Pivot and Gomory Cut

A MIP Feasibility Heuristic

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NSERC

problem

- **given a MIP, find a feasible solution**

MIP

- **mixed integer program**
- **min cx s.t. $Ax \geq b, \;\; u \leq x \leq l$**
  - **A,b,c,u,l integer**
  - **some x_j integer**

Gomory cut

$$\sum_{k\in N_Z:f_k\le f_j} f_k x_k + \sum_{k\in N_Z:f_k>f_j} \frac{f_j(1-f_k)}{1-f_j} x_k + \sum_{k\in N_*:a_{jk}\le 0} \frac{-f_j a_{jk}}{1-f_j} x_k + \sum_{k\in N_*:a_{jk}>0} a_{jk} x_k \quad \ge \quad f_j$$

- **wrt basic feasible x^* of simplex tableau**
- $x_j^* = x_j + \sum_{k\in N} a_{jk} x_k$ — **row corresponding to x_j**
- B**,**N — **basic,non-basic variables**
- N_Z**,**N_* — **integer,continous non-basic variables**
- j — **index of infeasible basic variable**
- f_k — $a_{jk} - \lfloor a_{jk} \rfloor$
- f_j — $x_j^* - \lfloor x_j^* \rfloor$

Gomory's algorithm

- **solve LP relaxation (simplex alg'm)**
- **while not done**
  - **add Gomory cut to formulation**
  - **(continue with dual simplex pivots)**

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performance?

- **correct**
- **finite**
- **inefficient (cuts usually not deep)**

Gomory's alg'm

- **tuned for optimum guarantee**
- **reoptimize after each cut added**
- **dual simplex pivots**
- **cuts do not guide pivoting**

Gomory's alg'm

- tuned for optimum guarantee
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- dual simplex pivots
- cuts do not guide pivoting

Pivot and Gomory Cut

- tuned for heuristic performance
- no reoptimization after adding cut
- primal simplex pivots
- cuts guide pivots/restarts

Pivot and Gomory Cut: toy example

$$\min\ [1\ \ 1]\ \boldsymbol{x} \quad \textbf{s.t.} \quad \begin{bmatrix} 6 & 4 \\ -3 & 4 \\ -3 & -4 \end{bmatrix} \boldsymbol{x} \ \geq \ \begin{bmatrix} 9 \\ -3 \\ -18 \end{bmatrix}$$

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integer: $\{x_1, x_2\}$ **slack/continuous:** $\{x_3, x_4, x_5\}$

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$$\begin{bmatrix} c \\ x_1 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 19/12 \\ 4/3 \\ 1/4 \\ 13 \end{bmatrix} + \begin{bmatrix} 7/36 & 1/18 \\ 1/9 & -1/9 \\ 1/12 & 1/6 \\ -2/3 & -1/3 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

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fantastic pivot? $x_1 \longleftrightarrow x_4$

after pivot: $\boldsymbol{x} = e_2 = (0, 9/4)$ **. . . infeasible**

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crossed Gomory cut G_1, so slack variable x_6

$$\begin{bmatrix} c \\ x_2 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 9/4 \\ 9/4 \\ 12 \\ 9 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/2 & 1/4 \\ -3/2 & 1/4 \\ -9 & 1 \\ 3 & -1 \\ -1 & 1/6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$$

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x_2 **cut:** $\frac{1}{6}x_1 + \frac{1}{12}x_3 \geq \frac{1}{4}$

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good pivot? $x_5 \longleftrightarrow x_3$

