

# Fast Exact MultiConstraint Shortest Path Algorithms

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**Abstract**—QoS routing has been shown to be NP-hard. A recent study of its hardness shows that the “worst-case” may not occur in practice [13]. This suggests that there may exist fast exact algorithms for the multi-constraint shortest path (MCSP) problem, an instance of QoS routing.

Search techniques such as A\* and IDA\* may solve hard problems exactly in polynomial time. In [14], we deploy the idea of iterative deepening search to design IDA\*\_MCSP, and show its efficiency by extensive empirical study. In this paper, we show that for infeasible cases, where there is no feasible solution, IDA\*\_MCSP may not be as efficient as A\*Prune. This motivates us to design an algorithm that is efficient in both feasible and infeasible cases.

We design an exact MCSP algorithm A\*\_MCSP, which introduces the state notion and dominance relationship between states. Furthermore, we design an exact MCSP algorithm FringeMCSP. It can be regarded as an integration of IDA\*\_MCSP and A\*\_MCSP. Extensive empirical study shows that FringeMCSP has good performance in both feasible and infeasible cases; while IDA\*\_MCSP still shows its superiority among the proposed MCSP algorithms in feasible cases.<sup>1</sup>

## I. INTRODUCTION

The multi-constraint path (MCP) problem is to find one or several feasible paths subject to multiple constraints on a given network topology with link weights, such as delay, loss, administrative cost, etc. For instance, we may want to find the shortest path in a DiffServ network or an overlay network with the delay and the loss ratio below certain level. The multi-constraint shortest path (MCSP) problem is to find the shortest path with respect to hop count that satisfies the constraints. The MCP and the MCSP problems are instances of QoS routing. Their NP-hardness property [21] led to the proposals of many heuristic algorithms or approximate algorithms. See [6] and [12] for overviews. On the other hand, Cheeseman et al. found that typical cases of many NP-hard problems are tractable in practice [5]. Kuipers et al. studied that the hardness of QoS routing occurs in topologies with special characteristics, and the problem may be easy to solve in realistic communication networks [13]. This suggests that there may exist fast exact algorithms for MCP and MCSP problems. Recently, there are research efforts on designing exact algorithms for them. See [12] for an overview.

QoS routing has two components: QoS information collection and QoS routing computation. We assume QoS information collection has been done by a mechanism such as the

OSPF extension in [3] or by a measurement approach such as that in [2]. We concentrate on QoS routing computation. A link may be a physical link in a network, or a virtual link in a Virtual Private Network or an overlay network. The information gathered may be related to delay, loss rate or bandwidth. Link weights are defined by these metrics. Link weights can be additive, multiplicative or concave. By additive weight, we mean the weight of a path is the sum of the weights of the links on the path. For a multiplicative weight, the weight of a path is the product of the weights of the links on the path. Multiplicative weights such as loss rate can be transformed to additive weights by taking the logarithm of the path weight. A concave weight of a path is the minimum of the link weights of the path. Concave weights such as bandwidth can be dealt with by using a preprocessing procedure to remove ineligible links. Without loss of generality, we focus on additive weights.

The constraints may be specified by the Service Level Agreements (SLAs) or customers' requirements. Given the link weights and constraints, each node can compute the shortest path to a destination subject to multiple constraints. In a DiffServ architecture, paths may be computed centrally at the Bandwidth Broker. In an overlay network [2], paths are computed at overlay nodes. Once a path is computed, source routing or MPLS may be used to forward packets on that route.

Search techniques from the artificial intelligence community such as A\* [9] and IDA\* [10] have shown their strength in solving some hard problems in practice [19]. A search algorithm usually uses look-ahead information to speed up the search. When the look-ahead information always underestimates the solution cost, the search algorithm is guaranteed to find the optimal solution. With the assistance of a good look-ahead function to predict solution length, A\* and IDA\* may solve hard problems exactly in polynomial time [19], [11].

A\*Prune [16] and SAMCRA [17] are two exact QoS routing algorithms. A\*Prune is designed for the MCSP problem and SAMCRA is designed for the MCP problem, although it can also be used for MCSP. They borrow ideas from A\* and conduct best first searches using a priority queue. They use look-ahead information to cut off the part of search space that won't lead to a feasible solution. In [14], we design a fast exact MCSP algorithm, IDA\*\_MCSP, and show its superiority over A\*Prune in feasible cases. We also show the high accuracy of look-ahead information in [14], which determines the efficiency of a search-based MCSP algorithm.

Previous work in QoS routing focuses on designing efficient algorithms when a problem is feasible, i.e., there is at least a solution. However, an MCSP may be infeasible, due to

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one or more constraints. This can happen in the service negotiation phase. Thus, it is desirable to design an efficient algorithm for both feasible cases and infeasible cases, which can either find the feasible path fast or can determine quickly that there is not a feasible path. In this paper, we first design A\*\_MCSP, which introduces the state notion to replace partial paths in A\*Prune and SAMCRA. Furthermore, we design FringeMCSP, which can be regarded as an integration of IDA\*\_MCSP and A\*\_MCSP. Extensive performance study shows that FringeMCSP performs well in both feasible and infeasible cases; while IDA\*\_MCSP still shows its superiority among the proposed MCSP algorithms in feasible cases.

