New Winning and Losing Positions for 7×7 Hex

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Abstract. In this paper we apply a decomposition method to obtain a new winning strategy for 7x7 Hex. We also show that some positions on the 7x7 Hex board, called trivial positions, are never occupied by Black among all of the strategies in the new solution. In other words, Black can still win the game by using the strategies described in this paper even if White already has pieces placed on those positions at the start of the game. Considering the symmetry properties of a Hex board for both players, we also derive 14 losing positions for Black's first move on a 7x7 Hex board.

1 Introduction

Hex is an interesting board game that was invented in 1942 by Piet Hein, a Danish mathematician. The game was also reinvented independently by the American mathematician John Nash in 1948. Hex is played by two opponents, Black and White, where Black moves first. Each player owns the two opposite edges of the board that bear his color. The object of the game is to build a connected chain of pieces across opposite sides of the board. The Hex board is a hexagonal tiling of n rows and m columns, where n is usually equal to m. Figure 1 shows an empty 7x7 Hex board. The rules of the game are relatively simple:

– The players take turns playing a piece of their color on an unoccupied hexagon.
– The game is won by the player that establishes an unbroken chain of his or her pieces connecting the player's sides of the board.

For example, Fig. 2 shows a Hex game in progress, in which it is Black's turn to play. If Black plays a piece at position "A", Black wins the game. However, if Black plays at any other position rather than "A", White can play a piece at position "A" and declare the win.

In 1949, John Nash proved that there is a winning strategy for the first player, but did not indicate what that play might be. A winning strategy for 7x7 Hex based on a decomposition method was declared in 2001 [1]. According to [2], this is the largest board size for which a solution has been found. In the solution,

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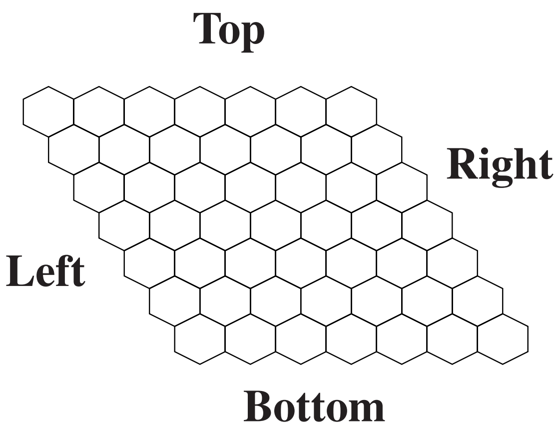

Fig. 1. An empty 7x7 Hex board.

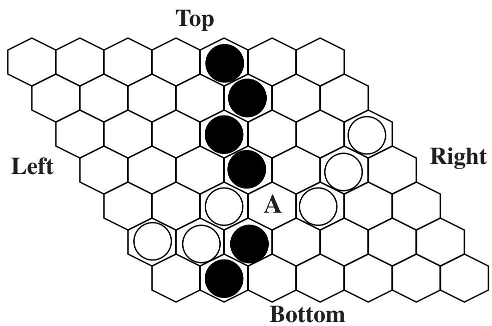

Fig. 2. Play on position A to win the game.

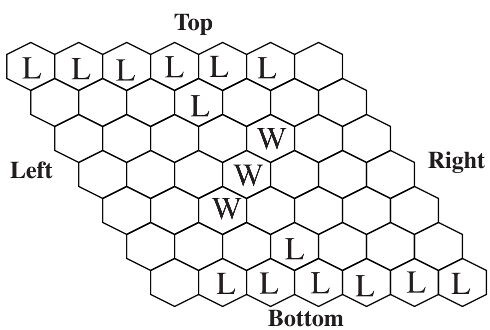

Fig. 3. Hexagons marked by W and L indicate the winning or losing opening positions. The empty hexagons remain unsolved opening positions.

the first Black piece is played at the center of the Hex board in order to take advantage of the symmetry properties. In this paper a new solution for 7x7 Hex

