Imaging Geometry
1. Basic Geometric Coordinate Transforms

C306
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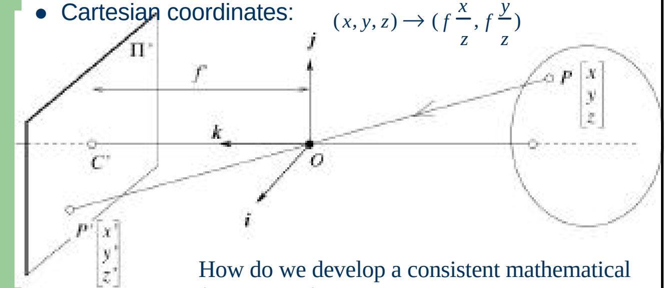
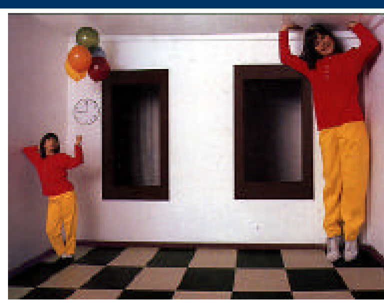
Remember: The equation of projection

- Cartesian coordinates: $(x, y, z) \rightarrow (f\frac{x}{z}, f\frac{y}{z})$

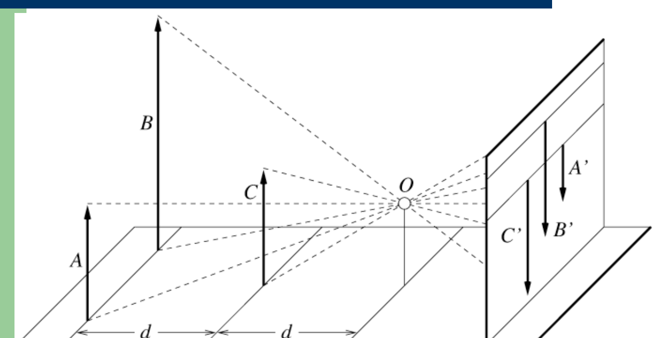


How do we develop a consistent mathematical framework for projection calculations?

Size illusions

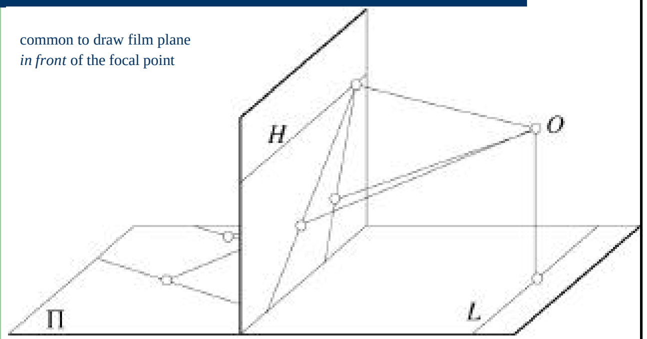


Distant objects are smaller



Parallel lines meet

common to draw film plane
in front of the focal point

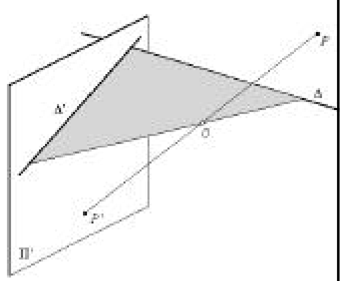


Vanishing points

- each set of parallel lines (=direction) meets at a different point
  - The *vanishing point* for this direction
  - How would you show this?
- Sets of parallel lines on the same plane lead to *collinear* vanishing points.
  - The line is called the *horizon* for that plane

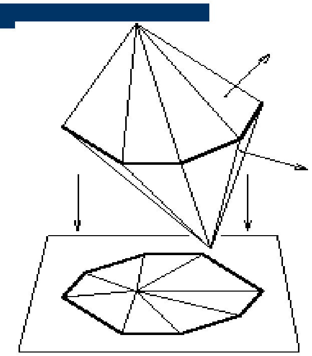
Geometric properties of projection

- Points go to points
- Lines go to lines
- Planes go to whole image
- Polygons go to polygons
- Degenerate cases
  - line through focal point to point
  - plane through focal point to line