Our work of designing exact MCSP algorithms is different from the work in QoS routing that proposes a heuristic and looks for a close to optimal solution. Our algorithms IDA\*\_MCSP, A\*\_MCSP and FringeMCSP are guaranteed to find the exact solution once it terminates (if it is feasible). Their optimality makes them different from the polynomial time approximation algorithms.

We give the notation used in the paper. A network is represented by a graph  $G = (V, E)$ , where  $V$  is the set of nodes and  $E$  is the set of edges. Each edge  $(u, v) \in E$  is associated with  $m$  non-negative additive weights,  $w_k(u, v)$ ,  $k = 1, 2, \dots, m$ . The  $m$  weights on edge  $(u, v)$  form an  $m$ -dimensional weight vector  $W(u, v) = \{w_1(u, v), w_2(u, v), \dots, w_m(u, v)\}$ . We denote the source node as *src* and the destination node as *dst*. We have an  $m$ -dimensional constraint vector  $C = \{c_1, c_2, \dots, c_m\}$ , corresponding to the  $m$  weights. The  $i$ -th weight of a path  $P$  is  $w_i(P) = \sum_{(u,v) \in P} w_i(u, v), \forall 1 \leq i \leq m$ . The weight vector of a path  $P$  is  $W(P) = \{w_1(P), w_2(P), \dots, w_m(P)\}$ . When we say *shortest path*, we mean the shortest path with respect to hop count, if not explicitly stated. The path length is the number of hops of a path. The definition of the MCSP problem follows.

*Definition: MCSP problem:* Given a network represented by a graph  $G$ ,  $m$  associated non-negative additive weights on edges, the source node *src*, the destination node *dst*, and the  $m$ -dimensional constraint vector  $C$ , the MCSP problem is to find the shortest path  $P$  such that  $w_i(P) \leq c_i, \forall 1 \leq i \leq m$ .

We present an overview of search techniques in Section II and an approach to compute look-ahead information [14] in Section III. We investigate the performance of IDA\*\_MCSP for infeasible cases in Section IV. We present A\*\_MCSP in Section V, FringeMCSP in Section VI and results for performance study in Section VII. Then we draw conclusions.

## II. OVERVIEW OF SEARCH TECHNIQUES

### A. Look-ahead

A search algorithm usually deploys some look-ahead function to predict the quality of a potential solution. A good look-ahead function plays an important role in enhancing the performance of a search algorithm, by facilitating decision making of whether to further search a branch. We use the Dijkstra's algorithm to calculate the shortest paths for *hop* or for each of the weight metrics to obtain look-ahead information.

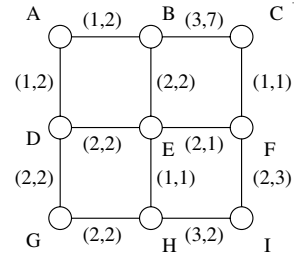


Fig. 1. Example Graph

We give an example for illustration. In Figure 1, we are to find the shortest path from  $A$  to  $C$  subject to the constraint  $\{7, 8\}$ . The vectors on the edges are edge weight vectors, such as  $W(A, B) = (1, 2)$ . Denote these two edge weights  $w_1, w_2$ , and the two constraints,  $c_1, c_2$ . Denote the look-ahead information for node  $u$  and metric  $w$  as  $lb(w, u)$ , where  $w$  can be the hop count or one of the two edge weights  $w_1$  and  $w_2$ . It is easy to compute that for node  $A$ ,  $lb(hop, A) = 2$ ,  $lb(w_1, A) = 4$ , and  $lb(w_2, A) = 6$ ; while for node  $E$ ,  $lb(hop, E) = 2$ ,  $lb(w_1, E) = 3$ , and  $lb(w_2, E) = 2$ , etc.

### B. A\* Search

A\* algorithm is a well-known search strategy in artificial intelligence. It is a best-first search algorithm. For each node in the search tree, A\* maintains the *distance traversed* so far and the *estimate distance*, which is the sum of the distance traversed and the distance of look-ahead.

A\* maintains an *openlist* and a *closedlist*. It takes the first node from the *openlist* and expands it to the neighbors. Newly expanded nodes are put into *openlist*, sorted in increasing order by the estimate distance. Ties are broken with traversed distance, giving preference to larger values, since a larger value means more reliable information. The state just removed from *openlist* is put in *closedlist*. A\* terminates if the goal node is found or there is no node on the *openlist*. A\* always finds the shortest path given that the heuristic does not over-evaluate the distance from a node to the destination. That is, admissibility of heuristics guarantees the optimality of the solution [19]. However, it requires exponential space since it stores all the temporary search results.

Liu and Ramakrishnan [16] extend A\* to A\*Prune for the problem of finding  $K$  shortest paths subject to multiple constraints. It maintains partial paths on a list, in increasing order of the estimates of distances to the destination, breaking ties with the lengths of partial paths. The first path on the list is expanded to the neighbors of the tail node. The resulting partial paths that pass the eligibility tests against all the constraints are put into the list for further expansion. The algorithm terminates once it finds  $K$  shortest paths or there are no more partial paths on the list.

We will introduce a new exact MCSP algorithm based on A\*, A\*\_MCSP, which introduces the state notion and dominance relationship between states. It eliminates the cost to maintain partial paths as in A\*Prune.

