A Handicap Hex Strategy

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Hex Basics

**Rules**

- Two players alternate turns playing on any empty cell
- Stones are permanent (no moving or capturing)
- Goal is to connect your two sides of the board
Hex Basics

Theoretical Properties

- An extra stone of your color is never a disadvantage
- Hex cannot end in a draw
- First-player win: strategy-stealing argument
- PSPACE-complete to determine winner in arbitrary position
Motivation

- Know only that there exists a winning first-player strategy
- How many stones do we need to place initially to guarantee a win, and where should these handicap stones be placed?
- Claude Berge would give three stone handicaps on the $11 \times 11$ board
On irregular Hex boards, player traversing shorter distance can win even as second-player using a simple pairing strategy.

Idea: use handicap stones to essentially reduce a regular Hex board to an irregular one.
Using graph-theoretic properties, can determine stones that can be added to Hex positions without changing its value

Two types of fill-in: dead cells and captured cells
Dead Cells

- A cell is dead if it is not on any minimal winning path (for either player)
- Dead cells can be filled-in with stones of either colour
Captured Cells

- A set of cells $S$ is $P$-captured if player $P$ has a second-player strategy to make all $\overline{P}$-claimed $S$-cells dead.
- $P$-captured sets of cells can be filled-in with $P$-coloured stones.
Have identified a new type of inferior cell: permanently inferior cells

- Adds a single stone for one particular player \( P \)
- Unlike fill-in, the strategy set is larger than the filled-in set
If $\overline{P}$ moves first in pattern, claim $\overline{P}$ must play at shaded cell

Any other move allows $P$ to capture all four cells
If $\overline{P}$ plays at shaded cell, dotted cell is dead
If $P$ plays first, captures all four cells
In all cases, $P$ can claim the dotted cell
Can assign $P$ the dotted cell without changing position’s value
Permanently Inferior Cell Patterns

- All three permanently inferior patterns
Placement of stones begins on second row, main diagonal
Stones placed every 6 spaces until last stone is at most
distance four from the edge
On an $n \times n$ Hex board, this requires $\left\lceil \frac{n+1}{6} \right\rceil$ stones
Handicap Reduction (1)

- Capture cells below handicap stones
Apply permanently inferior cell patterns next
Fill-in remaining gaps via dead and/or captured
Handicap Strategy

- Essentially reduced to $n - 1 \times n$ board, so proven existence of winning strategy
- Can easily make strategy explicit by enforcing inferior cell strategies
Summary

- Identified new form of inferior cell: permanently inferior cells
- Developed efficient and explicit handicap strategy for $n \times n$ Hex

Open Questions:

- Are there more permanently inferior cell patterns?
- Can the number of handicap stones be further reduced while maintaining an explicit strategy?
- Can the number of handicap stones be reduced if we only desire an existence proof (while still specifying initial stone placement)?
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