

# Hex and Mathematics

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*Natural Sciences and Engineering  
Research Council of Canada*

*University of Alberta  
Games Research*

- history
- properties
- recent progress
  - game-playing programs
  - solving
    - \* dead cell analysis

## **Hex and Mathematics**

H, van Rijswijck

to appear in *Discrete Mathematics*

## **Solving 7x7 Hex**

H, Björnsson, Johanson, Kan, Po, vR

to appear in *Theoretical Computer Science*

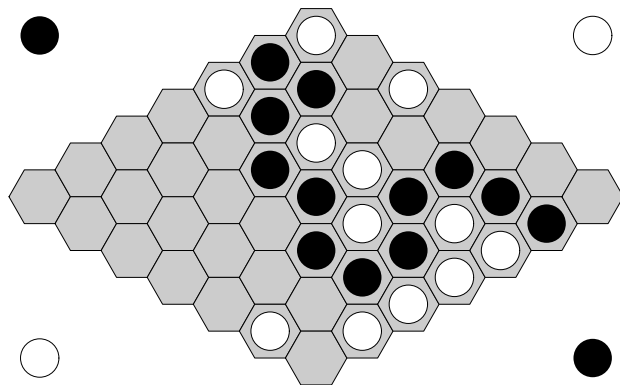
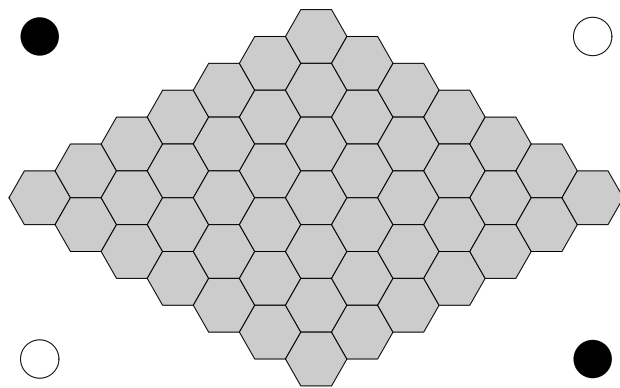
## **Dead Cell Analysis in Hex and the Shannon Game**

H, B, J, vR

submitted to *Proc. Graph Theory 2004*

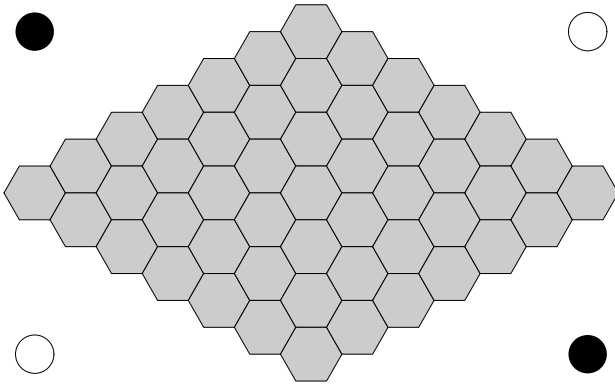
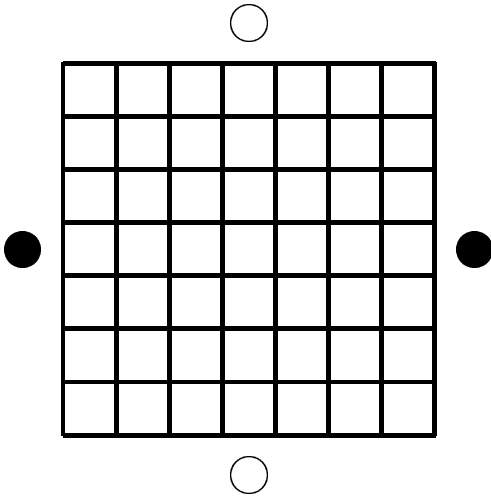
1942 Hein

“Matematik betragtet som ’Spil  
– Spillenes Matematik”



1948 Nash (& Gale)

“connecting topology and game theory”



**1950 Parker Brothers**

**1952 Nash**

**Some Games and Machines for Playing Them**

- no draws
- extra stones don't hurt
- exists 1st-player-win strategy

1953 Shannon (& Moore)

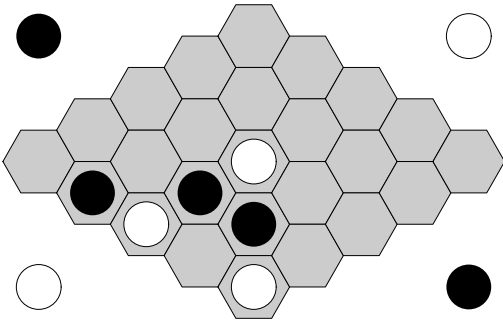
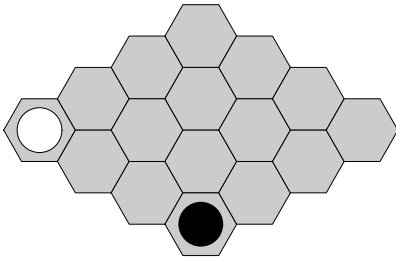
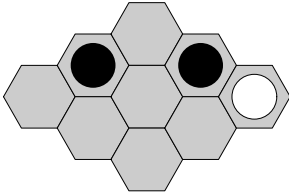
## Computers and Automata

“the problem of designing game-playing machines is fascinating . . .

“paradoxically, the positional judgment of this machine was good; its chief weakness was in end-game combinatorial play”

1957 Gardner

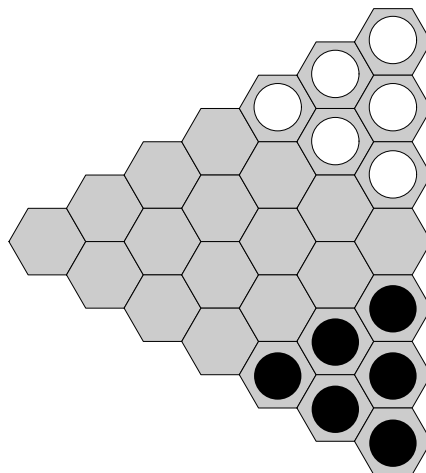
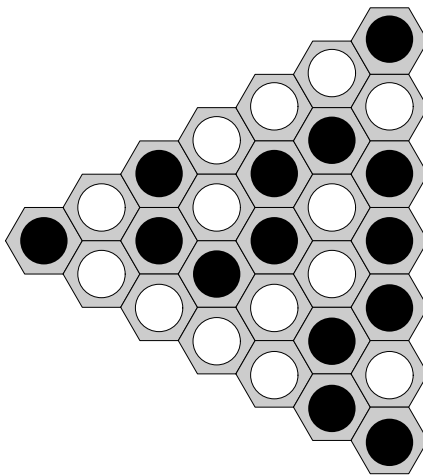
“played on the tiles of the bathroom floor”