after pivot: $\boldsymbol{x} = e_3 = (0, \frac{9}{2})$ **. . . infeasible**

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fantastic pivot? no

good pivot? $x_7 \longleftrightarrow x_5$

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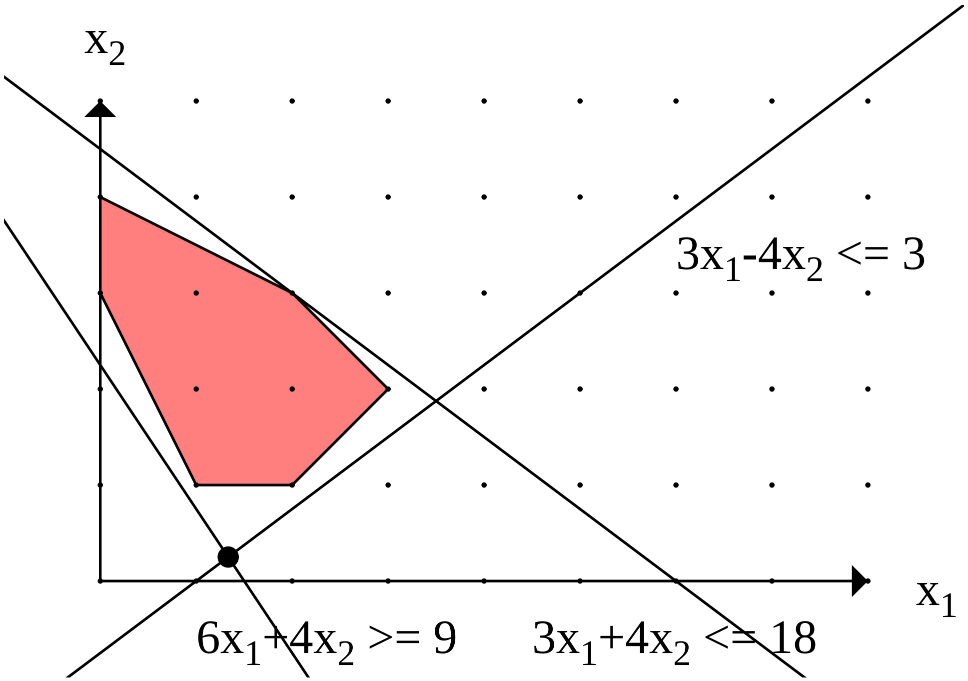
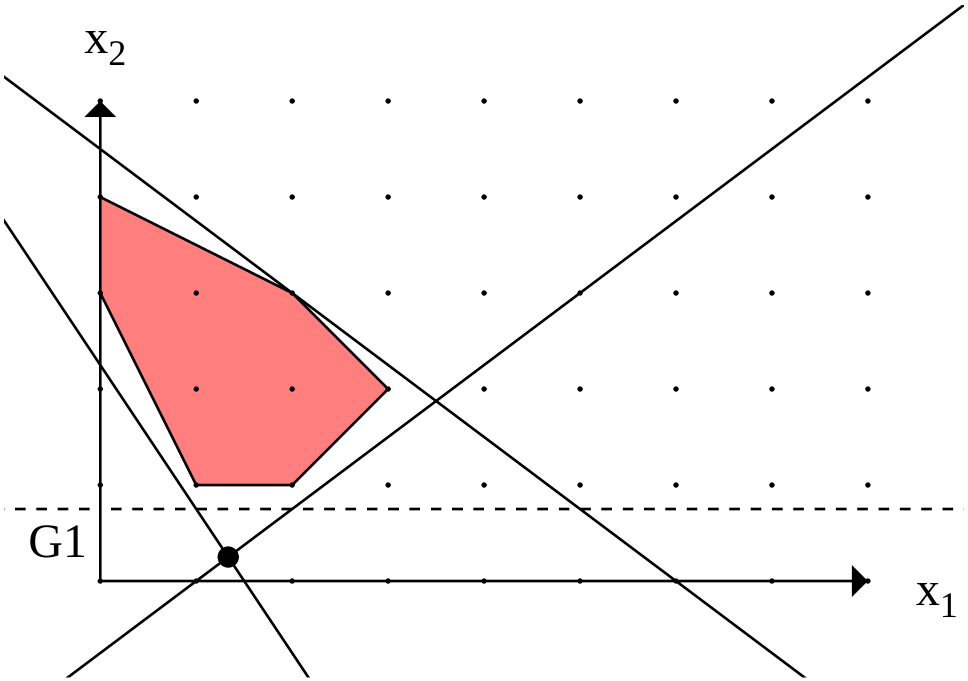
