TWIST AND TURN: THE STORY OF Hex

THE CLASSIC 2-PLAYER CONNECTION GAME

hayward@ualberta.ca

computing university of alberta

2011 march
1. PRE-

2. EARLY

3. MID

4. RECENT
THANK YOU

- joint: Arneson, Henderson, Toft [Rex]
- also van Rijswijck, Bjönnson, Johanson, Kan
- also UofA GAMES, Schaeffer, Müller, Stewart
- Natural Sciences and Engineering Research Council of Canada
PREHISTORY

- 15,000 years ago, last ice age ends
- 10,000 years ago, agriculture
  - hunt/gather → reap/sow
  - stories around fire → games around hearth
PREHISTORY

- 5,000 years ago: pre-??? Iraq
- 4,000 years ago: pre-go China
- 3,400 years ago: pre-checkers Egypt
- 1,500 years ago: pre-chess India
1736 Euler Königsberg bridge
1736 Euler Königsberg bridge
1750 Euler polyhedron $V - E + F = 2$
1840 Möbius 5 princes
GRAPH THEORY: 4-COLOR [TO 1940]

1840 Möbius 5 princes

K5 not planar
1852 Guthrie → brother → de Morgan
1879 Kempe ‘proof’
1890 Heawood: Kempe counterexample, 5 color theorem
...
1940 known for maps with ≤ 35 regions
HEIN’S GAME

RULES

- 2 players, alternate moves
- win: connect your two sides
HEIN’S GAME

RULES

- 2 players, alternate moves
- win: connect your two sides
Hein’s inspiration

Game criteria

- fair
- progressive
- final
- easy to comprehend
- strategic
- decisive
- no explicit math’l sol’n
- !!! 2 opposing-side chords of quadrilateral must cross
- no draw ⇒ hexagonal grid
- balanced/equal
- no state repetition
- finite
- clear state evaluation
- no draw
- not easy to solve

1942 Hein game
1948 Nash game
1952 Shannon computer
1957,58 Gardner columns
Hein’s inspiration

1942 Hein game
1948 Nash game
1952 Shannon computer
1957, 58 Gardner columns

Hayward@ualberta.ca
Twist and Turn: The story of Hex
Hein’s inspiration

- 1942 Hein game
- 1948 Nash game
- 1952 Shannon computer
- 1957, 1958 Gardner columns

Hayward@ualberta.ca

Twist and Turn: The Story of Hex
Hein’s inspiration
Hein’s inspiration

Outline
Pre-
Early
Mid
Recent

1942 Hein game
1948 Nash game
1952 Shannon computer
1957,58 Gardner columns

Hayward@ualberta.ca

Twist and turn: the story of Hex
**Hein’s columns**

**26. December 1942**

**Vil De lære Polygon?**

Piet Hein har konstrueret et Spil, der med lige stor Glæde kan dyrkes af Skakeksperten og den, der blot kan holde en Blyant

„Politiken“ udskriver i Dag en Præmieopgave, der vil volde Hovedbrud for Begyndere

—

繁荣 can afryde forbindelsen ved at besætte det mellemliggende Felt, derved anvendelse af den i Forklaringen noterede Regel.

**1942 Hein game**

1948 Nash game

1952 Shannon computer

1957,58 Gardner columns

Hayward@ualberta.ca

Twist and turn: the story of Hex
HEIN’S COLUMNS

1942 Hein game
1948 Nash game
1952 Shannon computer
1957,58 Gardner columns

Hein’s columns
Hein’s columns

1942 Hein game
1948 Nash game
1952 Shannon computer
1957,58 Gardner columns

Hayward@ualberta.ca
Hein’s columns

1942 Hein game
1948 Nash game
1952 Shannon computer
1957,58 Gardner columns

Hayward@ualberta.ca

Twist and turn: the story of Hex
Hein’s columns
HEIN’S COLUMNS

1942 Hein game
1948 Nash game
1952 Shannon computer
1957, 58 Gardner columns

HAYWARD@UALBERTA.CA

TWIST AND TURN: THE STORY OF HEX
PROBLEMS

Problems worthy of attack prove their worth by hitting back.
Hein’s gruks

CONSOLATION GROOK

Losing one glove is certainly painful, but nothing compared to the pain, of losing one, throwing away the other, and finding the first one again.
Nash’s game

**Princeton 1948-9**

- Nash new game $\rightarrow$ Gale
- Gale board $\rightarrow$ Fine Hall
Nash’s game
Nash’s game