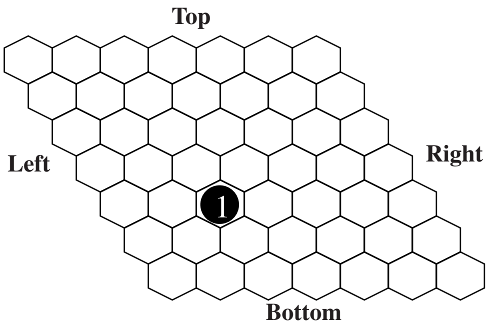

Fig. 4. The first move in the new solution. This is also LocalPattern 1 in [3].

is described that is based on the same decomposition method. This research also leads to the discovery of several losing first moves for Black, that is, if Black plays any of these moves White simply adopts Black's known winning strategy. As a result of this research, Fig. 3 shows all of Black's first moves that are known to lead Black either to win or to lose the game. The hexagons containing the letter "W" indicate that Black has a winning strategy if the first Black piece is played there. The hexagons containing the letter "L" indicate that if Black plays the first piece there he loses, that is, White has a winning strategy.

2 A New Solution for Hex on a 7×7 Board

In the already published winning strategy for 7x7 Hex [1], the first Black piece is played at the center of the board. In this paper we present a new solution for the 7x7 Hex game, in which the first piece is played at position ❶ as shown in Fig. 4.

The decomposition method is inspired by the concept of subgoals in AI planning. Finding winning strategies for a Hex game can be viewed as a Markov Decision Process (MDP). If a MDP can be broken up into several sub-MDPs, then there exists a parallel decomposition for the process. Each of the sub-MDPs has its own action space, which forms either a product or join decomposition of the original state space. Under a parallel decomposition, the sub-MDPs can be processed completely independently, and the original (global) MDP may be solved exponentially faster [4].

In a Hex game, the goal for Black is to form a connected chain from the top side to the bottom side of the board. This goal can be viewed as a sum of several subgoals. A subgoal may be "one Black piece is connected to another Black piece", "one Black piece is connected to Top", "one Black piece is connected to Bottom", "Top is connected to Bottom", or a combination of their OR/AND logical expressions. For example, "one Black piece is connected to another Black piece" OR "this Black piece is connected to Top" is one of the typical cases. The successes of achieving all of the subgoals in a game will lead to the success of accomplishing the goal of winning the Hex game. An important characteristic of

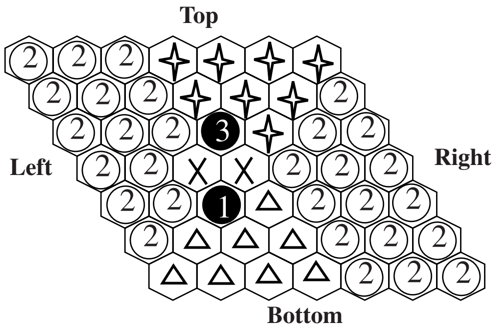

Fig. 5. ✦s establish LocalPattern 5, whose subgoal is to connect ❸ to Top; LocalPattern 2 is composed of two X s and its subgoal is to connect ❸ to ❶; LocalPattern 5 is marked by △s and its subgoal is to connect ❶ to Bottom. Black can win the game by playing ❸ if White plays in any of the ② positions in response to ❶.

a Hex game is that the success of a subgoal may be determined by only a small empty local region, which is called the influence region of the subgoal. If each of those subgoals has an influence region for the subgoal's success and all of the influence regions in a game are independent from each other, we can decompose the entire board into several local patterns. For example, in Fig. 5, the game is decomposed into three different local regions. The subgoal of the local pattern, whose influence region is covered by ✦s is to connect ❸ to the top row. The subgoal of another local pattern, whose influence region is over the two hexagons marked by X , is to connect ❶ to ❸. The subgoal of the third local pattern is to connect ❶ to the bottom row, its influence region is marked by △s.

