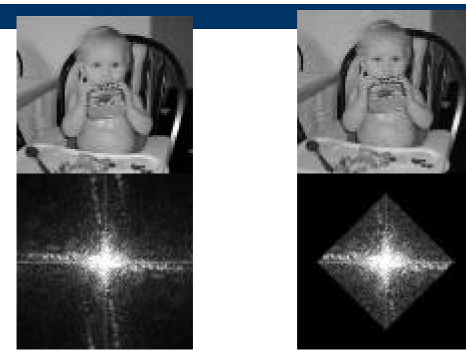


Manmade object images:



- What does the F-t express?

Cropping in Fourier space



Data Reduction: only use *some* of the existing frequencies

Vector spaces
Important properties from lin alg

1. Product
   1. With scalar
   2. “dot” product
   3. Vector product (for 3-vectors)
2. vector sum, difference
3. Norm
4. Orthogonality
5. Basis

See Shapiro-Stockman p. 178-181

Basic definitions for a vector space with a defined vector length.

Let U and V be any two vectors, u_i and v_i be real numbers denoting the coordinates of these vectors; and let a, b, c, etc. be any real numbers denoting scalars.

48 Definition *For vectors* $U = [u_1, u_2, \ldots, u_n]$ and $V = [v_1, v_2, \ldots v_n]$ *their* **vector sum** *is the vector* $U \oplus V = [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n]$.

49 Definition *For vector* $V = [v_1, v_2, \ldots v_n]$ *and real number (scalar) a the* **product of the vector and scalar** *is the vector* $aV = [av_1, av_2, \ldots, av_n]$.

50 Definition *For vectors* $U = [u_1, u_2, \ldots, u_n]$ *and* $V = [v_1, v_2, \ldots, v_n]$ *their* **dot product, or scalar product** *is the real number* $U \circ V = u_1v_1 + u_2v_2 + \ldots + u_nv_n$.

51 Definition *For vector* $V = [v_1, v_2, \ldots v_n]$ *its* **length, or norm,** *is the nonnegative real number* $\| V \| = V \circ V = (v_1v_1 + v_2v_2 + \ldots + v_nv_n)^{1/2}$.

52 Definition *Vectors U and V are* **orthogonal** *if and only if* $U \circ V = 0$.

53 Definition *The* **distance between vectors** $U = [u_1, u_2, \ldots, u_n]$ *and* $V = [v_1, v_2, \ldots v_n]$ *is the length of their difference* $d(U, V) = \| U - V \|$.

54 Definition *A* **basis** *for a vector space of dimension n is a set of n vectors* $\{ w_1, w_2, \ldots, w_n \}$ *that are independent and that span the vector space. The spanning property means that any vector V can be expressed as a linear combination of basis vectors:* $V = a_1w_1 \oplus a_2w_2 \oplus \ldots \oplus a_nw_n$. *The independence property means that none of the basis vectors w_i can be represented as a linear combination of the others.*

Properties of vector spaces that follow from the above definitions.

1. $U \oplus V = V \oplus U$
2. $U \oplus (V \oplus W) = (U \oplus V) \oplus W$
3. There is a vector O such that for all vectors V, $O \oplus V = V$
4. For every vector V, there is a vector $(-1)V$ such that $V \oplus (-1)V = O$
5. For any scalars a, b and any vector V, $a(bV) = (ab)V$
6. For any scalars a, b and any vector V, $(a + b)V = aV \oplus bV$
7. For any scalar a and any vectors U, V, $a(U \oplus V) = aU \oplus aV$
8. For any vector V, $1V = V$
9. For any vector V, $(-1V) \circ V = -\| V \|^2$

Important properites

Cachy-Schwarz inequality:

$$\text{For any two nonzero vectors } U \text{ and } V, \quad -1 \le \frac{U \circ V}{\| U \| \ \| V \|} \le +1 \qquad (5.24)$$

55 Definition *Let U and V be any two nonzero vectors, then the* **normalized dot product of** U *and* V *is defined as* $(\frac{U \circ V}{\|U\| \ \|V\|})$

56 Definition *Let U and V be any two nonzero vectors, then the* **angle between** U *and* V *is defined as* $\cos^{-1}(\frac{U \circ V}{\|U\| \ \|V\|})$

Finding local image properties:

Roberts basis: $W_1 = \frac{1}{2}\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 1 \\ \hline\end{array}$ $W_2 = \frac{1}{\sqrt{2}}\begin{array}{|c|c|}\hline 0 & 1 \\ \hline -1 & 0 \\ \hline\end{array}$ $W_3 = \frac{1}{\sqrt{2}}\begin{array}{|c|c|}\hline +1 & 0 \\ \hline 0 & -1 \\ \hline\end{array}$ $W_4 = \frac{1}{2}\begin{array}{|c|c|}\hline -1 & 1 \\ \hline 1 & -1 \\ \hline\end{array}$

Constant region: $\begin{array}{|c|c|}\hline 5 & 5 \\ \hline 5 & 5 \\ \hline\end{array} = \frac{20}{2}(\frac{1}{2}\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 1 \\ \hline\end{array}) = 10W_1 \oplus 0W_2 \oplus 0W_3 \oplus 0W_4$