1975 Schensted & Titus

Mudrack Y and Poly-Y

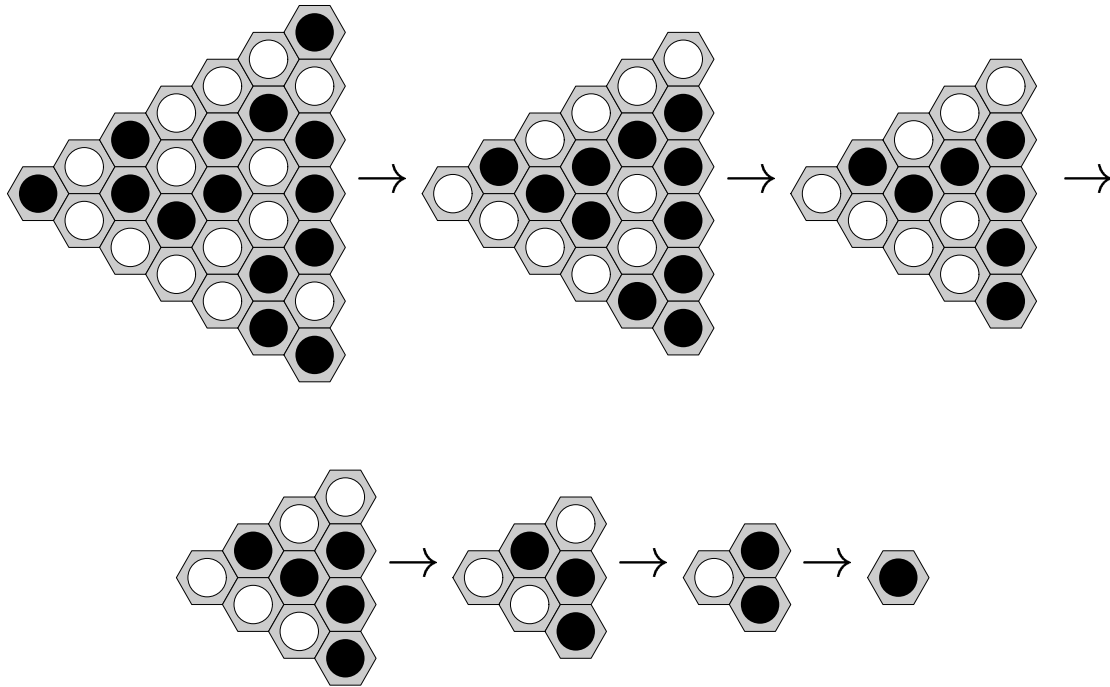
“whenever you feel you must use a car,  
try playing Y until the feeling passes”



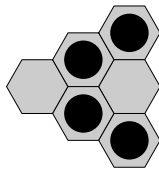
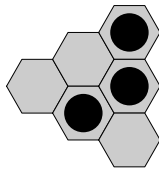
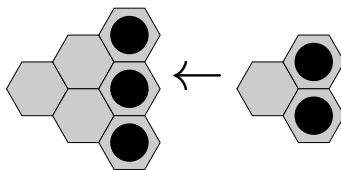
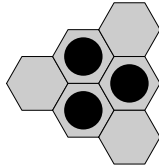
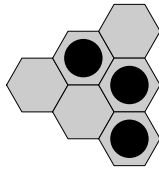


no draws in Y (or Hex)