Obviously, the three influence regions shown in Fig. 5 do not overlap, and Black can win the game by forming a connected chain from Top to Bottom if all of the three subgoals are reached. If we can find a strategy on each of the local influence regions for Black to reach its subgoal, summing up all of the local winning strategies would form the winning strategy for Black to win the game. Since White can make a move in only one of the local pattern regions at a time, Black only needs to play the next piece following the strategy for the corresponding local pattern. In most cases, a local pattern can be further decomposed into smaller local patterns by applying the same decomposition technique, though there do exist some cases in which a parallel decomposition is not possible. A big advantage of the parallel decomposition process is that most local patterns appear repeatedly in different games. Therefore, it is possible to represent the whole winning-strategy tree by recursively using those local patterns. In fact, the new winning-strategy tree can be represented by 63 different local patterns [3]. Although the number of local patterns in the new winning strategy discussed in this research is higher than that of the previous solution [1], the new solution introduces some more interesting results, which are discussed in Sect. 3.

The new winning strategy is outlined in the Appendix. Figures 11 to 24, in combination with Fig. 5 above, show the winning move for Black against all possible White first move replies. The figures also show the local patterns involved. The remaining figures in the appendix further decompose the higher level patterns as follows:

- Figures 25 and 26 are used to portray Fig. 24, the strongest possible defense by White.
- Figures 27 to 35 describe one layer further the local pattern (LocalPattern 19) covered by X s in Figs. 14, 15, and 16.
- Figures 36 to 42 describe one layer further the local pattern (LocalPattern 11) marked by X s in Fig. 17.

The local patterns depicted in the Appendix are representative of the new solution. For the complete winning strategy, including all of the 63 local patterns, see [3]. In each figure, different local influence regions are labeled by different symbols, for example, X , △, ✦, ❋, and ⟁. A local influence region, combined with several played pieces, forms a local pattern. Each of the local patterns is indexed by a local pattern number in the form of "LocalPattern n", which is the same terminology as used in [3]. Each of the local patterns ensures the success of reaching a subgoal. Therefore, the sum of all local patterns' successes in a given game will lead to a win for Black. There is no overlap between the local influence regions in a game, and all White's defense moves in a local influence region must have been covered by the winning strategy of its corresponding local pattern. From a logical point of view, the winning strategy has been proved. Considering that the winning-strategy tree is very complicated with many local patterns, a computer program was developed for its verification. The verification program can generate all possible moves by White in a local pattern, and test the subgoal by playing with the winning strategy. For testing larger local patterns, a so called "virtual connection" technique [5] has been applied in order to eliminate the obvious winning cases by forcing a game to end early.

3 Losing Opening Moves for Black

Although there are winning strategies for the first player to win a Hex game, a "bad" opening move can lead to a loss. Beck proved that the acute corner moves in any n by n Hex is a first-player-loss [6]. Jack van Rijswijck outlines all winning strategies for 6x6 Hex [7]. In a solution for a Hex game, if there are some hexagons that Black never needs to occupy for winning the game, we define those hexagons as "trivial positions". In other words, even if White has some pieces on those "trivial positions", Black can still play with the strategies in the solution in order to win the game.

For example, in Fig. 6, all positions marked by X s are "trivial positions", which are never needed by Black in the aforementioned solution. Black can win

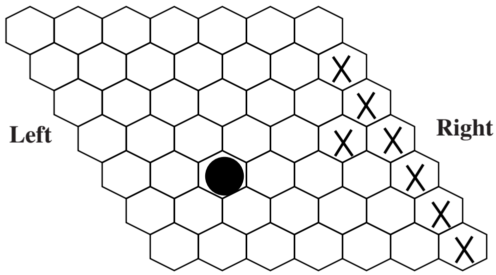

Fig. 6. All of X s are trivial positions. Even if they are all occupied by White's pieces, Black still can form a connected chain from Top to Bottom.

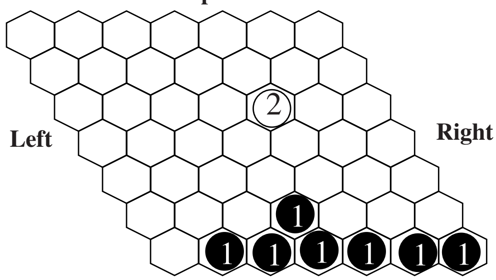

Fig. 7. ➊s are losing opening positions for Black. If Black plays its first move on one of ➊s, White can play ② and win the game. Note that, White will follow exactly what Black does in the new solution described in Sect. 2 except White's goal is to form a connected chain form Left to Right. If the board is reflected in the long diagonal and colours are interchanged, the situation will be the same as that of Fig. 6.

the game by following the winning strategies in the solution regardless of if those positions are occupied by White at the beginning of the game.