### C. Iterative Deepening Search

An iterative deepening search algorithm [10] conducts a series of depth-first searches. Different from depth-first search, it has a depth bound for each iteration. That is, when the search algorithm has traveled as far as the depth bound, or it predicts that there wouldn't be a solution within the search bound, it stops searching from that node. An iterative deepening search algorithm updates the depth bound after each iteration, until the solution is found, or it determines that there is no feasible solution. Bounding the search depth avoids searching too deeply. Moreover, by updating the search bound properly, the algorithm guarantees to find the shortest path. The cost to pay is that part of the search space has to be searched redundantly. We design IDA\*\_MCSP in [14]. Here we give an example for the problem using Figure 1.

Recall that we are to find the shortest path from  $A$  to  $C$  subject to the constraint  $\{7, 8\}$ . IDA\*\_MCSP starts with depth bound 2, since the shortest path from  $A$  to  $C$  has length 2. At iteration 1, the algorithm will reach nodes  $B$  and  $D$ . After reaching node  $B$ , it does not expand node  $C$ , since using the eligibility test (Figure 3),  $c_2 = 8$  will be violated. Neither will node  $E$  be expanded, since with the look-ahead information on hop count, the length of the shortest path from  $E$  to  $C$ , is 2; and  $1 + 1 + 2 = 4$  exceeds the depth bound 2 for this iteration. Node  $D$  won't be further expanded due to the reason that the depth bound will be violated.

An iteration with depth bound 3 won't find the solution either. In fact, with proper update, this iteration can be skipped to make the search more efficient. The depth bound can be set as 4 for the second iteration, since a potential solution via node  $E$  has at least  $2 + 2 = 4$  hops, where the 2's are the distance traversed so far and the length of shortest path from  $E$  to  $C$ , the look-ahead information on hop count.

In the iteration with depth bound 4, again, after reaching  $B$ ,  $C$  won't be expanded. After reaching  $E$ , it will expand  $F$  and reaches  $C$ . The shortest path  $P = ABEFC$  will be found, with the weight vector  $W(P) = \{6, 6\}$ . Path  $ADEFC$  may be found, if we expand neighbors in a different way.

### D. Fringe Search

A\* has to keep a sorted openlist of all the frontier nodes in the search tree, which may be costly. IDA\* has to make iterations with updated search threshold, unless the source has perfect lookahead information of the destination. Many nodes will be searched redundantly. Björnsson et al. [4] recently introduce Fringe search to attempt to overcome shortcomings of A\* and IDA\*, and show its efficiency on pathfinding on game maps. The basic idea follows. The algorithm keeps *nowlist* and *laterlist*, and iterates on a depth bound. In each iteration, the algorithm searches on *nowlist*, and put nodes to be considered later on *laterlist*. The head of *nowlist* is removed from the list and examined. If its estimate distance is greater than the depth threshold, then the head element is put into *laterlist*. Otherwise, add the children of the head to the front of *nowlist*. When *nowlist* is empty, update the depth threshold, copy *laterlist* to *nowlist*, and clear *laterlist*.

Search on *nowlist* iteratively, until the goal is found or it is determined that the goal is not reachable. Fringe search can be regarded as an integration of A\* and IDA\*. It attempts to take advantage of both A\* and IDA\*; at the same time, it attempts to avoid their shortcomings. We extend Fringe search to MCSP in Section VI.

### III. LOOK-AHEAD

We discuss the approach we use to compute look-ahead information. The input to a search-based MCSP algorithm is the graph representation of the network topology, the weights on each link, and the constraints.

A 2-dimensional array  $lb[m+1][n]$  is used to store the lower bounds, i.e. the look-ahead values, for the  $m+1$  metrics  $\{hop, w_1, w_2, \dots, w_m\}$  (hop count and  $m$  weights) for each of the  $n$  nodes. We use *computeLowerbound()* in Figure 2 to set the lower bounds at the initialization stage using the *lookahead()* function, which calls the Dijkstra's algorithm to calculate the shortest paths. Function *lookahead(metric)* calculates the shortest paths for all the nodes to the destination, with respect to *metric*, which can be the hop count or one of the weights. The results are recorded in the array  $lb[][]$ . The destination has lookahead values of 0 for each metric.

```

1: computeLowerbound()
2:   for each metric  $metric \in \{hop, w_1, w_2, \dots, w_m\}$ 
3:      $lb[metric][i] \leftarrow lookahead(metric);$ 

```

Fig. 2. Function: *computeLowerbound()*

Dijkstra's algorithm is used for calculating the respective shortest path due to its efficiency. More importantly, Dijkstra's algorithm makes underestimates, which is a necessary property for the look-ahead function in order to help a search-based MCSP algorithm find the optimal solution. By an underestimate, we mean that the least cost (length) path found by Dijkstra's algorithm won't exceed the cost (length) of the shortest constrained path. We assume the topology is symmetric, thus we can use Dijkstra's algorithm on the single-source shortest path problem to compute the look-ahead values of all the nodes to the destination. For an asymmetric topology, we may first compute the transpose of the graph [7], and then apply Dijkstra's algorithm. We assume link weights are static, so we need to calculate the lower bounds only once. If there are changes, an efficient, incremental algorithm can be designed based on the work in Ramalingam and Reps [18] to incrementally recalculate lowerbounds after changes.

