Imaging geometry

C306
Fall 2001
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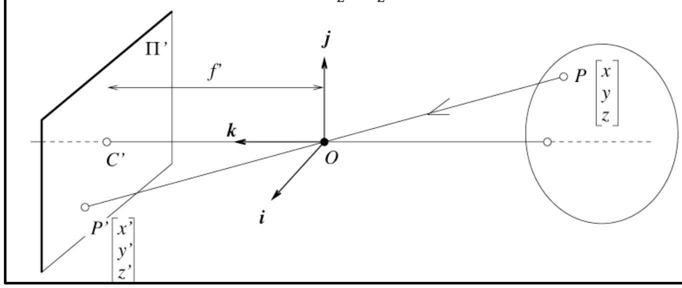
The equation of projection

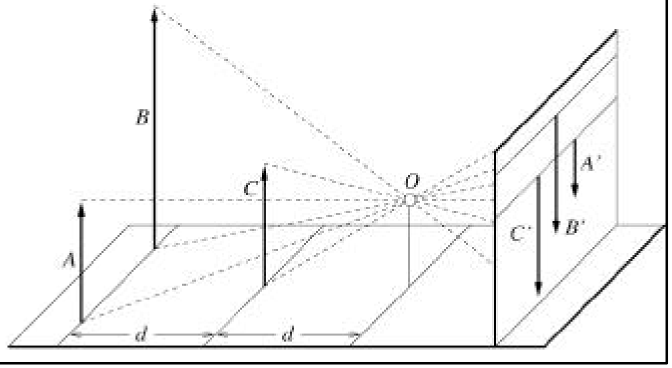
- Cartesian coordinates:
  - We have, by similar triangles, that (x, y, z) -> (f x/z, f y/z, -f)
  - Ignore the third coordinate, and get

$$(x, y, z) \rightarrow (f\frac{x}{z}, f\frac{y}{z})$$



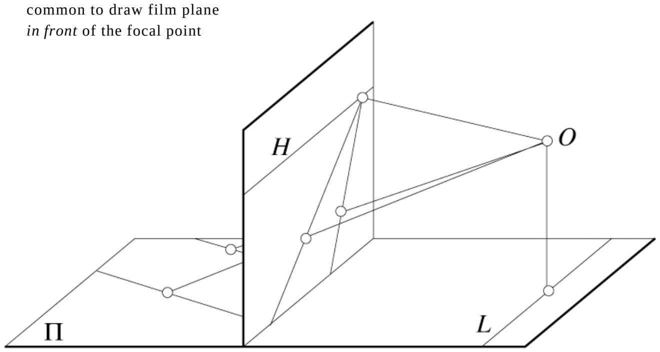
Size illusions

Distant objects are smaller



Parallel lines meet

common to draw film plane
in front of the focal point

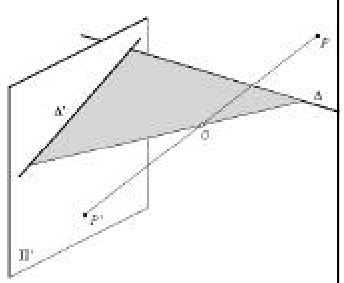


Vanishing points

- each set of parallel lines (=direction) meets at a different point
  - The *vanishing point* for this direction
  - How would you show this?
- Sets of parallel lines on the same plane lead to *collinear* vanishing points.
  - The line is called the *horizon* for that plane

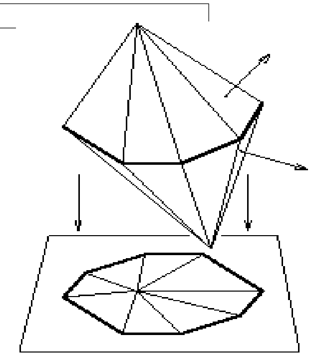
Geometric properties of projection

- Points go to points
- Lines go to lines
- Planes go to whole image
- Polygons go to polygons
- Degenerate cases
  - line through focal point to point
  - plane through focal point to line



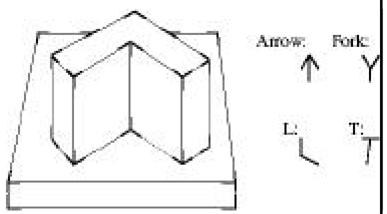
Polyhedra project to polygons

- (because lines project to lines)



Junctions are constrained

- This leads to a process called “line labelling”
  - one looks for consistent sets of labels, bounding polyhedra
  - disadv - can’t get the lines and junctions to label from real images



Basic transformations
Translation

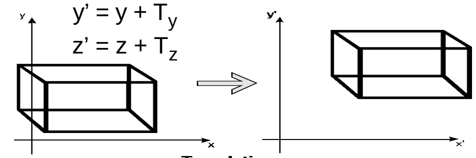
- A translation is a straight line movement of an object from one postion to another.