If the Hex board is turned over along the diagonal (from the left-top corner to the right-bottom corner), we can find that all of these marked positions in Fig. 6 then become Black's losing positions. These positions are marked by ➊s in Fig. 7. If Black occupies one of those positions with the first piece, White can play at ② and follow the winning strategy proposed in this paper to win the game.

4 Conclusions and Remarks

In this paper, we described a new winning strategy for 7x7 Hex where the first piece is played at ➊ in Fig. 8. In order to do so we decompose the game into 63

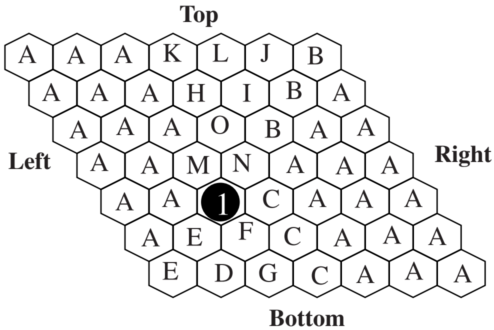

Fig. 8. White's next move on position As, Bs, Cs, ..., N, or O are discussed in Fig. 5, Fig. 11, Fig. 12, ..., Fig. 23, and Fig. 24, respectively. It is obvious that all White's defense moves are covered in the new solution.

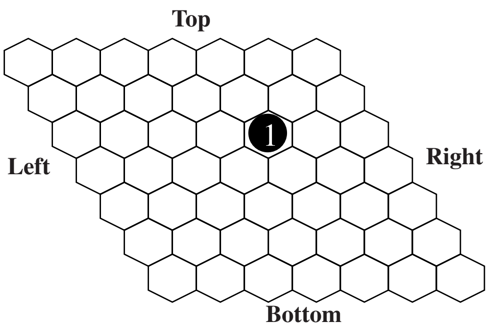

Fig. 9. Another winning position for Black due to the symmetry properties.

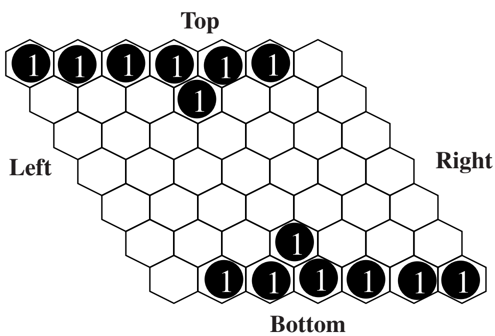

Fig. 10. All losing positions for Black derived from the new solution.

local patterns. Each local pattern will ensure a subgoal being achieved. Through Fig. 5, and Figs. 11 to 24 (position A to O in Fig. 8) we show that Black wins against all possible second level moves by White. We conclude that Black's ❶ in Fig. 8 is a winning move. Due to the symmetry properties of the 7x7 Hex board, we conclude that ❶ in Fig. 9 is also a winning move. The winning strategy described in this paper ensures the win with some simple coordinate transformations.

According to the "trivial positions" discovered in this research and the symmetry properties of a 7x7 Hex board, we also derived that there are 14 losing opening positions for Black, which are shown in Fig. 10. The newly discovered losing positions are especially valuable when the "swap rule", which gives the player to move second an option of swapping colors after Black's first move, is applied in a Hex game.

Figure 3 shows all solved winning and losing opening positions for 7x7 Hex. The empty hexagons in the figure still remain unsolved initial opening positions. However, with the decomposition method discussed in this paper, resolving those unsettled opening positions should be feasible. The decomposition method certainly can be applied to Hex games played on larger board, i.e., 8x8, 9x9, and beyond.

Parallel decomposition is a general approach, which has been applied successfully to AI planning [4, 8, 9]. Apart from Hex, it can possibly be applied to other games, for example, Go endgames. However, some modifications may be necessary because of the more complicated relationships between the subgoals and the final goal.