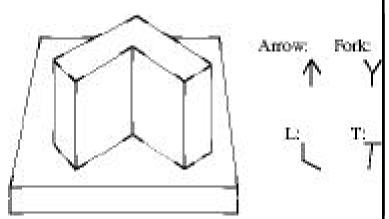
Polyhedra project to polygons

- (because lines project to lines)



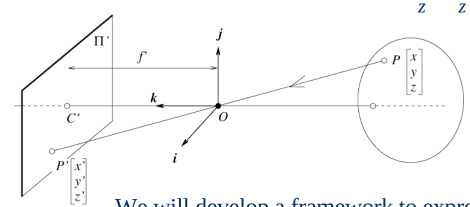
Junctions are constrained

- This leads to a process called "line labelling"
  - one looks for consistent sets of labels, bounding polyhedra
  - disadv - can't get the lines and junctions to label from real images



Back to projection

- Cartesian coordinates: $(x, y, z) \rightarrow (f\frac{x}{z}, f\frac{y}{z})$



We will develop a framework to express projection as x=PX, where x is 2D image projection, P a projection matrix and X is 3D world point.

Basic geometric transformations: Translation

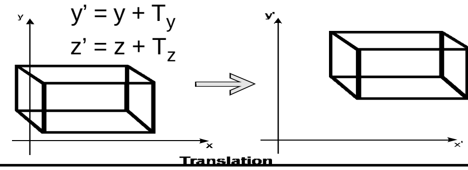
- A translation is a straight line movement of an object from one postion to another.

  A point (x,y) is transformed to the point (x',y') by adding the translation distances T_x and T_y:

  $x' = x + T_x$

  $y' = y + T_y$

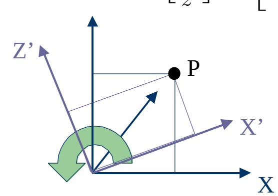
  $z' = z + T_z$



Coordinate rotation

- Example: Around y-axis

$$\mathbf{p}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\nu & 0 & \sin\nu \\ 0 & 1 & 0 \\ -\sin\nu & 0 & \cos\nu \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{R}_y\mathbf{p}$$

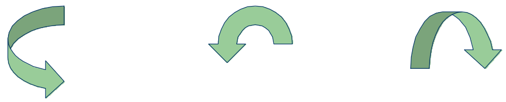


Euler angles

- Note: Successive rotations. Order matters.

$$\mathbf{R} = \mathbf{R_z R_y R_x}$$

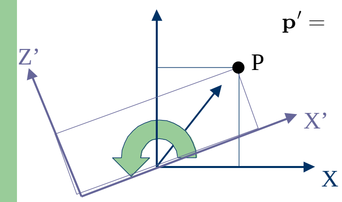
$$= \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\nu & 0 & \sin\nu \\ 0 & 1 & 0 \\ -\sin\nu & 0 & \cos\nu \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$



Rotation and translation

- Translation t' in new o' coordinates

$$\mathbf{p}' = \begin{bmatrix} \cos\nu & 0 & \sin\nu \\ 0 & 1 & 0 \\ -\sin\nu & 0 & \cos\nu \end{bmatrix} \mathbf{p} + \mathbf{t}'$$

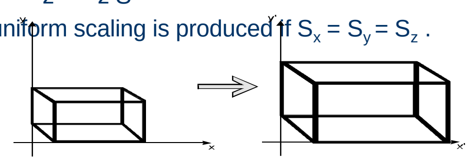


Basic transformations Scaling

- A scaling transformation alters the scale of an object. Suppose a point (x,y) is transformed to the point (x',y') by a scaling with scaling factors S_x and S_y, then:

$$x' = x\, S_x$$
$$y' = y\, S_y$$
$$z' = z\, S$$

- A uniform scaling is produced if $S_x = S_y = S_z$.