proof: Schensted's Y-reduction

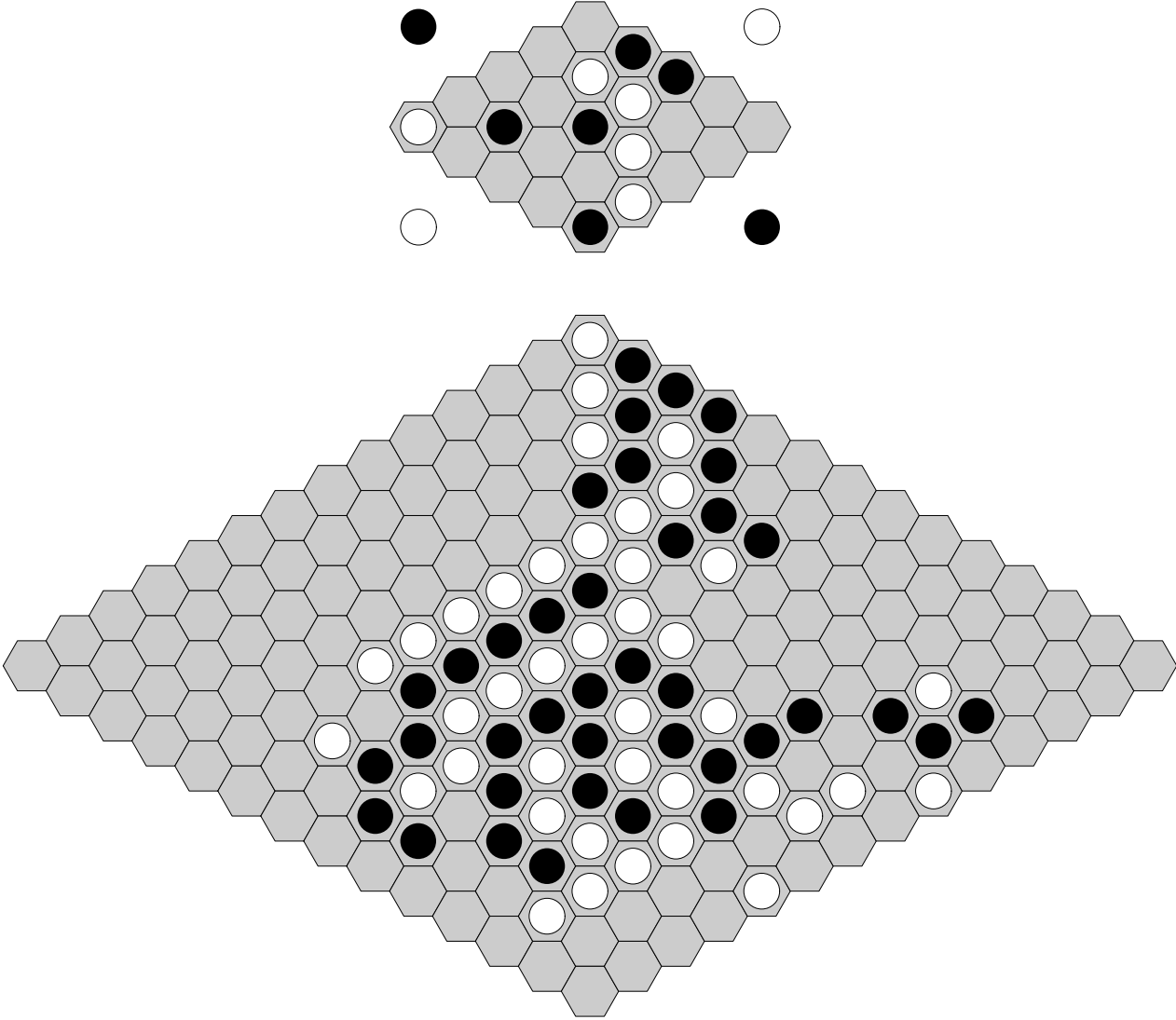


# reversing Y-reduction

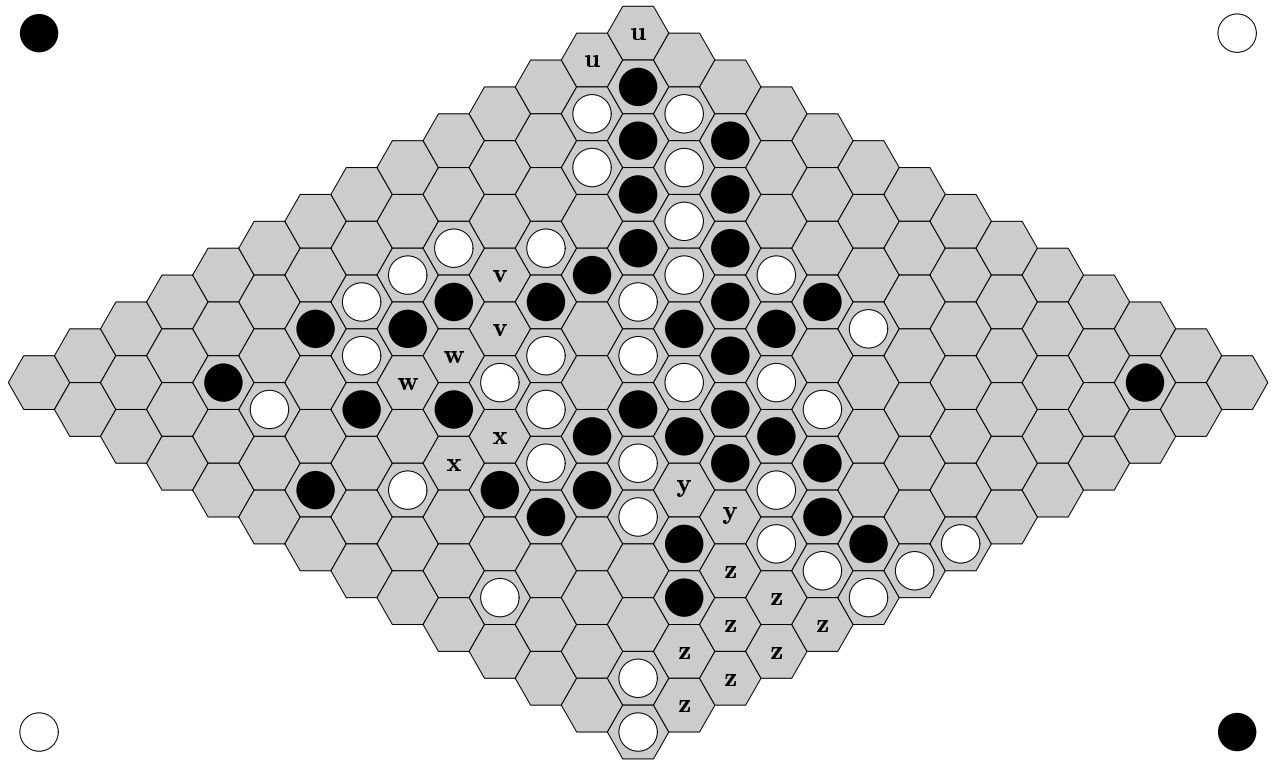


1977 Berge

“l’art subtil du Hex”