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Appendix: Detailed Descriptions of the New Winning Strategy

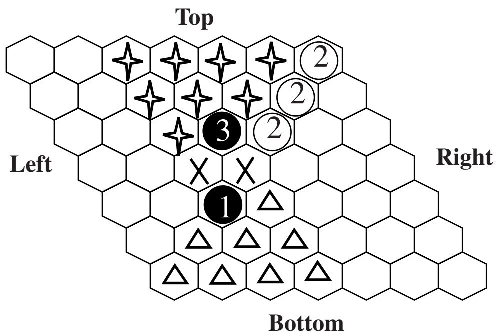

Fig. 11. The influence regions marked by ✦s, X s, and △s are discussed in LocalPattern 5, LocalPattern 2, and LocalPattern 5, respectively.

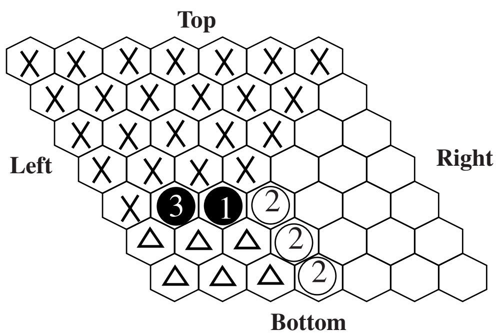

Fig. 12. X s and △s are covered by LocalPattern 20 and LocalPattern 6.

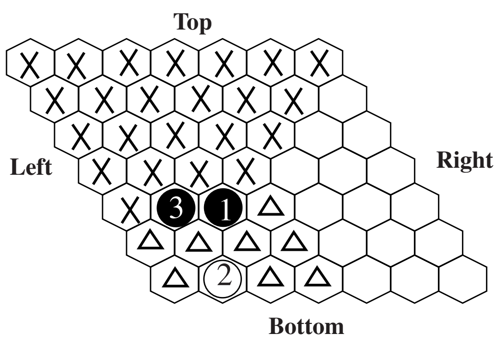

Fig. 13. The influence regions distinguished by X s and △s are explained by LocalPattern 20 and LocalPattern 4.

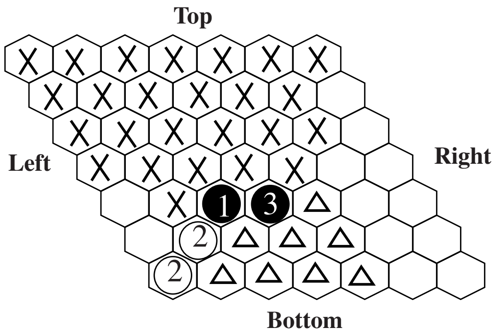

Fig. 14. The regions marked by X s and △s are clarified by LocalPattern 19 and LocalPattern 5. More discussions on LocalPattern 19 can be found in Fig. 27 to Fig. 35.

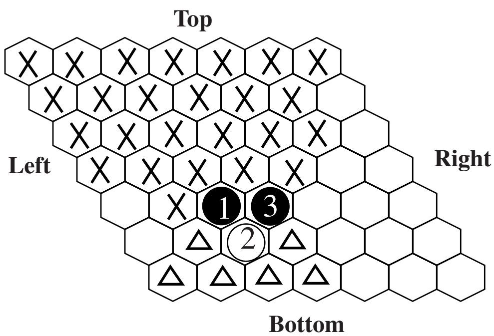

Fig. 15. The influence regions covered by X s and △s are discussed in LocalPattern 19 and LocalPattern 3.

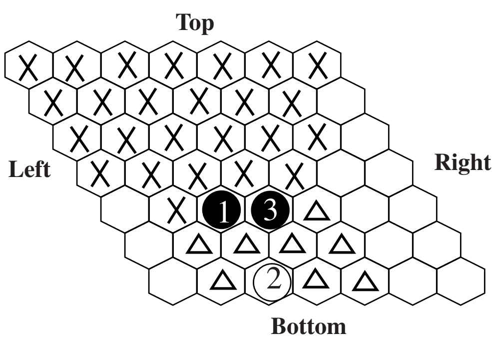

Fig. 16. The regions indicated by X s and △s form LocalPattern 19 and LocalPattern 4.