A point (x,y) is transformed to the point (x’,y’) by adding the translation distances T_x and T_y:

$x' = x + T_x$

$y' = y + T_y$

$z' = z + T_z$

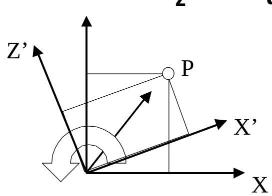


Translation

Coordinate rotation

- Example: Around y-axis

$$\mathbf{p}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{R}_y\mathbf{p}$$

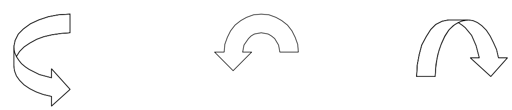


Euler angles

- Note: Successive rotations. Order matters.

$$\mathbf{R} = \mathbf{R_z R_y R_x}$$

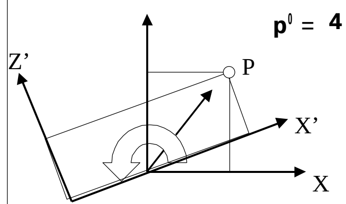
$$= \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos & -\sin \\ 0 & \sin & \cos \end{bmatrix}$$



Rotation and translation

- Translation t' in new o' coordinates

$$\mathbf{p}' = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \mathbf{p} + \mathbf{t}'$$



Basic transformations Scaling

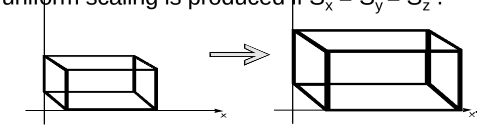
- A scaling transformation alters the scale of an object. Suppose a point (x,y) is transformed to the point (x',y') by a scaling with scaling factors S_x and S_y, then:

  $x' = x\,S_x$

  $y' = y\,S_y$

  $z' = z\,S$

- A uniform scaling is produced if $S_x = S_y = S_z$.



Scaling about the Origin

Basic transformations Scaling

The previous scaling transformation leaves the origin unaltered. If the point (x_f,y_f) is to be the fixed point, the transformation is:

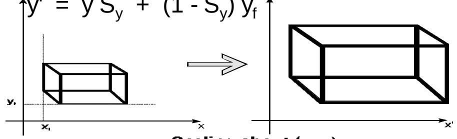
$x' = x_f + (x - x_f)\,S_x$

$y' = y_f + (y - y_f)\,S_y$

This can be rearranged to give:

$x' = x\,S_x + (1 - S_x)\,x_f$

$y' = y\,S_y + (1 - S_y)\,y_f$



Scaling about (x_f,y_f)

Matrix representation and Homogeneous coordinates

- Often need to combine transformations to build the total transformation. If all transformations could be represented as matrix operations then the combination of transformations simply involves the multiplication of the respective matrices
- As translations do not have a 2 x 2 matrix representation, we introduce homogeneous coordinates to allow a 3 x 3 matrix representation.

Matrix representation and Homogeneous coordinates

- The Homogeneous coordinate corresponding to the point (x,y,z) is the triple (x_h, y_h, z_h, w) where:

$$x_h = wx$$
$$y_h = wy$$
$$z_h = wz$$

We can (initially) set w = 1.

- Suppose a point P = (x,y,z,1) in the homogeneous coordinate system is mapped to a point P' = (x',y',z',1) by a transformations, then the transformation can be expressed in matrix form.

Matrix representation and Homogeneous coordinates

- For the basic transformations we have:
  - Translation

$$P' = \begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

  - Scaling

$$P' = \begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Geometric Transforms

Using the idea of homogeneous transforms, we can write:

$$p' = \begin{pmatrix} R & & & T \\ 0 & 0 & 0 & 1 \end{pmatrix} p$$

R and T both require 3 parameters.

$$R = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos & -\sin \\ 0 & \sin & \cos \end{bmatrix}$$

Geometric Transforms

If we compute the matrix inverse, we find that

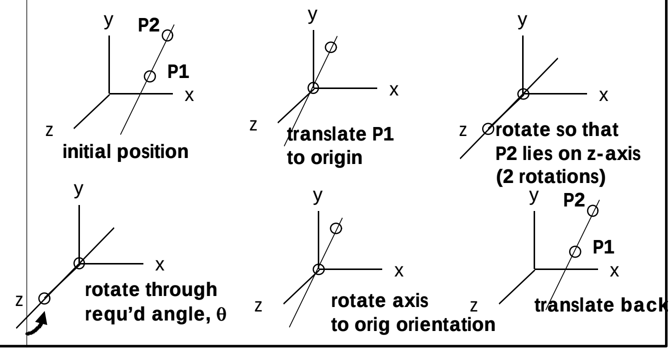
$$p = \begin{pmatrix} R' & & & -R'T \\ 0 & 0 & 0 & 1 \end{pmatrix} p'$$