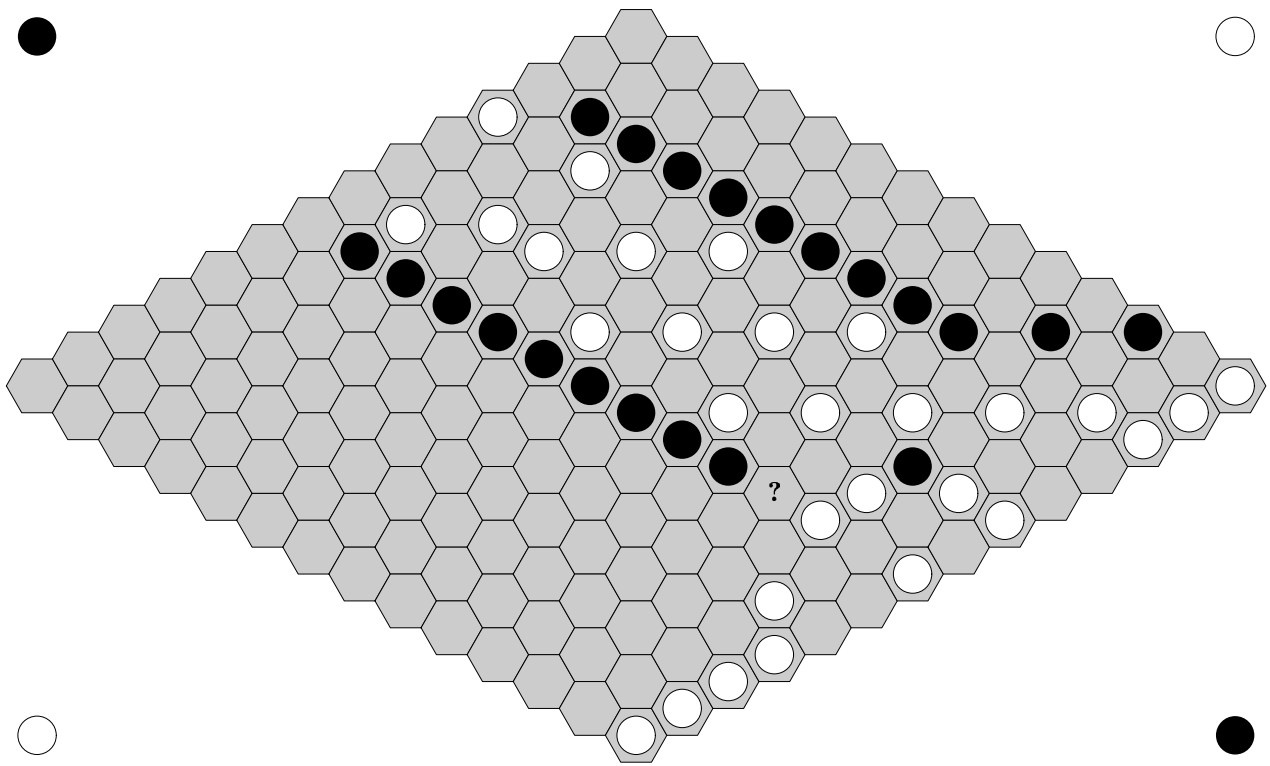


# virtual connections, mustplay



# 1981 Berge

“...to solve some Hex problem  
by using nontrivial theorems about  
combinatorial properties of sets ...”