Twist and turn: the story of Hex
Nash’s game
Nash’s game
Nash’s game
Nash’s game

Outline

- Pre-
- Early
- Mid
- Recent

1942 Hein game
1948 Nash game
1952 Shannon computer
1957,58 Gardner columns

Nash’s game

hayward@ualberta.ca

Twist and Turn: The Story of Hex
Nash’s theorem

N-by-n Hex is 1st-player win

- lemma: extra X-cell ok for player X
- lemma: no draws in Hex
- suppose P2 has win strategy S2
- then P1 can move anywhere, forget move, and follow S2
- thus P1 has win strategy, contradiction □
PROJECT RAND

SOME GAMES AND MACHINES FOR PLAYING THEM

John Nash

D-1164

2 February 1952

Assigned to
N-BY-(N+1) HEX IS LARGER-SIDE WIN
n-by-(n+1) Hex is larger-side win
1951 Bird Cage (a.k.a. Gale, Bridg-it) Machine

- Bird cage: players occupy edges
- Board ↔ electrical network
- Apply black side-to-side voltage
- Black edge ← short/contract edge
- White edge ← cut edge
- Computer move: take edge with max voltage drop
- Almost always won with first move
SHANNON’S ANALOG COMPUTERS

1952 Hex machine

- board $\leftrightarrow$ 2-dimensional potential field
- black cell $\leftarrow$ positive charge
- white cell $\leftarrow$ negative charge
- computer move: certain saddle point
- computer positionally strong, tactically weak
SHANNON’S ANALOG COMPUTERS

GAG MACHINE

- 7x8 board disguised to look regular
- played pairing strategy, always won
Shannon’s Analog Computers

Outline

Pre-

Early

Mid

Recent

1942 Hein game
1948 Nash game
1952 Shannon computer
1957,58 Gardner columns

Hayward@ualberta.ca

Twist and Turn: The Story of Hex
SHANNON’S ANALOG COMPUTERS

1942 Hein game
1948 Nash game
1952 Shannon computer
1957,58 Gardner columns

Hayward@ualberta.ca

Twist and turn: the story of Hex
Beyond Hex: Shannon Switching Game

- play on any graph
- two marked vertices
- black move: ‘short’ any vertex (make nbrs clique)
- white move: ‘cut’ any vertex (delete)
- black wins iff two marked vertices are shorted (connected)
- generalizes Hex
Beyond Hex: Shannon Switching Game
BEYOND HEX: SHANNON SWITCHING GAME
BEYOND HEX: SHANNON SWITCHING GAME

OUTLINE
PRE-EARLY
MID
RECENT

1942 Hein game
1948 Nash game
1952 Shannon computer
1957, 58 Gardner columns

T

T

HAYWARD@UALBERTA.CA
TWIST AND TURN: THE STORY OF HEX
Beyond Hex: Y

Shannon, Milnor, Schensted & Titus

- connect all 3 sides to win
BEYOND Hex: Y

OUTLINE
PRE-
EARLY
MID
RECENT

1942 Hein game
1948 Nash game
1952 Shannon computer
1957,58 Gardner columns

Hayward@ualberta.ca

Twist and Turn: The Story of Hex
Gardner’s columns

1957 July Scientific American Mathematical Games
- concerning the game of Hex, which may be played on the tiles of the bathroom floor

1958 Oct Scientific American Mathematical Games
- 4 mathematical diversions involving concepts of topology
Lehman’s Bridg-It solution

- poly-time algorithm to find winning move
- maintain 2 edge-disjoint spanning trees
- Bridg-It less fun
Beck opening: n-by-n acute corner loses
SCHENSTED & TITUS

MUDCRACK Y & POLY-Y

PROVERBS

- play the best . . . 1st player wins . . . handicapping
- two-way stretch . . . best offense is a good defense
- be relevant . . . double trouble
- you can’t see the whole sky through a bamboo tube
- waste not, want not . . . shun the worthless triangle
- beware the square . . . don’t trust the pentagon
- luck is a many-sided region . . . breaking mirrors is bad luck
- never try to cut a bamboo joint
- equivalent patterns . . . the aim of the game
SCHENSTED’S Y REDUCTION: NO DRAWS
SCHENSTED’S Y REDUCTION: NO DRAWS
SCHENSTED’S Y REDUCTION: NO DRAWS
Schensted’s Y reduction: no draws
SCHENSTED’S Y REDUCTION: NO DRAWS
SCHENSTED’S Y REDUCTION: NO DRAWS
SCHENSTED’S Y REDUCTION: NO DRAWS
Hex PSPACE-complete