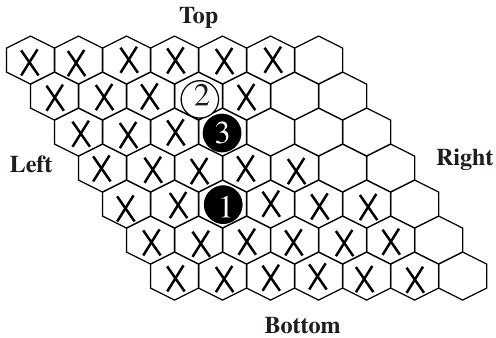

Fig. 17. ❶, ②, ❸, and all positions marked by X s are explained by LocalPattern 11, which will be further discussed in Fig. 36 to Fig. 42.

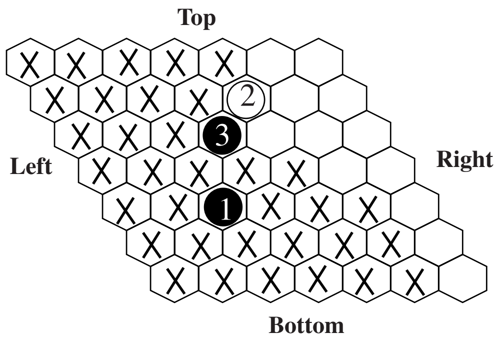

Fig. 18. LocalPattern 12 is slightly different from LocalPattern 11 shown in Fig. 17 in terms of pattern positions and winning strategy.

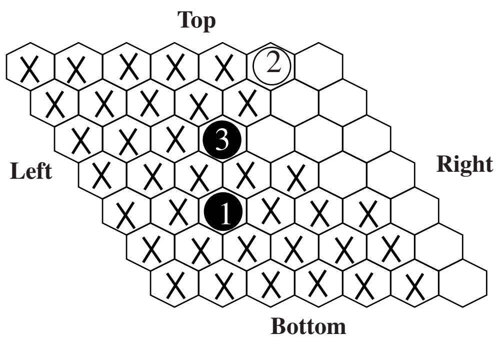

Fig. 19. The influence region marked by X s are explained by LocalPattern 14.

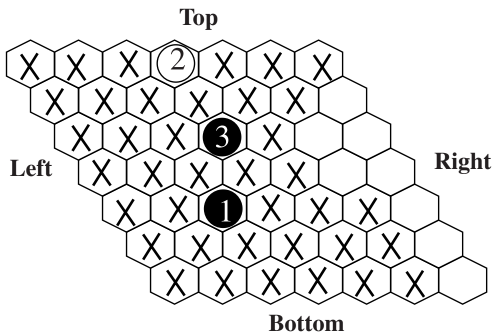

Fig. 20. This figure represents LocalPattern 15.

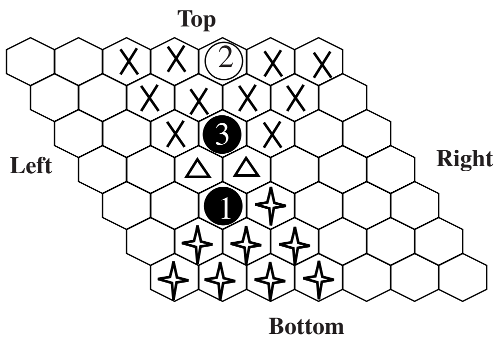

Fig. 21. The influence regions distinguished by X s, △s, and ✦s are explained in LocalPattern 9, LocalPattern 2, and LocalPattern 5, respectively.

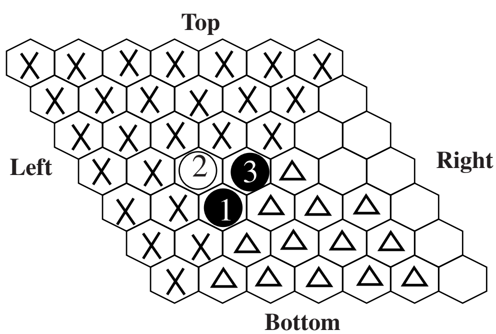

Fig. 22. Responding to ②, ❸ divides the board into two local patterns, LocalPattern 41 (marked by X s) and LocalPattern 21 (marked by △s).