**1976 Even & Tarjan**

**generalization of Hex PSPACE-complete**

**1981 Reisch**

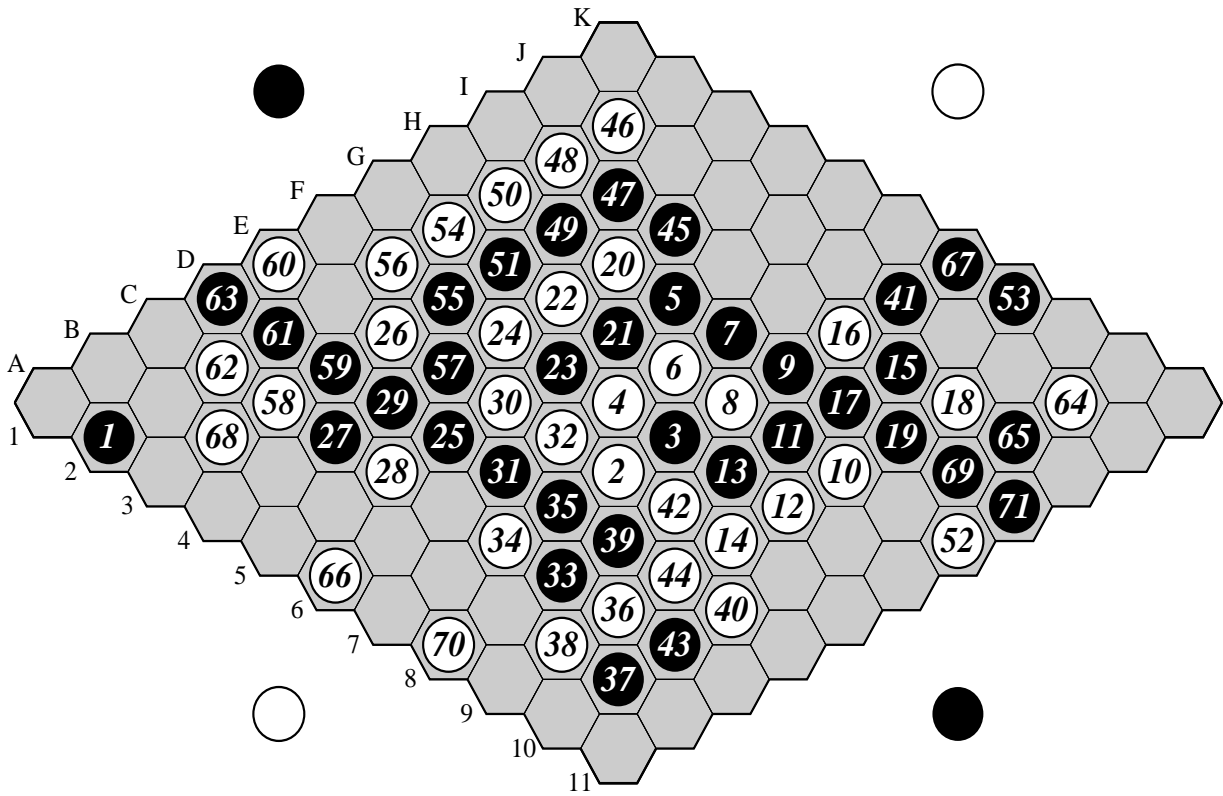
**Hex PSPACE-complete**

**1984 Berge**

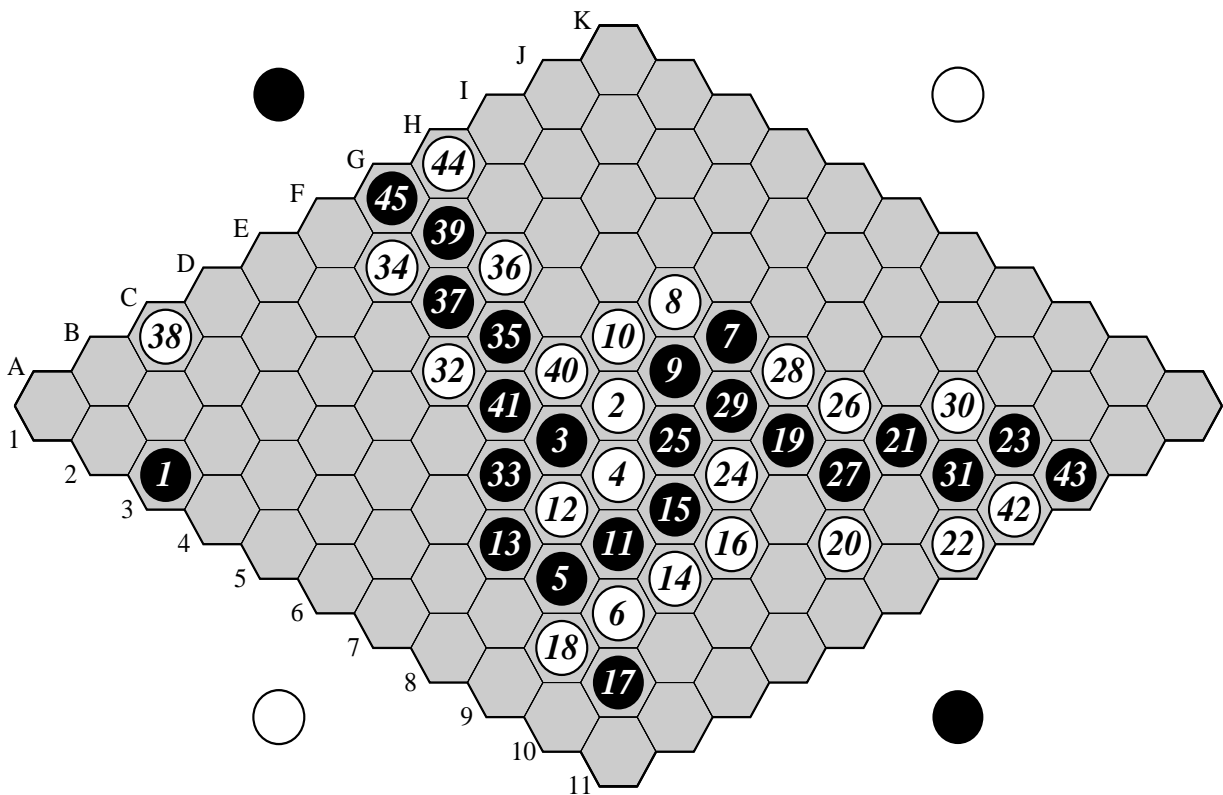
**“computers will never beat humans at Hex”**

H, Yngvi Björnsson, Mike Johanson, Maryia Kazekevich, Morgan Kan, Nathan Po, Jack van Rijswijck

- resistance network
- virtual connections
- dead cell analysis
- mustplay



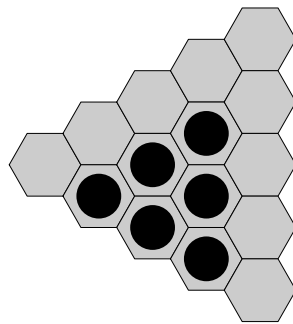
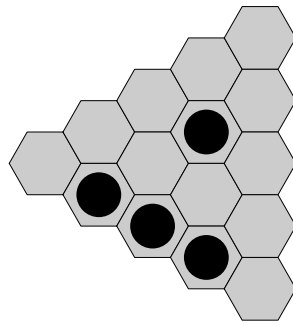
CGO 2004 Game 3. Six (black) defeats Mongoose.



Game 4. Mongoose (black) defeats Six.

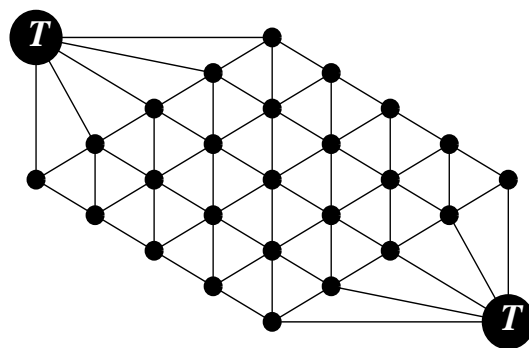
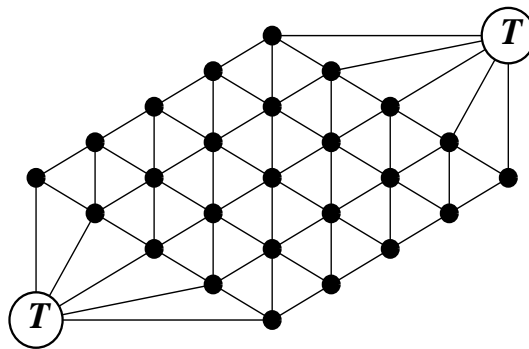


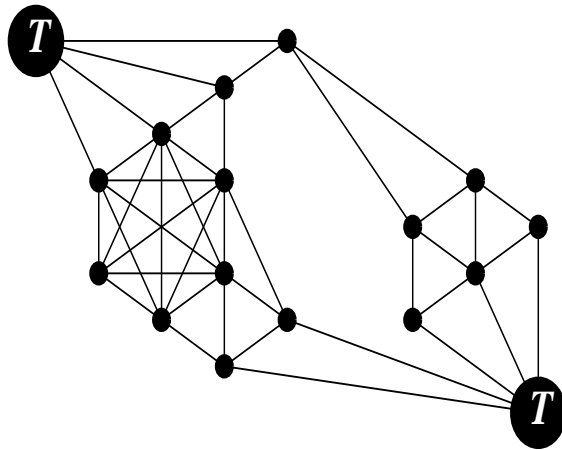
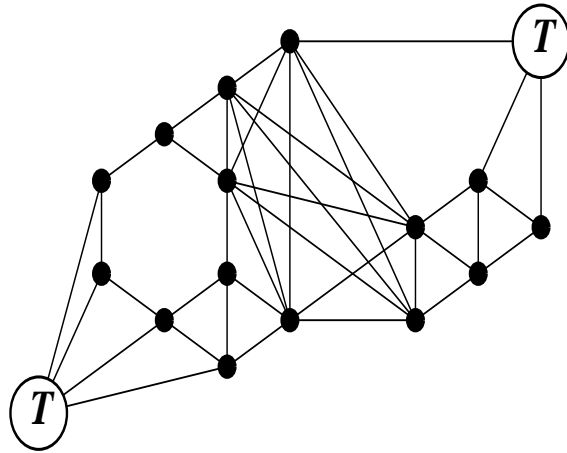
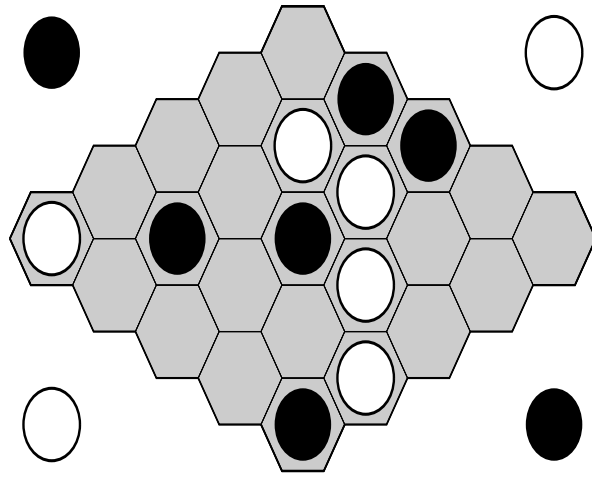
# dead cell analysis (H-vR-B-J)