Step Edge: $\begin{array}{|c|c|}\hline -1 & +1 \\ \hline -1 & +1 \\ \hline\end{array} = 0W_1 \oplus \frac{2}{\sqrt{2}}W_2 \oplus \frac{2}{\sqrt{2}}W_3 \oplus 0W_4$

Step Edge: $\begin{array}{|c|c|}\hline +1 & +1 \\ \hline -3 & +1 \\ \hline\end{array} = 0W_1 \oplus \frac{4}{\sqrt{2}}W_2 \oplus 0W_3 \oplus \frac{-4}{2}W_4$

Line: $\begin{array}{|c|c|}\hline 0 & 8 \\ \hline 8 & 0 \\ \hline\end{array} = 8W_1 \oplus 0W_2 \oplus 0W_3 \oplus 8W_4$

Standard basis

- e1 e2 e3
- e8 e9

$\begin{array}{|c|c|c|}\hline 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline\end{array}$ $\begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline\end{array}$ $\begin{array}{|c|c|c|}\hline 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline\end{array}$ ... $\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline\end{array}$ $\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ \hline\end{array}$

Nine "standard" basis vectors for the space of all 3x3 matrices.

$$\begin{array}{|c|c|c|}\hline 9 & 5 & 0 \\ \hline 5 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline\end{array} = 9\begin{array}{|c|c|c|}\hline 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline\end{array} +5\begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline\end{array} +5\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline\end{array}$$

= 9*e1 + 5*e2 + 5*e4

Frei-Chen basis

gradient: $\mathbf{W}_1 = 1/\sqrt{8}\begin{array}{|c|c|c|}\hline 1 & \sqrt{2} & 1 \\ \hline 0 & 0 & 0 \\ \hline -1 & -\sqrt{2} & -1 \\ \hline\end{array}$ $\mathbf{W}_2 = 1/\sqrt{8}\begin{array}{|c|c|c|}\hline 1 & 0 & -1 \\ \hline \sqrt{2} & 0 & -\sqrt{2} \\ \hline 1 & 0 & -1 \\ \hline\end{array}$

ripple: $\mathbf{W}_3 = 1/\sqrt{8}\begin{array}{|c|c|c|}\hline 0 & -1 & \sqrt{2} \\ \hline 1 & 0 & -1 \\ \hline -\sqrt{2} & 1 & 0 \\ \hline\end{array}$ $\mathbf{W}_4 = 1/\sqrt{8}\begin{array}{|c|c|c|}\hline \sqrt{2} & -1 & 0 \\ \hline -1 & 0 & 1 \\ \hline 0 & 1 & -\sqrt{2} \\ \hline\end{array}$

line: $\mathbf{W}_5 = 1/2\begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline -1 & 0 & -1 \\ \hline 0 & 1 & 0 \\ \hline\end{array}$ $\mathbf{W}_6 = 1/2\begin{array}{|c|c|c|}\hline -1 & 0 & 1 \\ \hline 0 & 0 & 0 \\ \hline 1 & 0 & -1 \\ \hline\end{array}$

Laplacian: $\mathbf{W}_7 = 1/6\begin{array}{|c|c|c|}\hline 1 & -2 & 1 \\ \hline -2 & 4 & -2 \\ \hline 1 & -2 & 1 \\ \hline\end{array}$ $\mathbf{W}_8 = 1/6\begin{array}{|c|c|c|}\hline -2 & 1 & -2 \\ \hline 1 & 4 & 1 \\ \hline -2 & 1 & -2 \\ \hline\end{array}$

constant: $\mathbf{W}_9 = 1/3\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline\end{array}$

Example of representing an intensity neighborhood using the Frei-Chen basis.

Consider the intensity neighborhood $\mathbf{N} = \begin{array}{|c|c|c|}\hline 10 & 10 & 10 \\ \hline 10 & 10 & 5 \\ \hline 10 & 5 & 5 \\ \hline\end{array}$

We find the component of this vector along each basis vector using the dot product as before. Since the basis is orthonormal, the total image energy is just the sum of the component energies and the structure of $\mathbf{N}$ can be interpreted in terms of the components.

$$\begin{aligned}
N \circ W_1 &= \frac{5+5\sqrt{2}}{\sqrt{8}} \approx 4.3;\ \ energy \approx 18 \\
N \circ W_2 &= \frac{5+5\sqrt{2}}{\sqrt{8}} \approx 4.3;\ \ energy \approx 18 \\
N \circ W_3 &= 0;\ \ energy = 0 \\
N \circ W_4 &= \frac{5\sqrt{2}-10}{\sqrt{8}} \approx -1;\ \ energy \approx 1 \\
N \circ W_5 &= 0;\ \ energy = 0 \\
N \circ W_6 &= 2.5;\ \ energy \approx 6 \\
N \circ W_7 &= 2.5;\ \ energy \approx 6 \\
N \circ W_8 &= 0;\ \ energy = 0 \\
N \circ W_9 &= 25;\ \ energy = 625
\end{aligned}$$

The total energy in $\mathbf{N}$ is $\mathbf{N} \circ \mathbf{N} = 675$, 625 of which is explained just by the