Scaling about the Origin

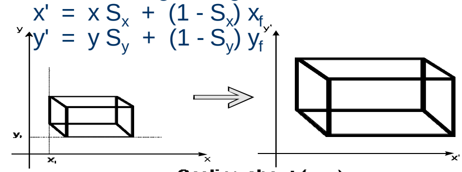
Basic transformations Scaling

The previous scaling transformation leaves the origin unaltered. If the point (x_f, y_f) is to be the fixed point, the transformation is:

$$x' = x_f + (x - x_f)\, S_x$$
$$y' = y_f + (y - y_f)\, S_y$$

This can be rearranged to give:

$$x' = x\, S_x + (1 - S_x)\, x_f$$
$$y' = y\, S_y + (1 - S_y)\, y_f$$



Scaling about (x_f, y_f)

Affine Geometric Transforms

In general, a point in n-D space transforms by

P' = rotate(point) + translate(point)

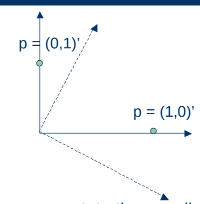
In 2-D space, this can be written as a matrix equation:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} Cos(q) & -Sin(q) \\ Sin(q) & Cos(q) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} tx \\ ty \end{pmatrix}$$

In 3-D space (or n-D), this can generalized as a matrix equation:

$$p' = R\,p + T \quad \text{or} \quad p = R^t\,(p' - T)$$

A Simple 2-D Example



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} Cos(p/4) & -Sin(p/4) \\ Sin(p/4) & Cos(p/4) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} Cos(p/4) \\ Sin(p/4) \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} Cos(p/4) & -Sin(p/4) \\ Sin(p/4) & Cos(p/4) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -Sin(p/4) \\ Cos(p/4) \end{pmatrix}$$

Suppose we rotate the coordinate system through 45 degrees (note that this is measured relative to the *rotated* system!

Matrix representation and Homogeneous coordinates

- Often need to combine transformations to build the total transformation. If all transformations could be represented as matrix operations then the combination of transformations simply involves the multiplication of the respective matrices
- As translations do not have a 2 x 2 matrix representation, we introduce homogeneous coordinates to allow a 3 x 3 matrix representation.

Matrix representation and Homogeneous coordinates

- The Homogeneous coordinate corresponding to the point (x,y,z) is the triple (x_h, y_h, z_h, w) where:

  $x_h = wx$
  $y_h = wy$
  $z_h = wz$

  We can (initially) set w = 1.
- Suppose a point P = (x,y,z,1) in the homogeneous coordinate system is mapped to a point P' = (x',y',z',1) by a transformations, then the transformation can be expressed in matrix form.

Matrix representation and Homogeneous coordinates

- For the basic transformations we have:
  - Translation

$$P' = \begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

  - Scaling

$$P' = \begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Geometric Transforms

Using the idea of homogeneous transforms, we can write:

$$p' = \begin{pmatrix} R & & & T \\ 0 & 0 & 0 & 1 \end{pmatrix} p$$

R and T both require 3 parameters.

$$\mathrm{R} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\nu & 0 & \sin\nu \\ 0 & 1 & 0 \\ -\sin\nu & 0 & \cos\nu \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$

Geometric Transforms

If we compute the matrix inverse, we find that

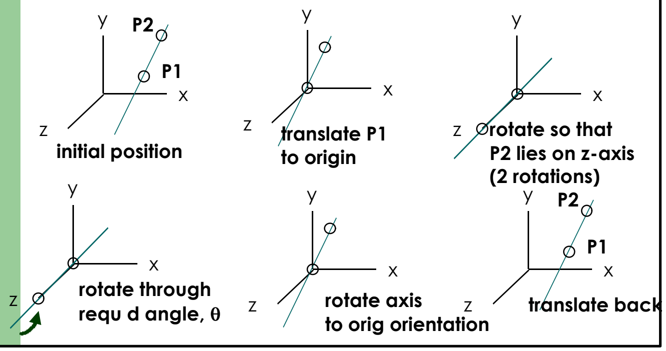
$$p = \begin{pmatrix} R' & & & -R'T \\ 0 & 0 & 0 & 1 \end{pmatrix} p'$$