## dead cells and induced paths

- node  $v$  is *dead* if,  
for every completion of  $G - v$ ,  
colour of  $v$  does not change winner
- *live* iff not dead
- Theorem: live iff on terminal-terminal induced path of reduced graph





death has consequences

- nodes *P*-captured:  $P$  has 2p-kill strategy

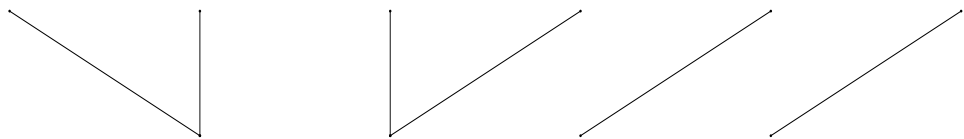
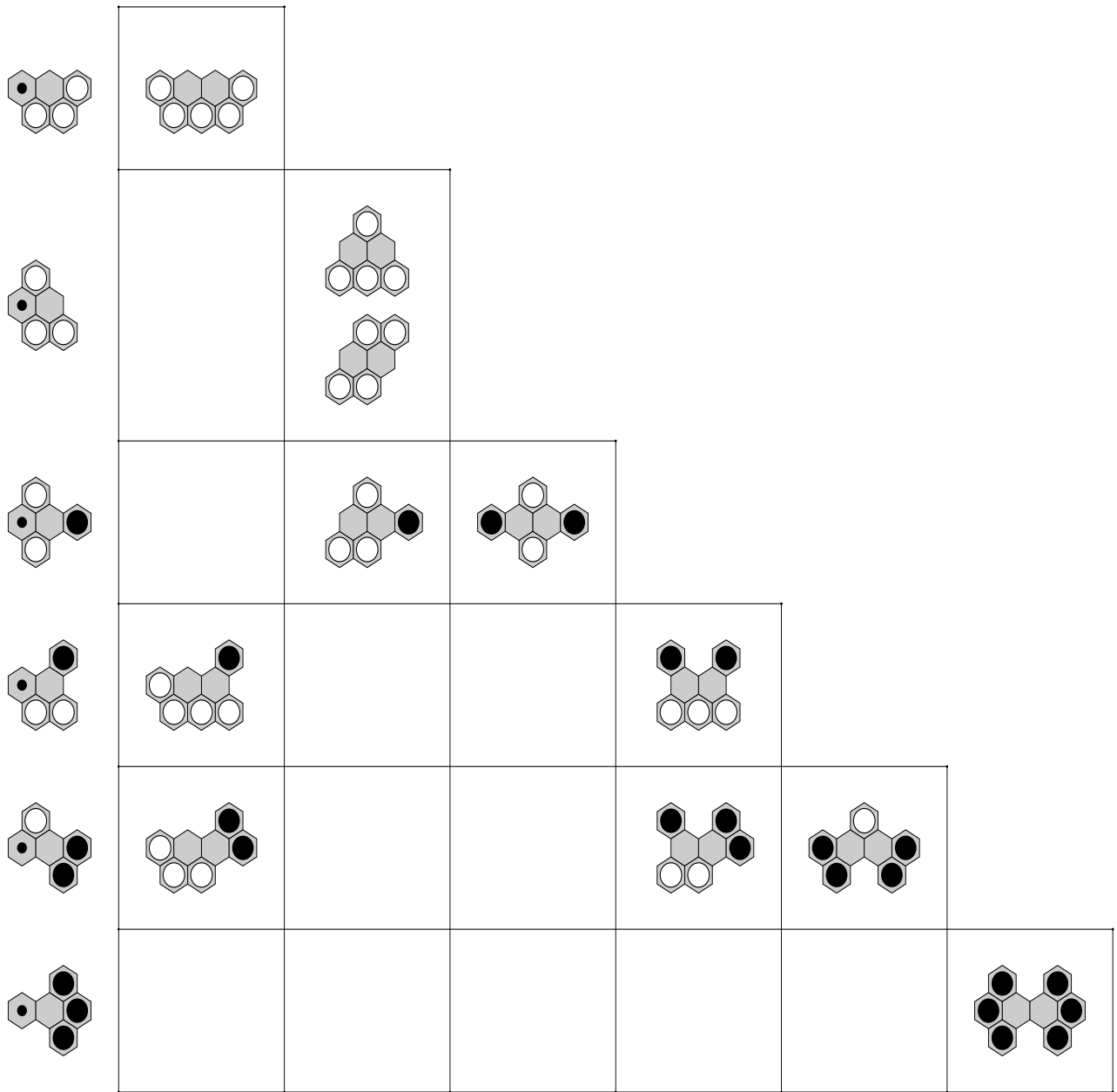
adding *P*-stones doesn't change game

- nodes *P*-dominated:  $P$  has 1p-kill strategy

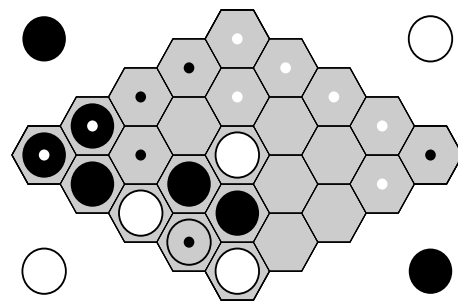
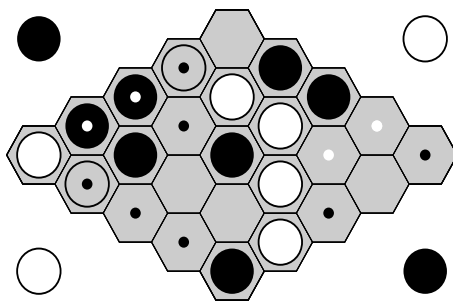
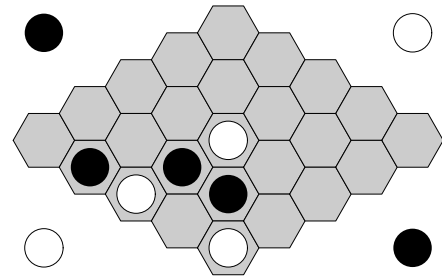
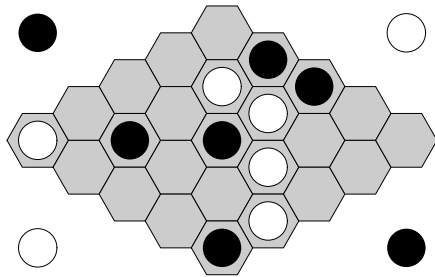
$P$  need only consider first move