We study the accuracy of look-ahead information on a wide range of topologies including inferred ISP topologies, Internet-like power-law topologies, and Waxman random topologies. The empirical study on the diverse topologies shows the high accuracy of look-ahead information using the Dijkstra's algorithm, which implies the potential efficiency of a search-based MCSP algorithm. Please see [14] for details.

#### A. Eligibility Test

A search-based MCSP algorithm can use look-ahead information to do eligibility tests to prune unnecessary portions of

the search space. Figure 3 presents the function that conducts an eligibility test at node  $node$  for the neighbor  $nbr$  according to the current accumulated weight vector  $curW$ . For each weight,  $w_i$ , it predicts the path weight from  $src$  to  $dst$  via  $node$  and  $nbr$  by adding up the following three components: the  $i$ -th element of  $curW$ ; the weight on the link from  $node$  to  $nbr$ ,  $w_i(node, nbr)$ ; and the look-ahead value for the predicted path from  $nbr$  to the destination,  $lb_i$ . If any constraint is violated, the test fails.

```

1: eligible(node, nbr, curW)
2:   for each weight  $w_i \in \{w_1, w_2, \dots, w_m\}$ 
3:      $lb_i = lb[w_i][node]$ ;
4:     if  $curW_i + w_i(node, nbr) + lb_i > C_i$ 
5:       return false;
6:   return true;

```

Fig. 3. Function:  $eligible(node, nbr, curW)$

#### IV. PERFORMANCE OF IDA\*\_MCSP IN INFEASIBLE CASES

In [14], extensive empirical study shows that, in feasible cases, IDA\*\_MCSP is much more efficient than A\*Prune. We now investigate the performance of IDA\*\_MCSP and A\*Prune in infeasible cases.

We use realistic ISP topologies from Rocketfuel project [20]. OSPF/IS-IS weights on the links (inferred weight and latency) are also provided. We also use synthetic topologies, including Internet-like topologies following power-laws [8], [1] and random graphs based on the Waxman model [22]. For power-law topologies, we use sizes of 3037, 5000, 7500 and 10000. For Waxman topologies, we use network sizes of 250, 500, 1000, 2500 and 5000. On these topologies, link weights are set uniformly in the range of  $(0, 1)$ . There are two link weights, i.e.  $m = 2$ . We use a *tightness factor*,  $\alpha$ , to set the constraint vector for each search. For each source-destination pair, each constraint is set as the tightness factor  $\alpha$  times the weight of the least weight path of the source-destination pair by Dijkstra's algorithm with respect to that weight.  $\alpha$  is set as 1.1, 1.2 or 1.5. As a consequence, the constraints may be different for different source-destination pairs.

Both IDA\*\_MCSP and A\*Prune do the identical preprocessing to calculate the lower bounds, thus we concentrate on the comparison of the computation time for the search process. For the inferred ISP topologies, we study each possible source-destination pair. In each synthetic topology, we choose 10 destinations randomly, and 100 random sources for each destination. The experiments are conducted on a Linux machine with 602 MHz CPU and 256MB memory. For each source-destination pair, we run 1000 repetitions and take the average for precision. For performance study for feasible cases in Section VII, in each synthetic topology, we choose 100 destinations randomly, and 100 random sources for each destination.

Figure 4 shows the results with scatter plots<sup>2</sup>. A point below the  $y = x$  line means for that instance, IDA\*\_MCSP is faster than A\*Prune. The closer to the  $x$ -axis, the faster IDA\*\_MCSP is, vice versa. For inferred ISP topologies, there are not many infeasible cases, esp. when the tightness factor  $\alpha$  is large. On ISP topologies and random topologies, IDA\*\_MCSP is faster than or comparable with A\*Prune. However, on power-law topologies, A\*Prune can be much faster than IDA\*\_MCSP in infeasible cases.

A\* maintains a priority list. IDA\* conducts iterative deepening search. In infeasible cases, the look-ahead information is not accurate (there is no solution at all). Therefore an A\*-based MCSP algorithm has the chance to narrow down the search space; while an IDA\*-based MCSP algorithm has to do redundant search. This is the reason A\*Prune may win IDA\*\_MCSP in some infeasible cases. It is interesting to design a MCSP algorithm that is efficient in both feasible and infeasible cases. An A\*-based algorithm is worth exploiting.

#### V. A\*\_MCSP

It is time-consuming to maintain partial paths in A\*Prune. Thus, it is desirable not to maintain partial paths during the search process. There may be several paths reaching a node with the same quality. By path quality, we mean both hop number and accumulated weights. We call such partial paths *equivalent partial paths*. Take the example in Figure 1 in Section II-A. When the search reaches node  $E$  by paths  $ABE$  and  $ADE$ , we have two equivalent partial paths, with the same hop number and the same accumulated partial path weights. We observe that it is a waste if the search algorithm processes two different paths further,  $ABEF$  and  $ADEF$ , since searching two different paths will not improve the path quality of searching only one of them.

We first introduce the concept of state and dominance, then we describe the A\*\_MCSP algorithm. State and dominance are also important components in designing FringeMCSP in Section VI.

