Playing Reverse Hex

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Jan 2011
proofs: allow holey boards
color symmetry
Some Complete Strategies

move only if $\geq 2$ empty cells

... so can end in draw
TRex from psn P with empty cell c
player Z = X or Y:

- X wins $(P + Y(c), Z)$ $\implies$ X not-loses $(P, Z)$
- Y wins $(P, Z)$ $\implies$ Y not-loses $(P + Y(c), Z)$
Lemma

TRex from psn P with empty cell c
player Z = X or Y:

• X wins \((P + Y(c), Z)\) \iff X not-loses \((P, Z)\)
• Y wins \((P, Z)\) \iff Y not-loses \((P + Y(c), Z)\)
Lemma

TRex from psn P with empty cell c
player Z = X or Y:

- X wins \((P + Y(c), Z)\) \(\iff\) X not-loses \((P, Z)\)
- Y wins \((P, Z)\) \(\iff\) Y not-loses \((P + Y(c), Z)\)
Theorem

TRex from color-symmetric psn P with \( \geq 2 \) empty cells:

- player to-move not-loses
- player not-to-move not-loses
Theorem

TRex from color-symmetric psn $P$ with $\geq 2$ empty cells:

- player to-move not-loses
- player not-to-move not-loses
Theorem

TRex from color-symmetric psn P with $\geq 2$ empty cells:

- player to-move not-loses
- player not-to-move not-loses
Some Complete Strategies

to-move not-loses

pf by cont'n
TRex
Rex
Some Complete Strategies

not-to-move not-loses

pf by cont’n
Theorem

Rex color-symmetric P $k$ empty cells:

- $k \geq 2 \implies$ exists non-losing move
- $k$ even $\implies$ 1st player not-loses
- $k$ odd $\implies$ 2nd player non-loses

* wins if P has no holes
Rex color-symmetric P $k$ empty cells:

- $k \geq 2 \iff$ exists non-losing move
- $k$ even $\iff$ 1st player not-loses
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Theorem

Rex color-symmetric P k empty cells:

• $k \geq 2 \implies$ exists non-losing move
• $k$ even $\implies$ 1st player not-loses*
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Theorem

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Theorem

Rex color-symmetric P \( k \) empty cells:

- \( k \geq 2 \implies \) exists non-losing move
- \( k \) even \( \implies \) 1st player not-loses\(*
- \( k \) odd \( \implies \) 2nd player non-loses\(*

\* wins if P has no holes
Lemma

Rex color-symmetric P \ k empty cells:

• \( k \geq 2 \) even:
  \( X \text{ wins } (P, Y) \iff \forall \text{ empty } c, X \text{ wins } (P + X(c), Y) \)

• \( k \geq 3 \) odd:
  \( X \text{ wins } (P, X) \iff \forall \text{ empty } c, X \text{ wins } (P + X(c), X) \)
Lemma

Rex color-symmetric P \( k \) empty cells:

- \( k \geq 2 \) even:
  \( X \) wins \((P, Y) \) \( \iff \) \( \forall \) empty \( c \), \( X \) wins \((P + X(c), Y) \)

- \( k \geq 3 \) odd:
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Lemma

Rex color-symmetric P $k$ empty cells:

- $k \geq 2$ even:
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- $k \geq 3$ odd:
  \[ X \text{ wins } (P, X) \iff \forall \text{ empty } c, X \text{ wins } (P + X(c), X) \]
Theorem

Rex $n \times n$ P $2t$ empty cells
long diagonal color-symmetry
c in corner empty wedge and X-peripheral:

- X not-loses $(P + X(c), Y)$
Theorem

Rex \( n \times n \) P \( 2t \) empty cells
long diagonal color-symmetry
c in corner empty wedge and X-peripheral:

- X not-loses \((P + X(c), Y)\)
X not-loses \((P + X(c), Y)\) pf by cont'n
some winning $2t \times 2t$ openings
Theorem

Rex $n \times n$ $P$

$P$ color-symmetric

$2t + 1$ empty cells

$P' = P + X(c)$:

- $c$-reflection $d \neq c \implies Y$ not-loses $P' + Y(d)$

- long diagonal symmetry, $d$ $Y$-peripheral in empty corner wedge of $P' \implies Y$ not-loses $P' + Y(d)$
Theorem

Rex \( n \times n \) P

P color-symmetric

2t + 1 empty cells

\( P' = P + X(c) \):

- \( c \)-reflection \( d \neq c \implies \)
  Y not-loses \( P' + Y(d) \)

- long diag’l symmetry, \( d \) Y-peripheral in empty corner wedge of \( P' \)
  \( Y \) not-loses \( P' + Y(d) \)
Theorem

Rex \( n \times n \) P

P color-symmetric

2\( t + 1 \) empty cells

\( P' = P + X(c) \):

- c-reflection \( d \neq c \implies Y \) not-loses \( P' + Y(d) \)

- long diag'l symmetry, \( d \) Y-peripheral in empty corner wedge of \( P' \implies Y \) not-loses \( P' + Y(d) \)
Theorem

Rex $n \times n$ P
P color-symmetric

$2t + 1$ empty cells

$P' = P + X(c)$:

• c-reflection $d \neq c \implies Y$ not-loses $P' + Y(d)$

• long diag’l symmetry, $d$ Y-peripheral in empty corner wedge of $P' \implies Y$ not-loses $P' + Y(d)$
Theorem

Rex \( n \times n \) P  
\[ P \text{ color-symmetric} \]
\[ 2t + 1 \text{ empty cells} \]
\[ P' = P + X(c) : \]

• c-reflection \( d \neq c \implies \)
Y not-loses \( P' + Y(d) \)

• long diag’l symmetry, \( d \) Y-peripheral
in empty corner wedge of \( P' \implies \)
Y not-loses \( P' + Y(d) \)
some winning $3 \times 3$ replies
some winning $5 \times 5$ replies
a $4 \times 4$ example

black-to-play wins from here so ...
a $4 \times 4$ example

...black-to-play wins from here
Some Complete Strategies

2×2

1st-player win

e.g. force 2nd-player this way
Some Complete Strategies

3 × 3

2nd-player win

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Some Complete Strategies

4 × 4

1st-player win

Playing Reverse Hex
4 × 4

another way to win
5 × 5

2nd-player win

Playing Reverse Hex
Some Complete Strategies

2×2

all winning openings
3×3
all winning replies

TRex
Rex
Some Complete Strategies

Playing Reverse Hex
Some Complete Strategies

4 × 4 all (but 2?) winning openings

Playing Reverse Hex
thanks to

- NSERC Alberta Ingenuity
- UofA GAMES UofA Hex
- M Mueller J Schaeffer