R and T both require 3 parameters. These correspond to the 6 extrinsic parameters needed for camera calibration

Rotation about a Specified Axis

- It is useful to be able to rotate about any axis in 3D space
- This is achieved by composing 7 elementary transformations (next slide)

Rotation through θ about Specified Axis



Comparison:

- Homogeneous coordinates
  - Rotations and translations are represented in a uniform way
  - Successive transforms are composed using matrix products: y = Pn*..*P2*P1*x
- Affine coordinates
  - Non-uniform representations: y = Ax + b
  - Difficult to keep track of separate elements

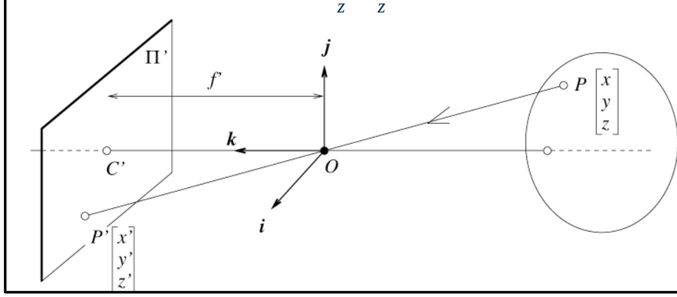
2. Camera models and projections

- Using geometry and homogeneous transforms to describe:
  - Perspective projection
  - Weak perspective projection
  - Orthographic projection

The equation of projection

- Cartesian coordinates:
  - We have, by similar triangles, that (x, y, z) -> (f x/z, f y/z, -f)
  - Ignore the third coordinate, and get

$$(x, y, z) \rightarrow (f\frac{x}{z}, f\frac{y}{z})$$



The camera matrix

- Homogenous coordinates for 3D
  - four coordinates for 3D point
  - equivalence relation (X,Y,Z,T) is the same as (k X, k Y, k Z,k T)
- Turn previous expression into HC's
  - HC's for 3D point are (X,Y,Z,T)
  - HC's for point in image are (U,V,W)

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/f & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix} \qquad (U,V,W) \to \left(\frac{U}{W}, \frac{V}{W}\right) = (u,v)$$

Camera parameters

- Issue
  - camera may not be at the origin, looking down the z-axis
    - extrinsic parameters
  - one unit in camera coordinates may not be the same as one unit in world coordinates
    - intrinsic parameters - focal length, principal point, aspect ratio, angle between axes, etc.

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} \text{Transformation} \\ \text{representing} \\ \text{intrinsic parameters} \end{pmatrix} \begin{pmatrix} \text{Transformation} \\ \text{representing} \\ \text{projection model} \end{pmatrix} \begin{pmatrix} \text{Transformation} \\ \text{representing} \\ \text{extrinsic parameters} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix}$$

Intrinsic Parameters

Intrinsic Parameters describe the conversion from metric to pixel coordinates (and the reverse)

$x_{mm} = - (x_{pix} – o_x)\, s_x$
$y_{mm} = - (y_{pix} – o_y)\, s_y$

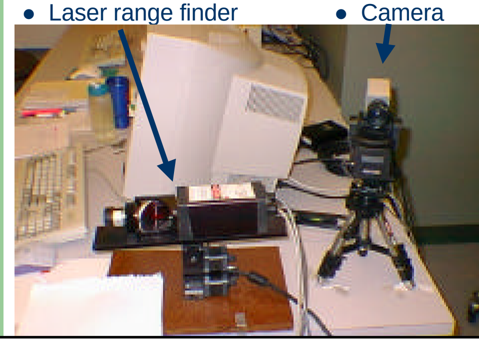
or

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix}_{pix} = \begin{pmatrix} -f/s_x & 0 & o_x \\ 0 & -f/s_y & o_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ w \end{pmatrix}_{mm} = M_{\text{int}}\, p$$