##### A. State and Dominance

The maintenance cost increases by maintaining partial paths formed from equivalent partial paths, without improving the quality of the searched paths. It suffices to remember the quality of paths that have reached a node. We use the *state* notation to represent a state in the search space in our A\* extension to MCSP, rather than *path* as in A\*Prune. In the following, when we mention a state, we mean the state in the state notation we use. A state is a tuple,

$$state = \{node, hop, estimate, W, parent\},$$

where  $node$  is the node in question,  $hop$  records the number of hops from the source to this state,  $estimate$  is the sum of  $hop$  and the look-ahead of  $node$ ,  $W$  is the weight vector

<sup>2</sup>To save space, we do not show the results for Rocketfuel and random topologies, for pair-wise comparisons, i.e., one algorithm versus another. The average search time over all instances for each topology can be found in Figure 9 and 10.

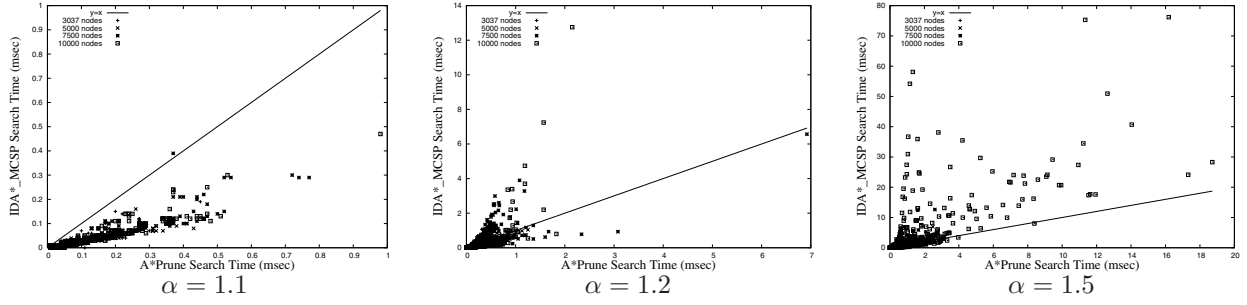


Fig. 4. Search Process Time (msec), Infeasible Case: A\*Prune ( $x$ -axis) vs. IDA\*\_MCSP ( $y$ -axis)

containing the weight values accumulated so far, and *parent* is a backward pointer to the state that leads to the current state.

Using the state notation, both path  $ABE$  and path  $ADE$  can be encoded as a state,  $S_E = \{E, 2, 4, \{3, 4\}, S_B\}$ , with *hop* and *estimate* as 2 and 4 respectively. Here we have the weight vector as  $\{3, 4\}$ . The *parent* field is set as  $S_B$ , the state notation of node  $B$ . We store the parent that gives the state the best quality. The quality of a state is measured by both the hop number and the accumulated weights. A state of the best quality dominates all the other states (of the same node). This suffices to find a single optimal path.

We have made the search space compact by using state notation. We can check dominance relation of two states of the same node,  $p$ , to make further enhancements.  $S_p^2 = \{p, hop_2, estimate_2, W_2, parent_2\}$  is dominated by  $S_p^1 = \{p, hop_1, estimate_1, W_1, parent_1\}$ , if

$$hop_1 \leq hop_2 \text{ and } W_1 \leq W_2,$$

where  $W_1 = \{w_{1,1}, w_{1,2}, \dots, w_{1,m}\}$ , and  $W_2 = \{w_{2,1}, w_{2,2}, \dots, w_{2,m}\}$ . Thus  $W_1 \leq W_2$  is defined by  $w_{1,i} \leq w_{2,i}, \forall 1 \leq i \leq m$ . Note that  $hop_1 \leq hop_2$  implies  $estimate_1 \leq estimate_2$ , since  $S_p^1$  and  $S_p^2$  are of the same node  $p$ , thus they have the same look-ahead value for hop count. It is unnecessary to further search a dominated state, since it is impossible for it to lead to a better solution than the state that dominates it. Discarding dominated states can prevent cycles, since a state associated with a path containing a cycle is dominated by a state for the path removing the cycle. Thus a search algorithm with this feature does not need to remove cycles explicitly. Note that a dominance relation is necessary for discarding a state. That is, we can not discard a state  $S$  which is not dominated by any other state, even if one or several of its constraints are much larger than that of the other states. In MCSP, such an  $S$  may lead to a better path (with respect to hop count) than the others.

In the above example, we set *parent* field as  $S_B$ , but not  $S_D$ , because a possible state  $S'_E = \{E, 2, 4, \{3, 4\}, S_D\}$  via path  $ADE$  is dominated by  $S_E = \{E, 2, 4, \{3, 4\}, S_B\}$  via path  $ABE$  (in fact, they have the same state quality).

We also conduct eligibility tests on a possible state expansion to a neighbor, by checking whether any constraint will be violated using look-ahead values of weights. In Figure 1 in Section II-A, state  $\{D, 1, 4, \{1, 2\}, S_A\}$  won't be expanded

to state  $\{G, 2, 6, \{3, 4\}, S_D\}$ , since the weight metric  $w_1$  evaluated for node  $D$  is  $4 + 6 = 10$ , which violates constraint  $c_1 = 7$  ( $c_2$  is also violated).

We observe that the number of states will not exceed the number of partial paths. Free of path operations is another benefit of the state approach, which maintains backward pointers to construct the entire path after it is found. However, it needs to check dominance relationship between states.