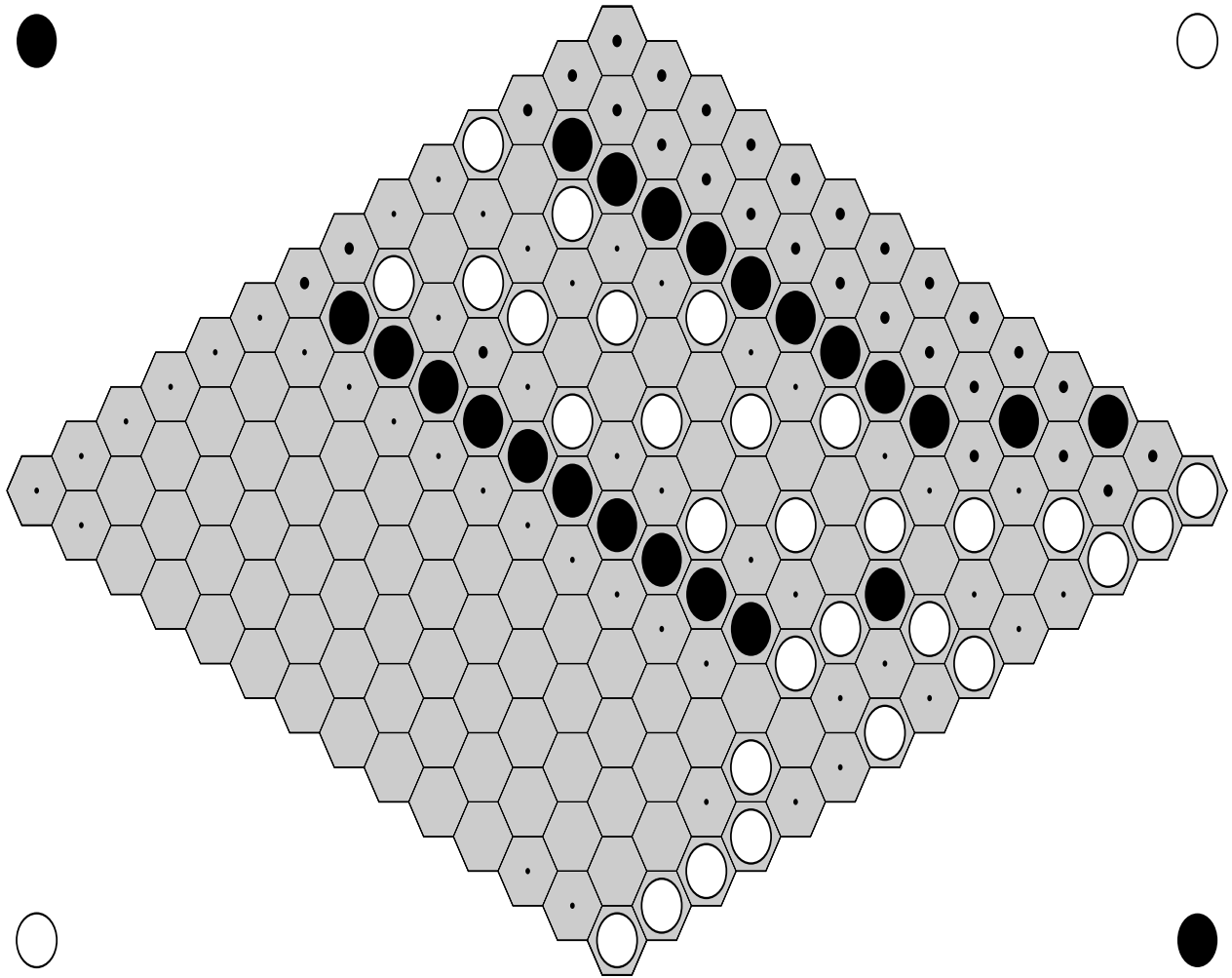
algorithmic considerations

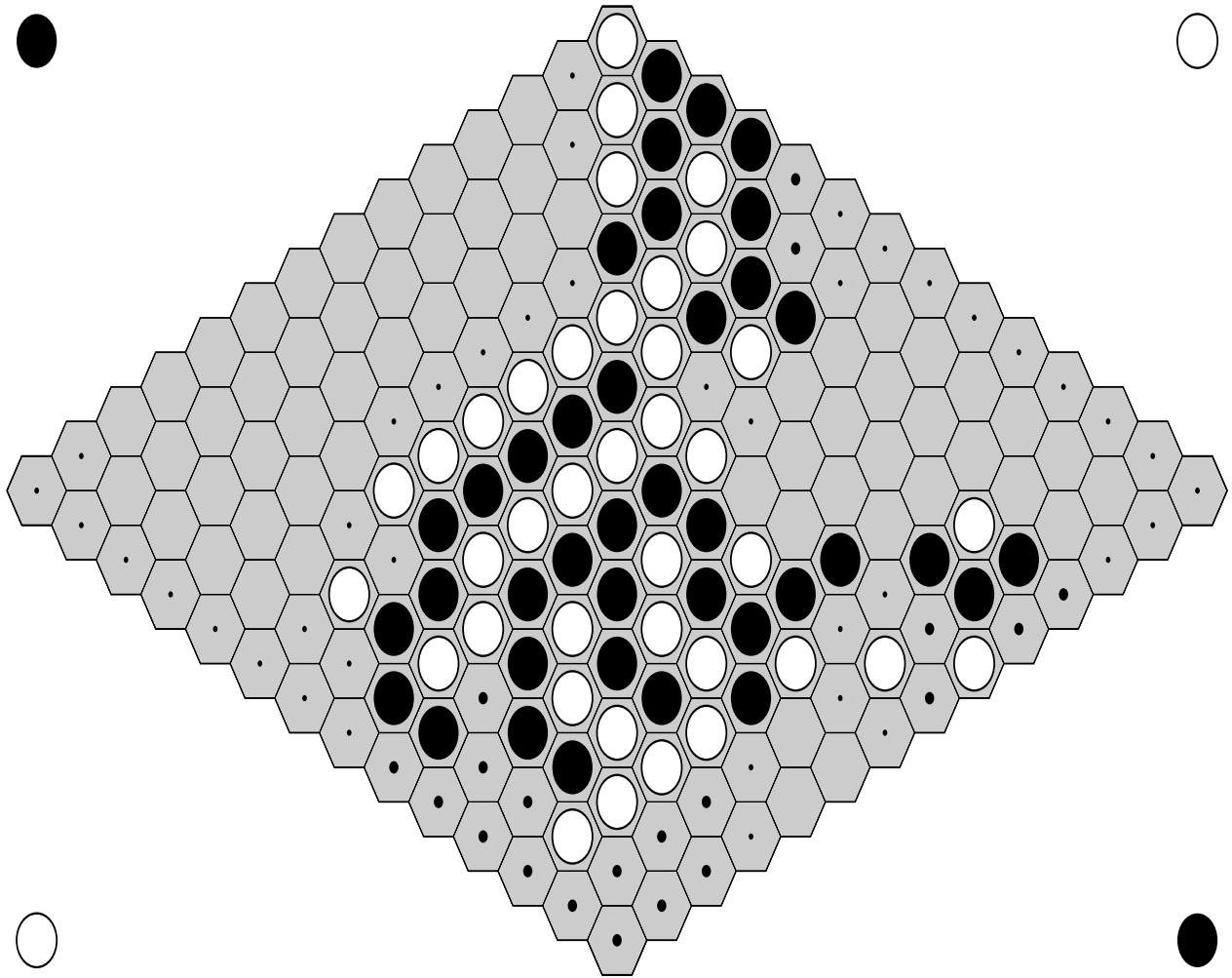
- induced path closure NP-complete (Fellows)
- dead nodes often simplicial