Note: Focal length is a property of the camera and can be incorporated as above

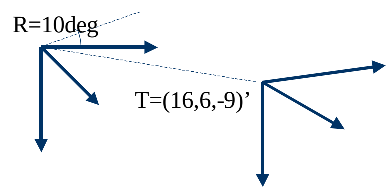
Example: A real camera

- Laser range finder
- Camera



Relative location Camera-Laser

- Camera
- Laser



In homogeneous coordinates

- Rotation:
- Translation

$$R = \begin{bmatrix} \cos -10 & 0 & \sin -10 \\ 0 & 1 & 0 \\ -\sin -10 & 0 & \cos -10 \end{bmatrix} \qquad \mathrm{T} = \begin{pmatrix} 1 & 0 & 0 & 16 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Full projection model

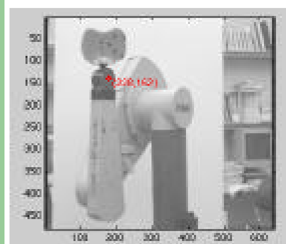
- Camera internal parameters
- Camera projection

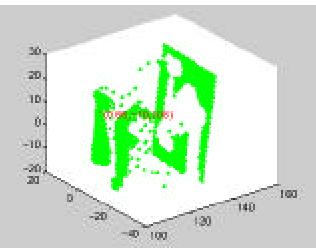
$$\mathrm{p_{camera}} = \begin{pmatrix} 1278.6657 & 0 & 256 \\ 0 & 1659.5688 & 240 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0.985 & 0 & -0.174 & 0 \\ 0 & 1 & 0 & 0 \\ 0.174 & 0 & 0.985 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 16 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.6612 \\ -10.55 \\ 108.0 \\ 1 \end{pmatrix} = \begin{pmatrix} 22262 \\ 16755 \\ 97.47 \end{pmatrix}$$

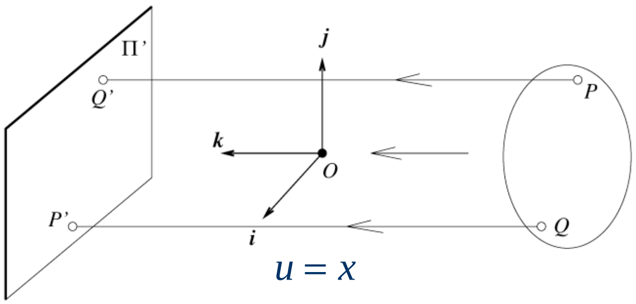
Result

- Camera image
- Laser measured 3D structure





Orthographic projection



$$u = x$$

$$v = y$$

The fundamental model for orthographic projection

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix}$$

Perspective and Orthographic Projection

perspective

Orthographic (parallel)

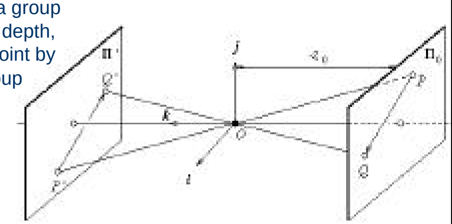
Weak perspective

$$u = Tx$$

$$v = Ty$$

$$T = f \,/\, Z$$

- Issue
  - perspective effects, but not over the scale of individual objects
  - collect points into a group at about the same depth, then divide each point by the depth of its group
  - Adv: easy
  - Disadv: wrong



The fundamental model for weak perspective projection

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & f/Z^* \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix}$$

Camera Model Structure

Assume R and T express camera *in world coordinates, then*

$$^{c}p = \begin{pmatrix} & R' & & -R'T \\ 0 & 0 & 0 & 1 \end{pmatrix} {}^{w}p$$

Combining with a perspective model (and neglecting internal parameters) yields

$$^{c}u = M\,{}^{w}p = \begin{pmatrix} -R'_x & R'_x T \\ -R'_y & R'_y T \\ \frac{R_z}{f} & \frac{-R_z T}{f} \end{pmatrix} {}^{w}p$$