R and T both require 3 parameters. These correspond to the 6 extrinsic parameters needed for camera calibration

Rotation about a Specified Axis

- It is useful to be able to rotate about any axis in 3D space
- This is achieved by composing 7 elementary transformations (next slide)

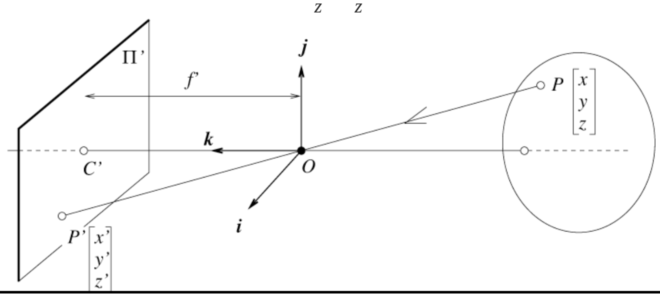
Rotation through θ about Specified Axis



The equation of projection

- Cartesian coordinates:
  - We have, by similar triangles, that (x, y, z) -> (f x/z, f y/z, -f)
  - Ignore the third coordinate, and get

$$(x, y, z) \rightarrow (f\frac{x}{z}, f\frac{y}{z})$$

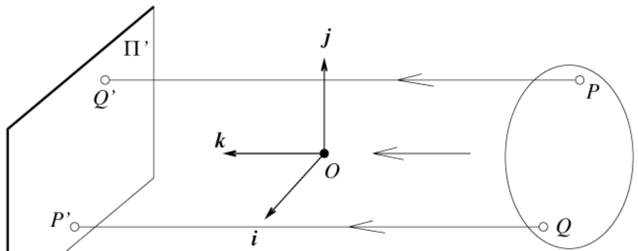


The camera matrix

- Homogenous coordinates for 3D
  - four coordinates for 3D point
  - equivalence relation (X,Y,Z,T) is the same as (k X, k Y, k Z,k T)
- Turn previous expression into HC's
  - HC's for 3D point are (X,Y,Z,T)
  - HC's for point in image are (U,V,W)

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{f} & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix} \qquad (U,V,W) \rightarrow (\frac{U}{W}, \frac{V}{W}) = (u,v)$$

Orthographic projection

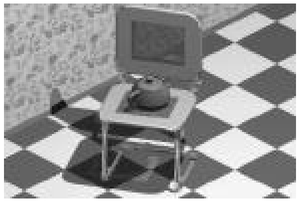


$$u = x$$

$$v = y$$

Perspective and Orthographic Projection

perspective



Orthographic (parallel)

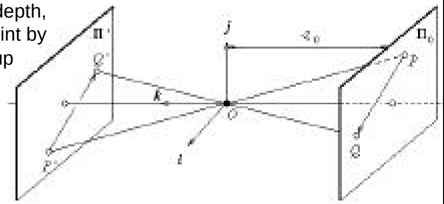
Weak perspective

$$u = Tx$$

$$v = Ty$$

$$T = f / Z$$

- Issue
  - perspective effects, but not over the scale of individual objects
  - collect points into a group at about the same depth, then divide each point by the depth of its group
  - Adv: easy
  - Disadv: wrong



The fundamental model for orthographic projection

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix}$$

Camera parameters

- Issue
  - camera may not be at the origin, looking down the z-axis
    - extrinsic parameters
  - one unit in camera coordinates may not be the same as one unit in world coordinates
    - intrinsic parameters - focal length, principal point, aspect ratio, angle between axes, etc.

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} \text{Transformation} \\ \text{representing} \\ \text{intrinsic parameters} \end{pmatrix} \begin{pmatrix} \text{Transformation} \\ \text{representing} \\ \text{projection model} \end{pmatrix} \begin{pmatrix} \text{Transformation} \\ \text{representing} \\ \text{extrinsic parameters} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix}$$

Intrinsic Parameters

Intrinsic Parameters describe the conversion from metric to pixel coordinates (and the reverse)

$x_{mm} = -(x_{pix} - o_x)\, s_x$
$y_{mm} = -(y_{pix} - o_y)\, s_y$

or

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix}_{pix} = \begin{pmatrix} -f/s_x & 0 & o_x \\ 0 & -f/s_y & o_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ w \end{pmatrix}_{mm} = M_{\text{int}}\, p$$