**Rex**, or **Reverse Hex**: whoever connects their two sides loses. For the $n \times n$ board, Winder gave a strategy-stealing proof that the first (second) player can win if $n$ is even (odd). Lagarias and Sleator showed that the loser can prolong the game so that the board is filled.

**Cylindrical Hex**, or **CylHex**: $c \times h$ board is wrapped around a cylinder with circumference $c$, height $h$. **Updown** wins by connecting top to bottom. **Around** wins by encircling cylinder.

**Random Hex**: each player, on each turn, makes uniform random move, among all possible moves. Let $p_1(n)$ be the probability that the first player wins random $n \times n$ Hex.

**Quiz**

- When did Piet Hein invent Hex?
- Did Gardner believe Nash’s invention was independent of Hein’s?
- For Hex, give the smallest board with no known particular winning opening move.
- For Rex, give the smallest board with no known particular losing opening move.
- Give $p_1(n)$ for $n$ up to 5.
- Give a strategy-stealing proof of L&S’s result.

**8 Puzzles**

- Hex/Rex, Black/White to play: Best move? Who wins?

- $3 \times 4$ CylHex: Black(Around)/White(UpDown) to play: Best move? Who wins?
- CylHex: Black/White to play. Best move? Who wins?

**Open**

- Who wins $c \times h$ Cylindrical Hex, for odd $c \geq 5$?
- Prove/disprove: for $n$ odd, $\lim_{n \to \infty} p_1(n) = .5$. 
Truncated Rex (TRex): game ends if only one empty cell (so, draw possible).

State $S = (P, X)$: board position $P$ and player-to-move $X$.

For a state $S = (P, X)$, define $S' = (P', X)$, where $P'$ is obtained from $P$ by either adding a stone of either color, or removing a stone of either color.

Lemma (Toft): For TRex, if player $Z$ wins $S = (P, X)$, then $Z$ not-loses $S'$.

Proof (sketch). Consider a winning strategy for $S$, and modify it for $S'$: play as in the corresponding position in $S$; if this is ever not possible (because the move is already occupied), then play anywhere, and argue by induction on the number of empty cells.

$Z$ wins, so the last move in every line of play is by $Z$. This is TRex, so there is an empty cell after each last move. Also, just before $Z$ makes their final move, every empty cell must be a losing move. So there are at least two such cells in every case, so changing the color of only one of these empty cells to $Z$’s color — which is what happens when we move this strategy to $S'$ — will not give $Z$ a losing position. So $Z$ can at least draw in $S'$. □

Theorem (Toft): In TRex, for $S = (P, Y)$ with $P$ player-symmetric, each player non-loses.

Proof. strategy stealing

- Assume $Z$ wins $S = (P, Y)$.
  
  $Z$ is $Y$ or $\overline{Y}$

- $Z$ wins $(P^+, Y)$ for a $P^+$ obtained by adding a $Y$-stone to $P$

- $Z$ not-loses $(P, \overline{Y})$ use the lemma

- $Z$ not-loses $(P, Y) = S$ player-symmetry

- contradiction

- $Z$ not-wins $S$ original assumption is false

- $\overline{Z}$ not-wins $S$ $Z$ is arbitrary

- in TRex, each player not-loses. □

Corollary. Winder, Lagarias/Sleator results empty board player-symmetric

Hayward/Toft/Henderson, How to Play Reverse Hex, DM 312-1 (6 Jan 2012) 148-156
Huneke/Hayward/Toft, A winning strategy for $3 \times n$ cylindrical Hex, DM 331 (2014) 93-97

http://webdocs.cs.ualberta.ca/~hayward/papers