### B. A\*\_MCSP

Figure 5 presents the function to check whether a state  $s'$  newly created for *node* is dominated by an old state stored for *node*. Figure 6 presents the main algorithm of A\*\_MCSP, our extension of A\* to MCSP. At the initialization stage, *computeLowerbound()* in Figure 2 calculates the lower bounds for the hop count and each link weight. We use *eligible()* in Figure 3 to make eligibility tests.

The search process from Line 4 to Line 14 of A\*\_MCSP() in Figure 6 finds the shortest constrained path. A\*\_MCSP has an *openlist* and a *closedlist*. New states are put into *openlist* for further expansion. On the *openlist*, states are sorted in increasing order by the *estimate* values, breaking ties with *hop* values, and giving preference to larger *hop* values. If a neighbor can not pass the eligibility tests, A\*\_MCSP will not further consider it. When constructing a new state for an eligible neighbor, A\*\_MCSP updates *estimate* using the look-ahead information. It also updates the weight vector  $W$  (plus the constraints on the link) and *hop* (plus 1). Function *isDominated(s)* checks whether state  $s$  is dominated by any state on *openlist* or *closedlist*. We use an efficient data structure to implement the *openlist* for A\*\_MCSP. We use a 2-dimensional bucket to store the states, with the assistance of an array and a variable to query proper state to expand.

```

1: isDominated( $s'$ )
2:   $node \leftarrow$  node field of  $s'$ ;
3:   $S \leftarrow$  all stored states of  $node$ ;
4:  for each  $s \in S$ 
5:    if  $s < s'$  return true;
6:  return false;

```

Fig. 5. Function: *isDominated(s')*

As the performance study will show, in general, A\*\_MCSP outperforms A\*Prune. However, in most feasible cases,

```

1: A*_MCSP()
2:   computeLowerbound();
3:   put state {src, 0, lb[hop][src], 0, null} into openlist;
4:   while openlist not empty
5:     s ← first state in openlist;
     /*take s away from openlist*/
6:     put s into closedlist;
7:     if s.node is dst
8:       construct path;
9:       return true;
10:    for each neighbor nbr of s.node
11:      if eligible(s.node, nbr, curW)
12:        construct state s' for nbr;
13:        if isDominated(s') is false
14:          put s' into openlist in order;
15:    return false;

```

Fig. 6. Algorithm: A\*\_MCSP()

```

1: FringeMCSP()
2:   computeLowerbound();
3:   put state {src, 0, lb[hop][src], 0, null} into laterlist;
4:   depth_bound ← lb[hop][src];
5:   done ← false
6:   while done is false
7:     nowlist ← laterlist
8:     while nowlist not empty and done is false
9:       s ← first state in nowlist;
       /*take s away from nowlist*/
10:      done ← expand(s);
11:      if done is false
12:        update depth_bound;
13:        if reach stopping condition
14:          report failure;
15:      else return true;

```

Fig. 8. Algorithm: FringeMCSP()

IDA\*\_MCSP is much more efficient than A\*\_MCSP. It is interesting to investigate whether there is still room to improve A\*\_MCSP in both feasible cases and infeasible cases. Fringe search is worth exploiting.

## VI. FRINGEMCSP

Björnsson et al. [4] recently introduce Fringe search to attempt to overcome the shortcomings of A\* and IDA\*, namely, sorting *openlist* and redundant search. We extend Fringe search to design an exact MCSP algorithm, FringeMCSP. It searches iteratively according to a threshold like IDA\*\_MCSP. It uses the state notation and dominance check like A\*\_MCSP. It uses *eligible()* in Figure 3 to make eligibility test. It uses *isDominated()* in Figure 5 to check dominance relationship of states of a node.

```

1: expand(s)
2:   if s.f > depth_bound
3:     insert s at the end of laterlist;
4:     return false;
5:   for each neighbor nbr of s.node
6:     if eligible(s.node, nbr, curW)
7:       if nbr is dst
8:         construct path;
9:         return true;
10:    construct state s' for nbr;
11:    if isDominated(s') is false
12:      if s'.f < depth_bound
13:        put s' into nowlist;
14:      else
15:        put s' into laterlist;
16:    return false;

```

Fig. 7. Function: expand(s)

Figure 8 presents the FringeMCSP algorithm. It keeps *nowlist* and *laterlist* of states. It initializes the lowerbounds for *hop* and each of the weight metrics. In each iteration, the algorithm searches on *nowlist*, and put states to be considered later on *laterlist*. The head state of *nowlist* is removed from the list and examined using the *expand()* function in Figure 7. We use *s.node* to denote the node of *s*, and *s.f* to denote the estimate distance of state *s*, which is the sum of the distance traversed so far and the look-ahead value of hop count. If the estimate distance is greater than the depth threshold, then the head state is put into *laterlist*. Otherwise, for each eligible neighbors, construct the state, and check its

dominance relationship with states of the same node. For an un-dominated state, if the estimate distance is less than the depth threshold, put it in *nowlist*<sup>3</sup>; otherwise, put it into *laterlist* to consider in the next iteration. When *nowlist* is empty, update the depth threshold, copy *laterlist* to *nowlist*, and clear *laterlist*. Search on *nowlist* iteratively, until the goal is found or it is determined that the goal is not reachable.