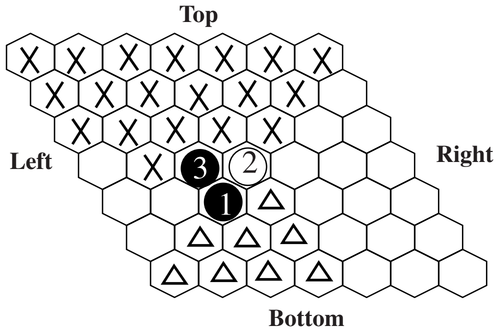

Fig. 23. After Black's ❸, the board can be split into two local patterns, LocalPattern 34 (marked by X s) and LocalPattern 5 (marked by △s).

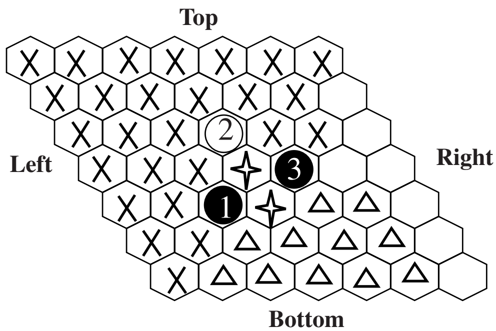

Fig. 24. The influence regions marked by X s, △s, and ✧s are discussed in LocalPattern 22, LocalPattern 21, and LocalPattern 2, respectively. White's ② is the strongest possible defense move in the new solution. Black's ❸ would divide the board into three local patterns. LocalPattern 21 would guarantee Black's ❶ and ❸ to connect to Bottom. LocalPattern 22 is responsible for connecting either ❶ and ❸ to Top, or connecting Top to Bottom directly.

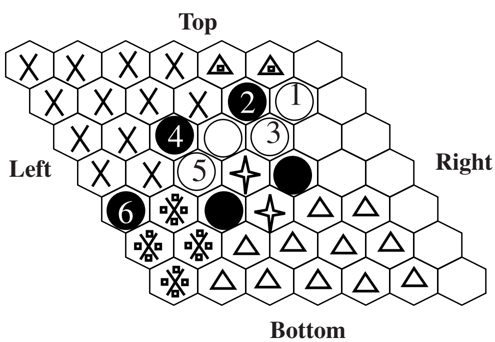

Fig. 25. Further development of Fig. 24. The influence regions marked by X s, ⧍s, ✧s, ⁂ s, and △s are explained by LocalPattern 25, LocalPattern 2, LocalPattern 2, LocalPattern 13, and LocalPattern 21, respectively.

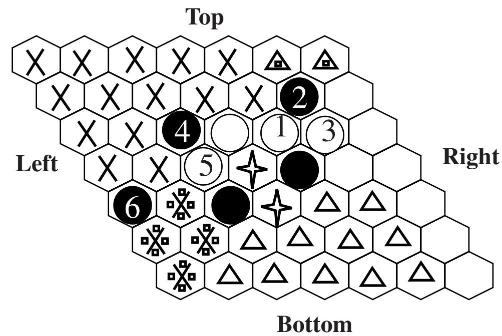

Fig. 26. Similar to Fig. 25, except the influence region marked by X s is covered by LocalPattern 30.

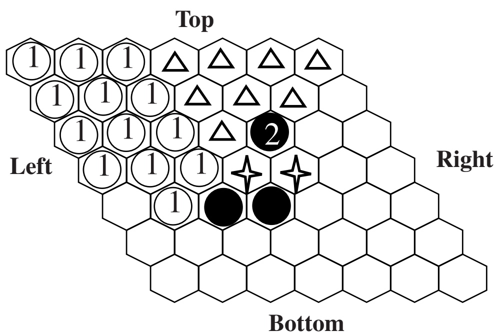

Fig. 27. Referenced to LocalPattern 19, whose influence region is distinguished by X s in Fig. 14, Fig. 15, and Fig. 16, respectively. Black's ❷ will ensure the connection to Top after White plays at one of the ① s.