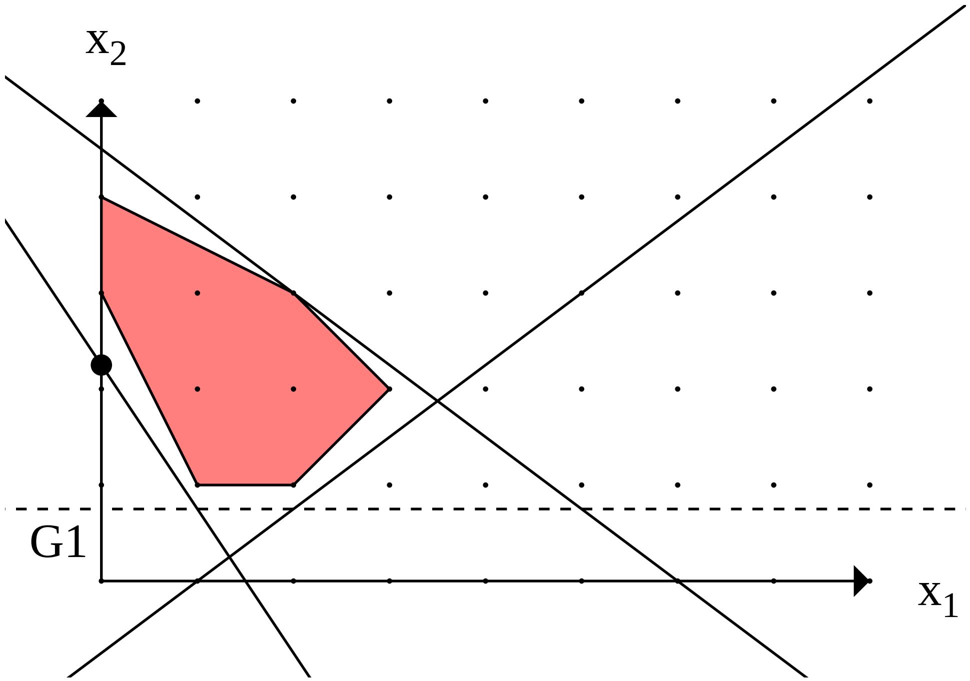
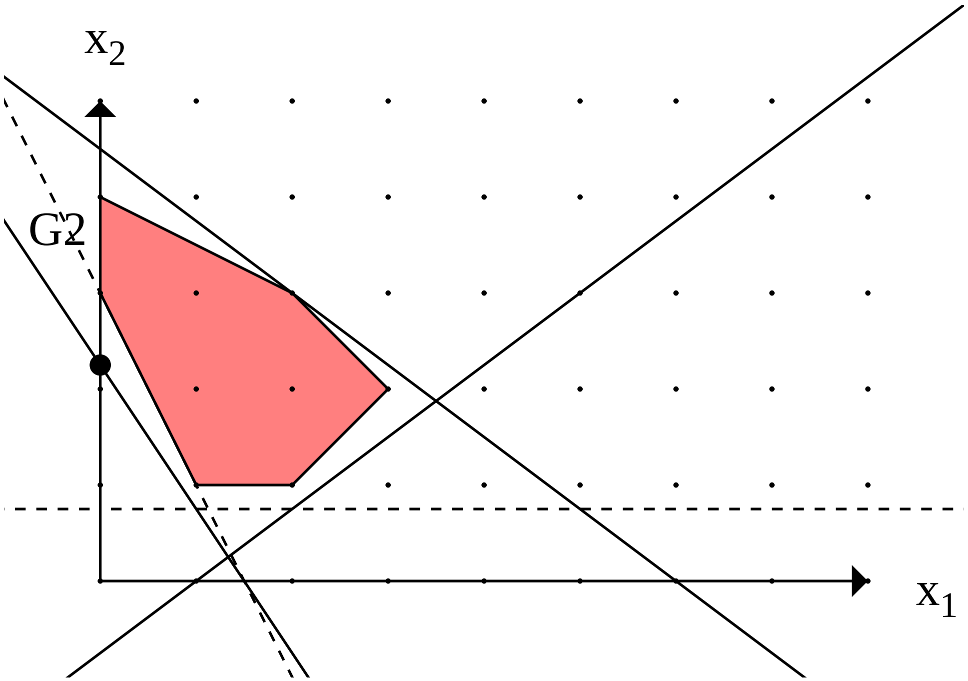
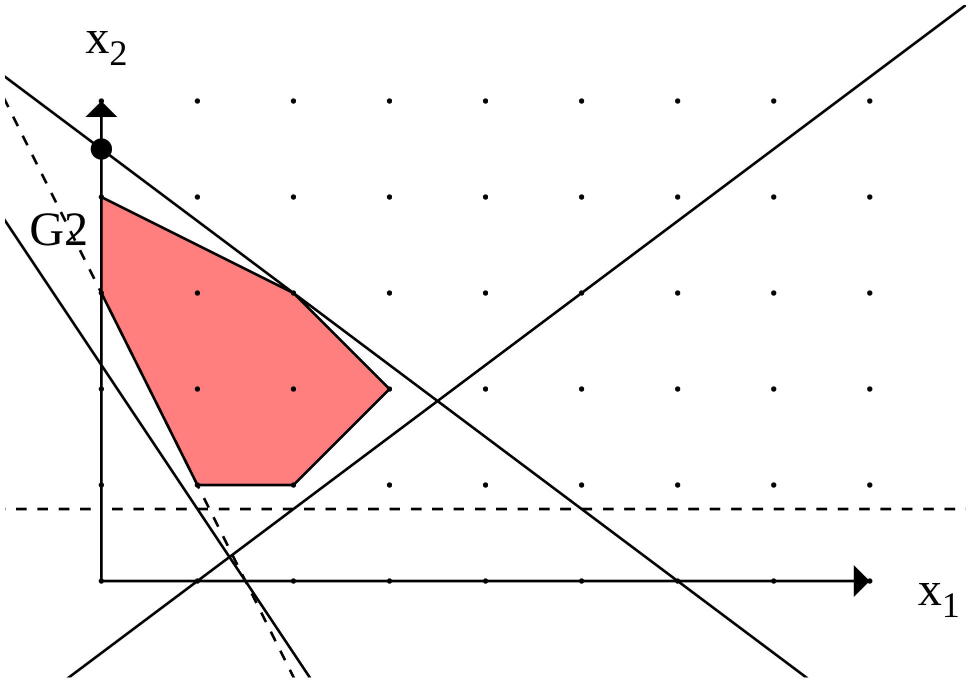
after pivot: $\boldsymbol{x} = e_4 = (0, 3)$ **. . . feasible**