The *search process* from Line 6 to Line 15 of FringeMCSP() in Figure 8 finds the shortest constrained path.

Take the example in Figure 1 in Section II-A again. FringeMCSP searches in a similar way as IDA\*\_MCSP does, with the exception that FringeMCSP won't search node *A*, *B* and *E* for the redundantly second time.

## VII. PERFORMANCE STUDY

The experiments are set up similarly to Section IV. We present the average and the confidence interval of the search time. Figure 9 shows that for infeasible cases, FringeMCSP has a good performance. IDA\*\_MCSP may not perform well in infeasible cases, due to the inaccurate look-ahead information, which causes lots of redundant search. FringeMCSP makes a good tradeoff. It uses the depth bound to restrict the search depth at each iteration. It uses *nowlist* and *laterlist* to avoid redundant search. It does not need to sort states on the lists.

Figure 10 shows that in feasible cases, IDA\*\_MCSP is the most efficient. The rank is, IDA\*\_MCSP, FringeMCSP, A\*\_MCSP and A\*\_Prune. FringeMCSP is close to IDA\*\_MCSP. For feasible cases, IDA\*\_MCSP is fast due to the accurate look-ahead information and the iterative deepening search. As a result, IDA\*\_MCSP does not do much redundant search. As well it does not need to maintain partial paths or states and their ordering.

In short, for a feasible MCSP problem, IDA\*\_MCSP is the best choice. For a MCSP problem uncertain about its feasibility, FringeMCSP may be a good option.

## VIII. CONCLUSIONS

We deploy search techniques to design fast exact algorithms for MCSP, which was shown as NP-hard. We use highly

<sup>3</sup>If the children states are put into the front of *nowlist* in order of their expansion, FringeMCSP mimics IDA\*\_MCSP, except for the elimination of redundant search. We follow this approach in implementation.

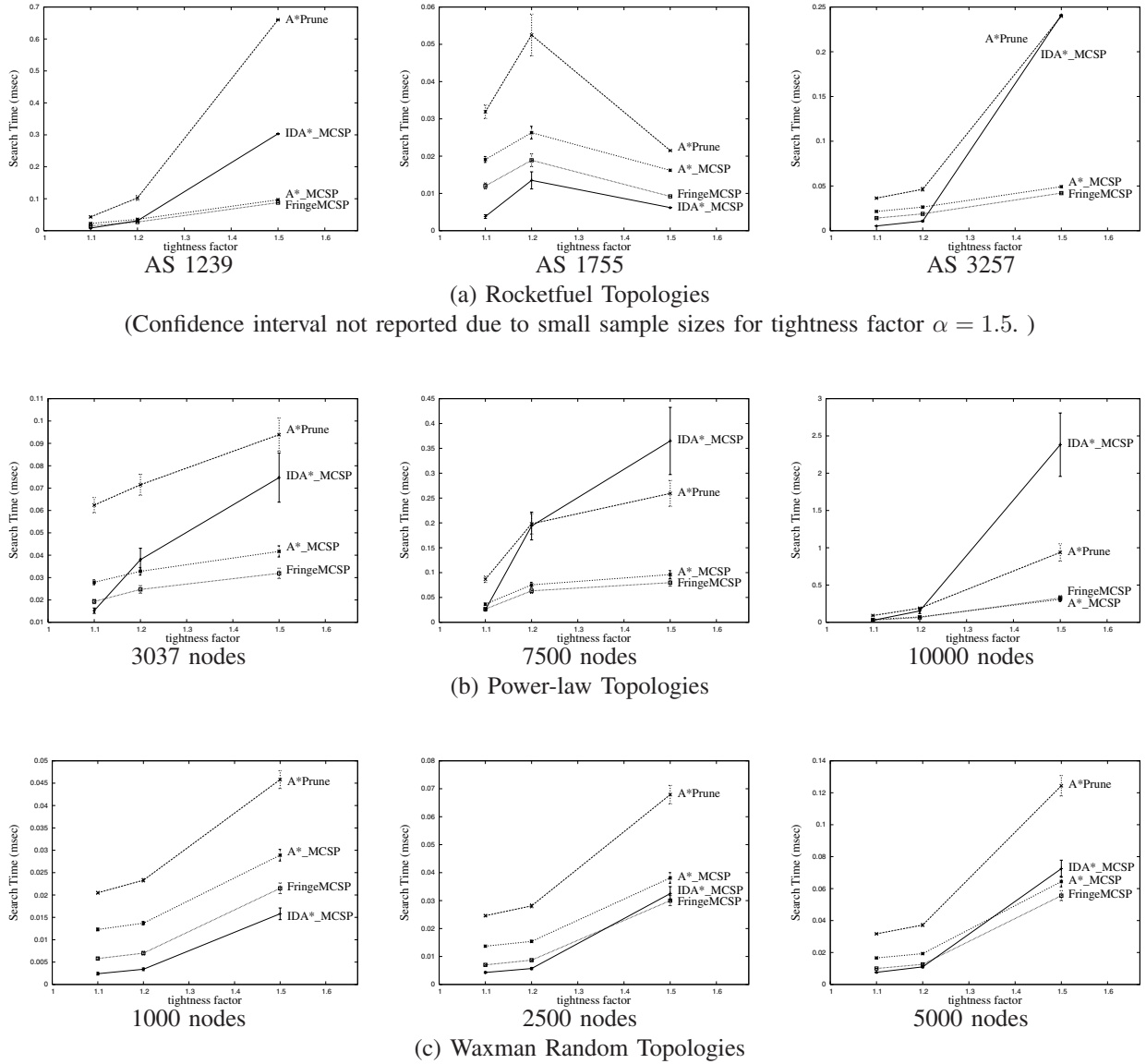


Fig. 9. Average Search Process Time (msec) with 95% Confidence Interval, Infeasible Case

accurate look-ahead information to predict path length, and use eligibility test to prune unnecessary search space.