- 1975 Even & Tarjan: Shannon v-switching PSPACE-complete
- 1981 Stefan Reisch: Hex ist PSPACE-vollstandig
- 2000 Clay Math Inst: $1 000 000  P vs NP
BERGE L’ART SUBTIL DU HEX

1963 Lehman solution
1969 Beck opening
1975 Schensted & Titus book
1975 Even & Tarjan PSPACE-complete
1977 Berge problems
1979 Gale no-draw proof
Berge L’Art Subtil du Hex

- 1963 Lehman solution
- 1969 Beck opening
- 1975 Schensted & Titus book
- 1975 Even & Tarjan PSPACE-complete
- 1977 Berge problems
- 1979 Gale no-draw proof

Hayward@ualberta.ca

Twist and turn: the story of Hex
Berge L’Art Subtil du Hex
Gale’s no-draw proof
Gale’s no-draw proof
ICGA CGO Hex Tournaments

- 2000 London: Hexy Queenbee Killerbee
- 2003 Graz: Six Mongoose
- 2004 Ramat-Gan: Six Mongoose
- 2006 Turin: Six Wolve HexKriger
- 2008 Beijing: Wolve MoHex Six Yopt
- 2009 Pamplona: MoHex Wolve Six Yopt
- 2010 Kanazawa: MoHex Wolve MIMHex Yopt
Computer Players

Shannon to Wolve

- Shannon bird-cage eval’n
- Hexy: + limited search, and-or virtual connection algebra
- Six: + vc lazy queue processing (+ vcs through edge)
- Wolve: + inferior cell engine, capture in vc engine, solver

MoHeX

- same ic engine, vc engine, solver as Wolve
- Monte Carlo tree search
# Solving Hex Openings

## Human Proofs

<table>
<thead>
<tr>
<th>Year</th>
<th>Author</th>
<th>Board Size</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>2001</td>
<td>Yang</td>
<td>17/49 7x7</td>
<td></td>
</tr>
<tr>
<td>2002</td>
<td>Yang</td>
<td>8x8</td>
<td></td>
</tr>
<tr>
<td>2003</td>
<td>Yang</td>
<td>9x9</td>
<td></td>
</tr>
<tr>
<td>2004</td>
<td>Noshita</td>
<td>7x7</td>
<td></td>
</tr>
<tr>
<td>2005</td>
<td>Noshita</td>
<td>8x8</td>
<td></td>
</tr>
<tr>
<td>2006</td>
<td>Mishima</td>
<td>8x8</td>
<td></td>
</tr>
</tbody>
</table>
# Solving Hex Openings

## Computer Proofs

<table>
<thead>
<tr>
<th>Year</th>
<th>Authors</th>
<th>Board Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>1995</td>
<td>Enderton</td>
<td>6x6</td>
</tr>
<tr>
<td>2000</td>
<td>van Rijswijck</td>
<td>6x6</td>
</tr>
<tr>
<td>2003</td>
<td>H Björnsson, Johanson, Kan, Po, van Rijswijck</td>
<td>all 7x7</td>
</tr>
<tr>
<td>2007</td>
<td>Rasmussen, Maire Hayward</td>
<td>all 7x7</td>
</tr>
<tr>
<td>2009</td>
<td>Arneson, H Henderson</td>
<td>all 8x8</td>
</tr>
<tr>
<td>2010</td>
<td>A H H</td>
<td>53/31</td>
</tr>
</tbody>
</table>

*Hayward@ualberta.ca*  
*Twist and Turn: the Story of Hex*
Solving Hex Openings
Solving Hex Openings
Solving Hex Openings
Solving Hex Openings
Solving Hex Openings
Solving Hex Openings
Solving Hex Openings
Solving Hex Openings
Solving Hex Openings
READING

- *Hex Strategy ...* Browne
- *Connection Games ...* Browne
- *Hexaflexagons, Probability, Paradoxes ...* Gardner
- *2nd SciAm Bk Math’l Puzzles & Diversions...* Gardner
- *Politiken columns* Hein
- *Everything You Always Wanted ... Hex ...* Maarup
- *A Beautiful Mind* Nasar
- *Mudcrack Y and Poly-Y* Schensted & Titus
- *Four Colors Suffice* Wilson