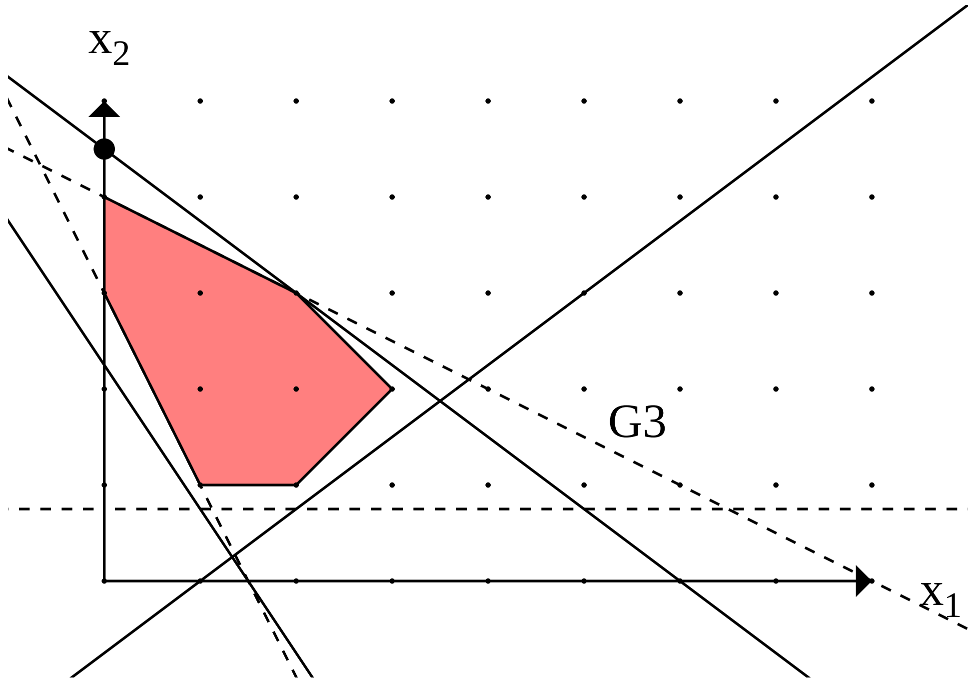

x2
3x1-4x2 <= 3
x1
6x1+4x2 >= 9
3x1+4x2 <= 18

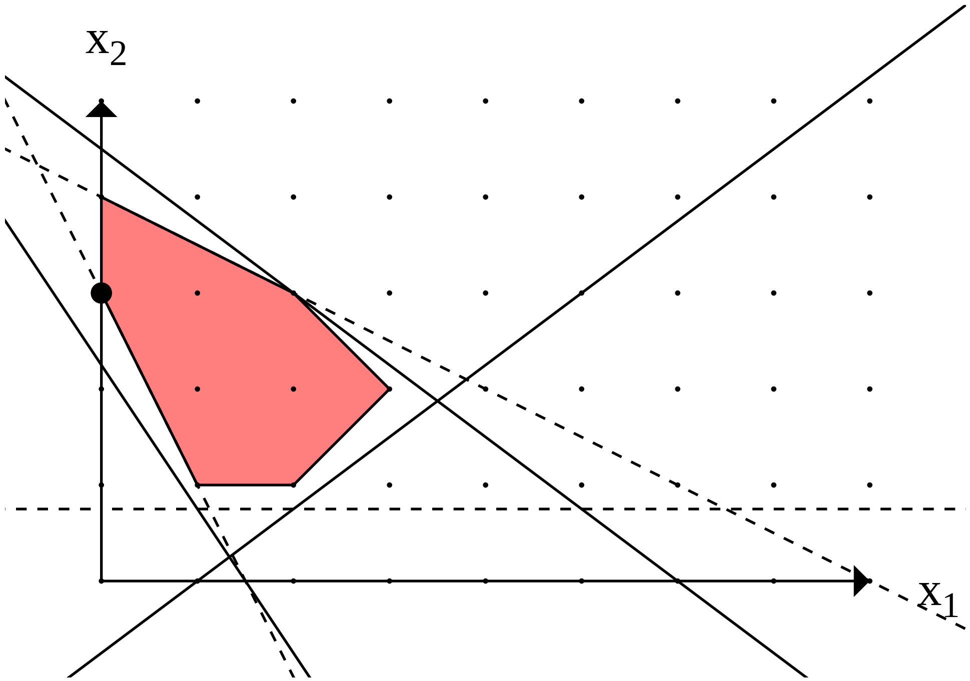

x2
G1
x1


x2
G1
x1


x2
G2
x1


x_2
G2
x_1


x_2
G3
x_1


x_2
x_1

Pivot and Gomory cut

- **solve LP relaxation (simplex alg'm)**
- **compute Gomory cut**
- **repeat**
  - **if fantastic pivot then pivot**
    if cut crossed, add to formulation and compute new cut
  - **else if good pivot then pivot**
    if cut crossed, add to formulation and compute new cut
  - **else cross cut (via temporary objective function)**
    add to formulation and compute new cut

background

53 Dantzig	simplex algorithm
58 Gomory	Gomory cuts
73 Chvátal	Chvátal-Gomory cuts
80 Balas Martin	pivot and complement
88 Bixby Lowe	Cplex
96 Balas Ceria Cornuéjols Natraj	Gomory cuts revisited
05 Fischetti Glover Lodi	feasibility pump
05 Ghosh-H	pivot and Gomory cut

comparing MIP feasibility heuristics

- **algorithms:** **PC, PGC, Cplex, FP**
- **data set 1:** **MIPLib et al. (77 instances)**
- **data set 2:** **randomly generated (500 instances)**

feasibility-hard market share instances

$$\min \sum_{i=1}^{m} s_i \quad s.t. \quad \sum_{j=1}^{n} a_{ij} x_j + s_i = b_i$$

$$\sum_{j=1}^{n} a_{ij} x_j - s_i = b_i \quad i = (m_1 + 1), ..., m$$

$$x_j \in \{0, 1\} \;\; [j = 1, ..., n] \qquad s_i \;\geq\; 0 \;\; [i = 1, ..., m]$$

if a_{ij} **uniform integer [0,99]**

$$b_i = \lfloor 0.5 \times \sum_{j=1}^{n} a_{ij} \rfloor$$

$$p = 0.5 \qquad m = \lfloor \tfrac{n}{1.5} \rfloor$$

$m_1 = \lceil pm \rceil$ **when** $0 > p \geq 0.5$

$m_1 = \lfloor pm \rfloor$ **when** $0.5 < p < 1$

then **hard with high probability**

feasible with high probability

77 MIPLib et al. instances

- **Cplex $>$ PGC,FP $>>$ PC**
- **Cplex LP solver $>>$ PGC/FP LP solver (Gnu LPK)**