Note the M is defined only up to a scale factor at this point! If M is viewed as a 3x4 matrix defined up to scale, it is called the *projection matrix.*

Camera Model Structure

Assume R and T express camera *in world coordinates, then*

$$^{c}p = \begin{pmatrix} & R' & & -R'T \\ 0 & 0 & 0 & 1 \end{pmatrix} {}^{w}p$$

Combining with a weak perspective model (and neglecting internal parameters) yields

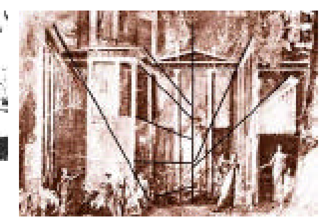
$$^{c}u = M\,{}^{w}p = \begin{pmatrix} -R'_x & R'_x T \\ -R'_y & R'_y T \\ 0 & \frac{R_z(\bar{P} - T)}{f} \end{pmatrix} {}^{w}p$$

Where $\bar{P}$ is the nominal distance to the viewed object

History of Perspective

Prehistoric:

Roman



Renaissance
Leonardo da Vinci: The last supper

Other Models

- The *affine camera* is a generalization of weak perspective.
- The *projective camera* is a generalization of the perspective camera.
- Both have the advantage of being linear models on real and projective spaces, respectively.
- But in general will recover structure up to an affine or projective transform only. (ie distorted structure)
- Learn about in cmput 615 3-Dimensional Computer Vision

Recall: Intrinsic Parameters

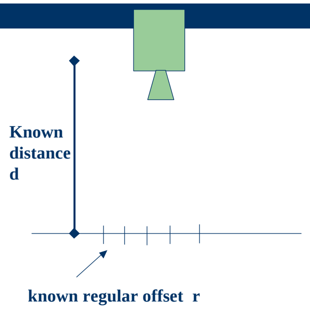
Intrinsic Parameters describe the conversion from metric to pixel coordinates (and the reverse)

$$x_{mm} = -(x_{pix} - o_x)\, s_x$$
$$y_{mm} = -(y_{pix} - o_y)\, s_y$$

or

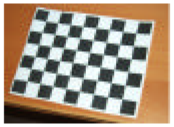
$$\begin{pmatrix} x \\ y \\ w \end{pmatrix}_{pix} = \begin{pmatrix} -1/s_x & 0 & o_x \\ 0 & -1/s_y & o_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ w \end{pmatrix}_{mm} = M_{\text{int}}\, p$$

CAMERA CALIBRATION: A WARMUP



$$\frac{rk_i}{d} = (x_i - o_x)s_x$$

$$\frac{r}{d} = (x_{i+1} - x_i)s_x$$



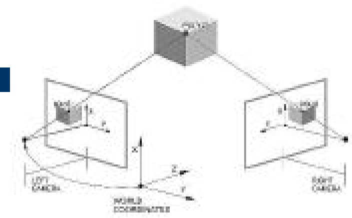
A simple way to get scale parameters; we can compute the optical center as the numerical center and therefore have the intrinsic parameters

Camera calibration

- Issues:
  - what are intrinsic parameters of the camera?
  - what is the camera matrix? (intrinsic+extrinsic)
- General strategy:
  - view calibration object
  - identify image points
  - obtain camera matrix by minimizing error
  - obtain intrinsic parameters from camera matrix
- Error minimization:
  - Linear least squares
    - easy problem numerically
    - solution can be rather bad
  - Minimize image distance
    - more difficult numerical problem
    - solution usually rather good, but can be hard to find
      - start with linear least squares
  - Numerical scaling is an issue

Stereo Vision

- **GOAL:** Passive 2-camera system for triangulating 3D position of points in space to generate a depth map of a world scene.
- Humans use stereo vision to obtain depth



Stereo depth calculation: Simple case, aligned cameras

DISPARITY= **(XL - XR)**

Similar triangles:

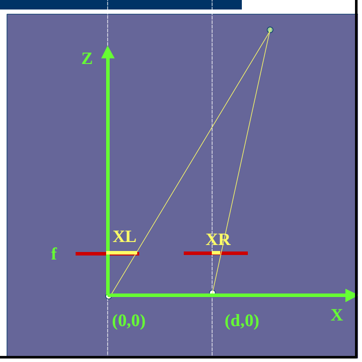
Z = (f/XL) X
Z= (f/XR) (X-d)

Solve for X:

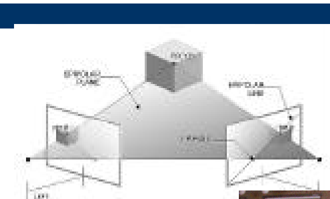
(f/XL) X = (f/XR) (X-d)
X = (XL d) / (XL - XR)

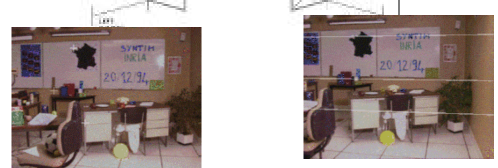
Solve for Z:

$$Z = \frac{d*f}{(XL - XR)}$$



Epipolar constraint





Special case: **parallel cameras** epipolar lines are parallel and aligned with rows

Stereo measurement example:

- Left image

Resolution = 1280 x 1024 pixels

f = 1360 pixels

- Right image

Baseline d = 1.2m

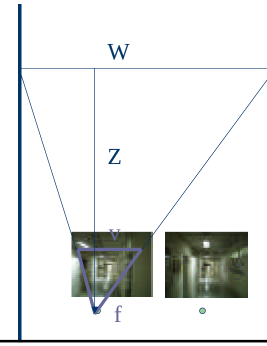
Q: How wide is the hallway

How wide is the hallway? General strategy

- Similar triangles:

$$\frac{W}{Z} = \frac{v}{f}$$

- Need depth Z
- Then solve for W



How wide is the hallway? Steps in solution:

1. Compute focal length f in meters from pixels
2. Compute depth Z using stereo formula (aligned camera planes)

$$Z = \frac{d*f}{(XL - XR)}$$

3. Compute width:

$$W = Z\frac{v}{f}$$

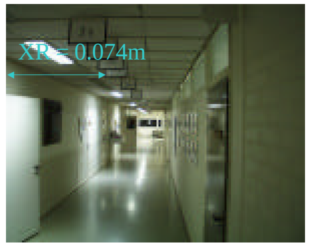
Focal length:

f = 1360 pixels

$$f = \frac{1360}{1280} * 0.224 = 0.238\text{m}$$

0.224m is 1280 pixels

How wide…
Depth calculation



Disparity: XL – XR = 0.07m

(Note in the disparity calculation the choice of reference (here the edge) doesn't matter. But in the case of say X-coordinate calculation it should be w.r.t. the center of the image as in the stereo formula derivation

- Depth

$$Z = \frac{1.2 * 0.238}{0.07} = 4.1m$$

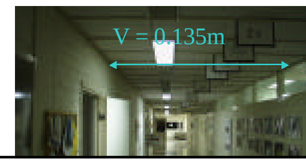
How wide…?
Answer:

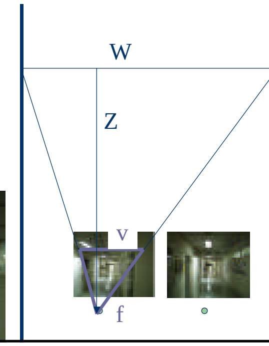
- Similar triangles:

$$W = Z\frac{v}{f}$$

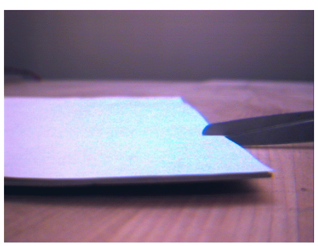
- The width of the hallway is:

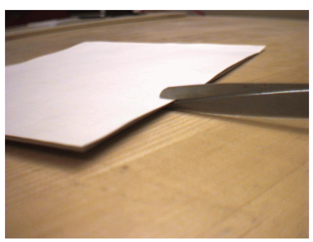
$$W = 4.1 * \frac{0.135}{0.238} = 2.3m$$





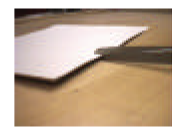
Visual ambiguity

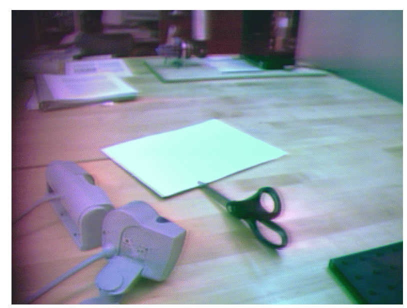




- Will the scissors cut the paper in the middle?

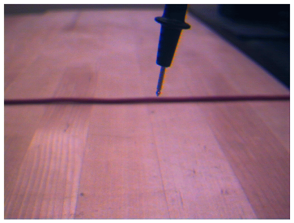
Ambiguity

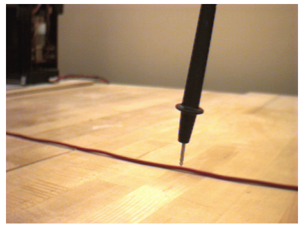




- Will the scissors cut the paper in the middle? **NO!**

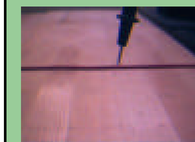
Visual ambiguity

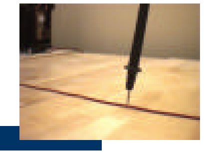


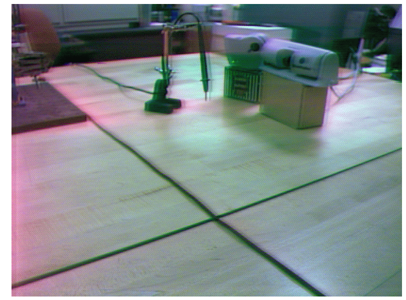


- Is the probe contacting the wire?

Ambiguity

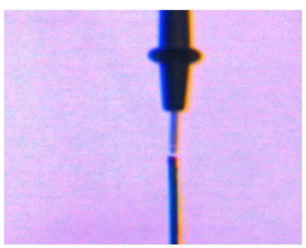


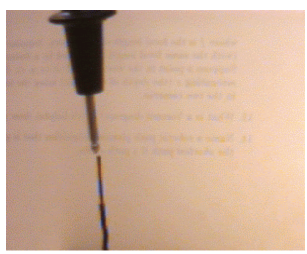




- Is the probe contacting the wire? **NO!**

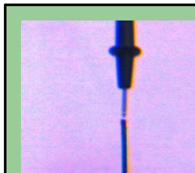
Visual ambiguity

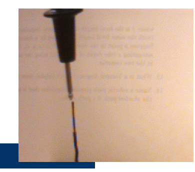


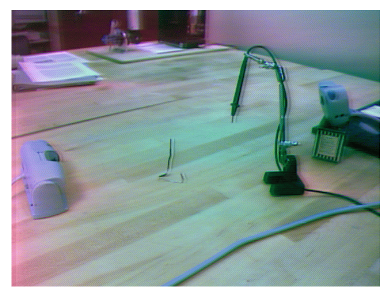


- Is the probe contacting the wire?

Ambiguity







- Is the probe contacting the wire? **NO!**

Camera Models

- Internal calibration:
- Weak calibration:
- Affine calibration:
- Stratification of stereo vision:
  - characterizes the reconstructive certainty of weakly, affinely, and internally calibrated stereo rigs

C_{sim} up to a similarity (scaled Euclidean transformation)

C_{aff} up to an affine transformation of task space

C_{proj} up to a projective transformation of task space

C_{inj} reconstruction up to a bijection of task space

Visual Invariance