We show that for infeasible cases, IDA\*\_MCSP [14] may not be as efficient as A\*Prune. This motivates us to design an algorithm that is efficient in both feasible and infeasible cases.

We design an A\*-based exact MCSP algorithm, A\*\_MCSP, which introduces the state notion and dominance relationship between states. Furthermore, we design an exact MCSP algorithm FringeMCSP. It can be regarded as an integration of IDA\*\_MCSP and A\*\_MCSP. Extensive empirical study shows that FringeMCSP has good performance in both feasible and infeasible cases; while IDA\*\_MCSP still shows its superiority among the proposed MCSP algorithms in feasible cases.

The longer version of this paper [15] provides more technical details and experimental results.

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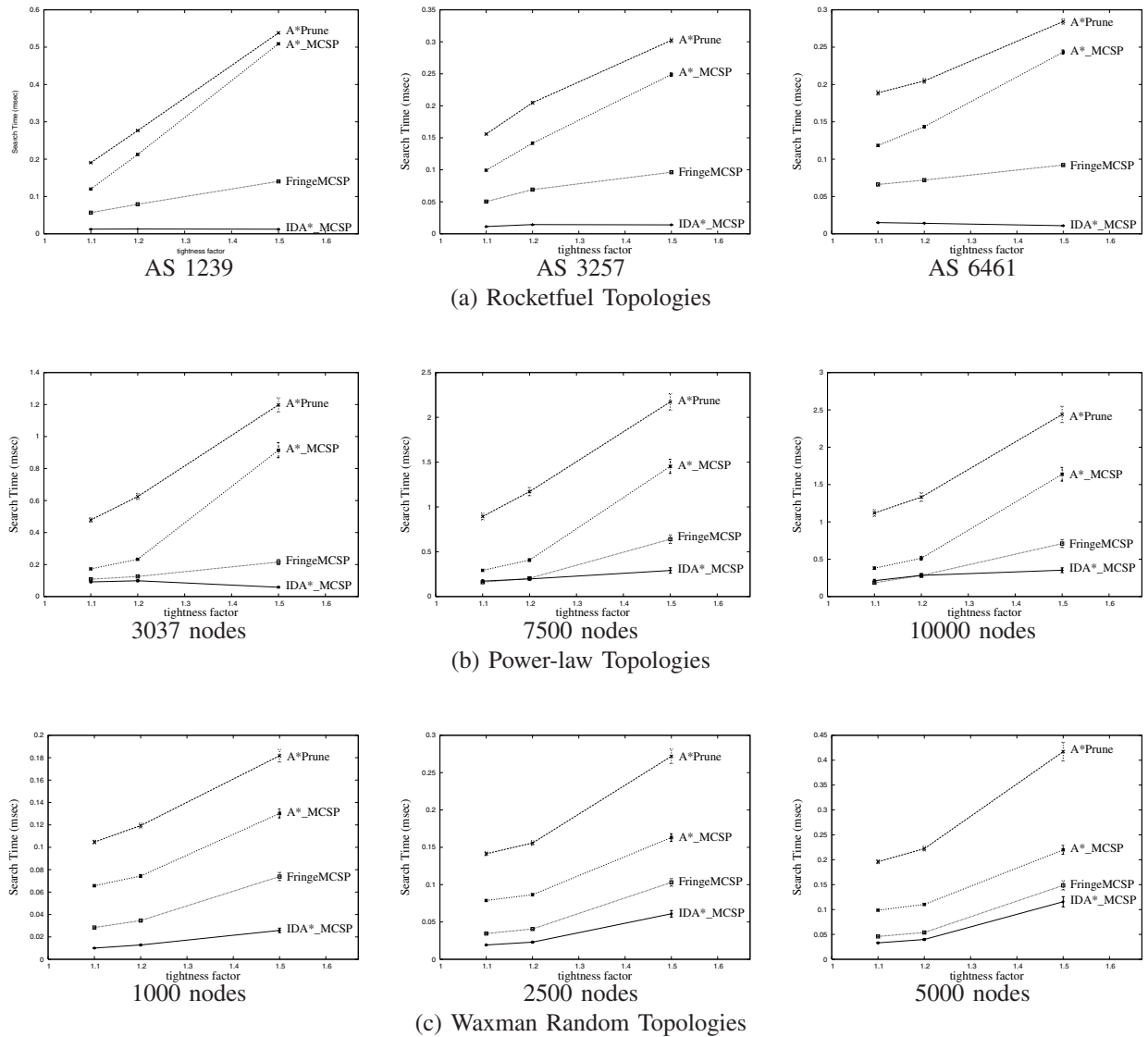


Fig. 10. Average Search Process Time (msec) with 95% Confidence Interval, Feasible Case

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