77 MIPLib et al. instances

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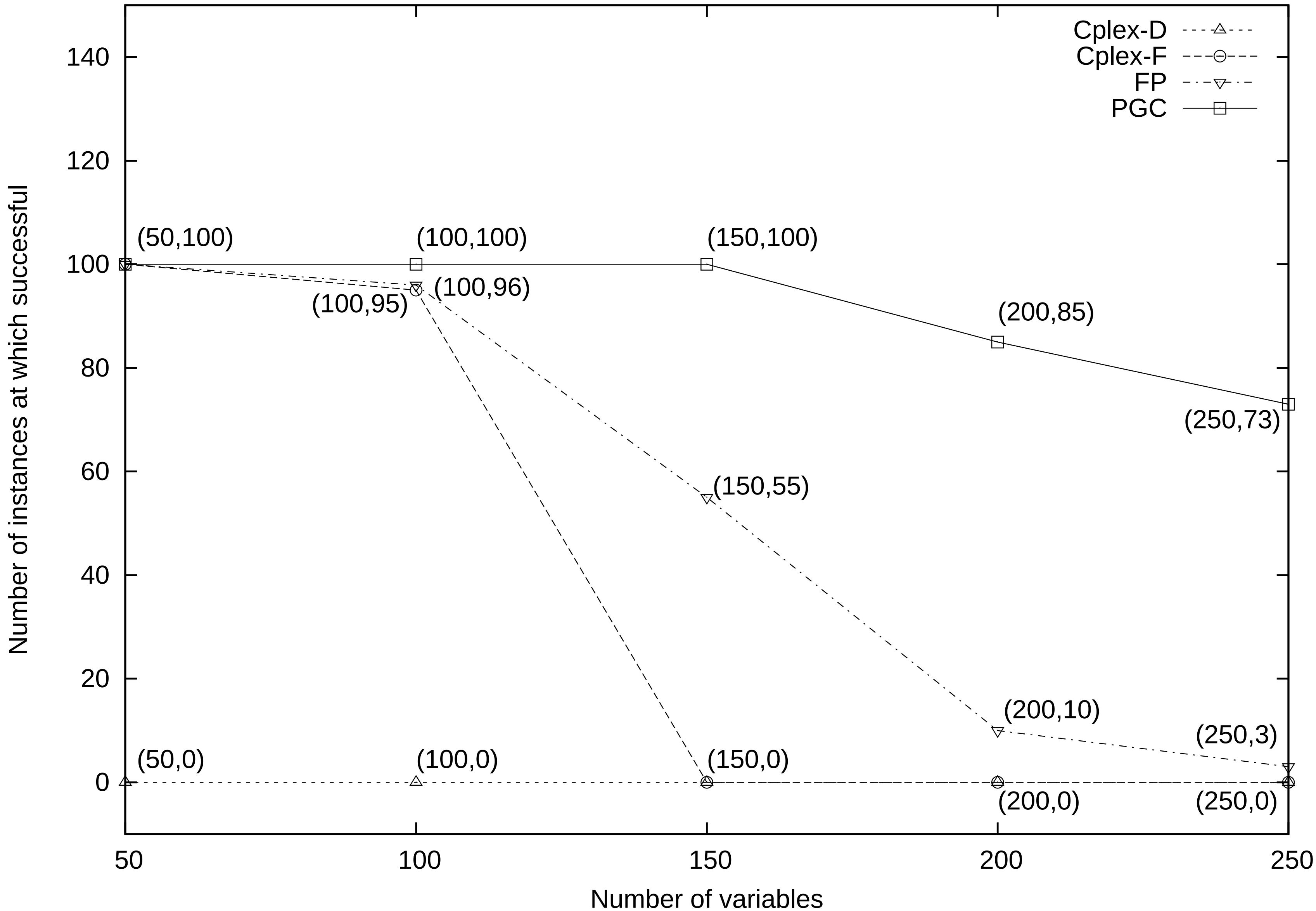
500 randomly generated instances

- **PGC >> FP >> Cplex**


Cplex-D
Cplex-F
FP
PGC
(50,100)
(100,100)
(150,100)
(100,95)
(100,96)
(200,85)
(250,73)
(150,55)
(200,10)
(250,3)
(50,0)
(100,0)
(150,0)
(200,0)
(250,0)
140
120
100
80
60
40
20
0
50
100
150
200
250
Number of instances at which successful
Number of variables

ARIGATO !