# dead cell analysis

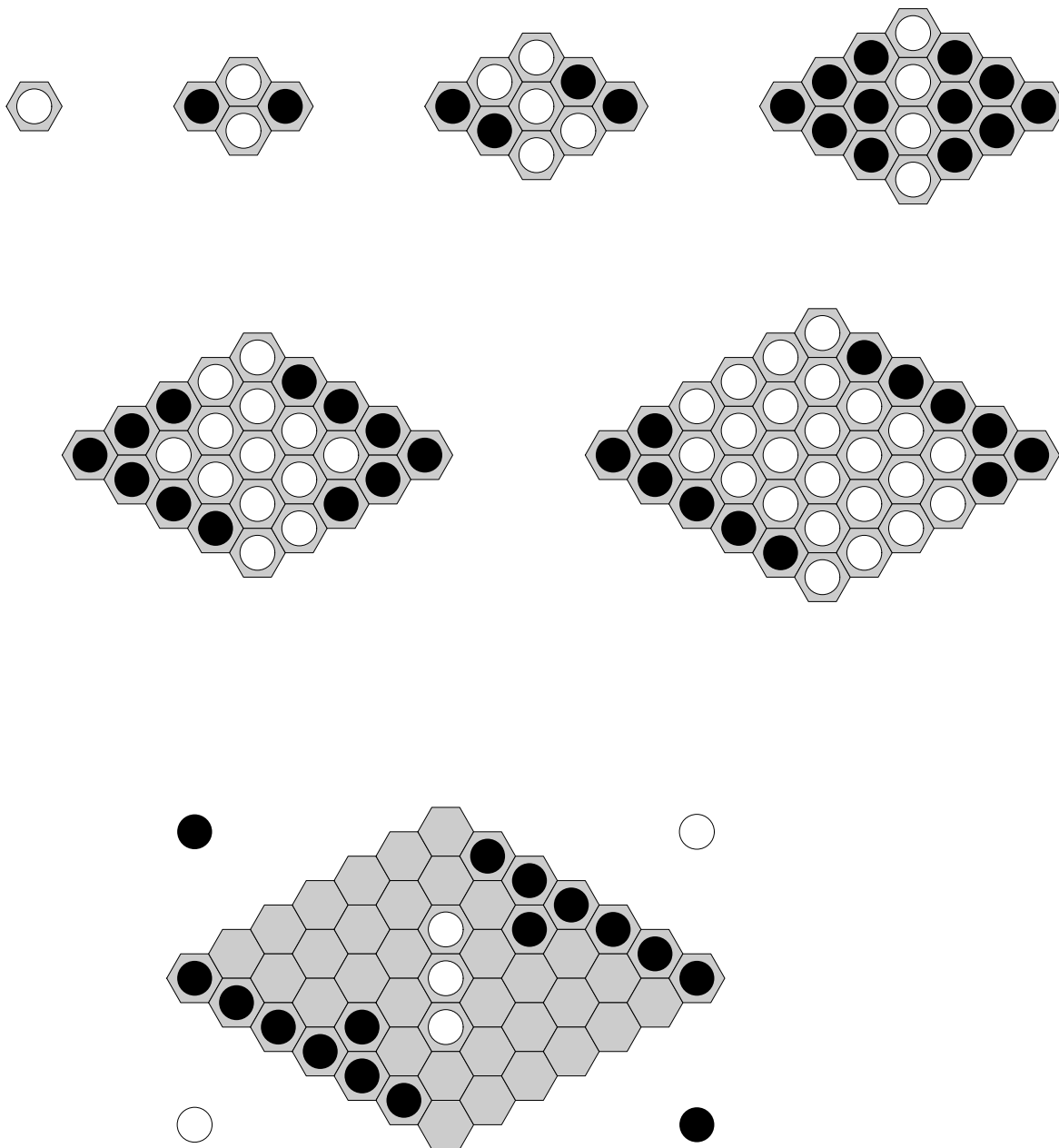








# solving Hex: opening winning moves



# 2002 H/Björnsson/Johanson/Kan/Po/Van Rijswijck