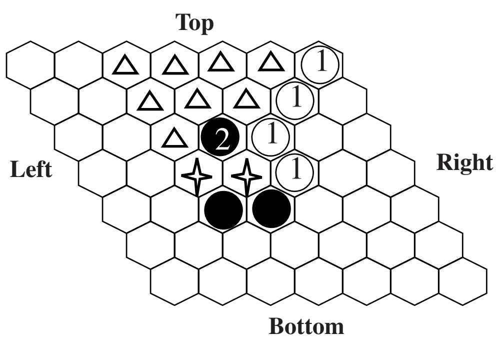

Fig. 28. Referenced to LocalPattern 19, if White plays at one of ①s, Black's ❷ will be able to ensure a connection to Top.

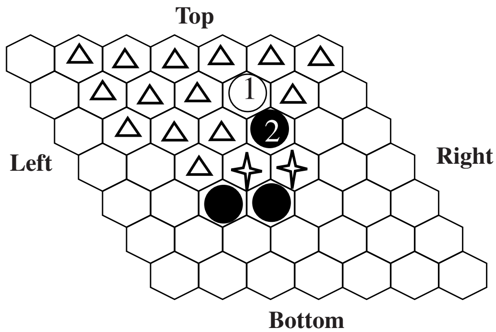

Fig. 29. If White plays at ①, Black's ❷ can ensure a connection to Top. The local influence region marked by △ is explained in LocalPattern 36.

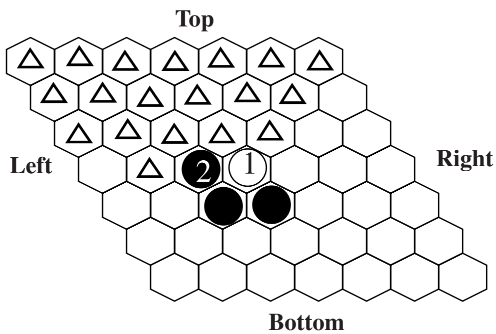

Fig. 30. This is another parallel case associated with the discussion of LocalPattern 19. If White plays at ①, Black's ❷ would ensure a connection to Top. The influence region distinguished by △ is explained in LocalPattern 34.

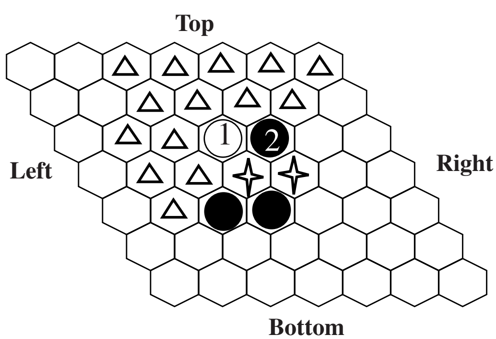

Fig. 31. LocalPattern 39, whose influence region is marked by △s, would ensure a connection between ❷ and Top.

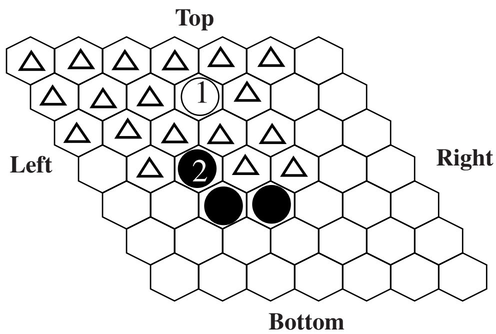

Fig. 32. Black's ❷ responds ① for making a connection to Top. The influence region is explained in LocalPattern 37.

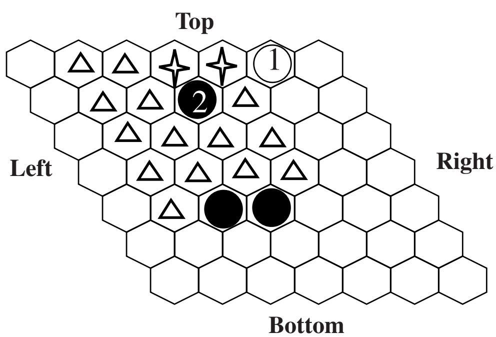

Fig. 33. If White plays at ①, Black can play ❷ to make a connection to Top. This case is discussed in LocalPattern 62.

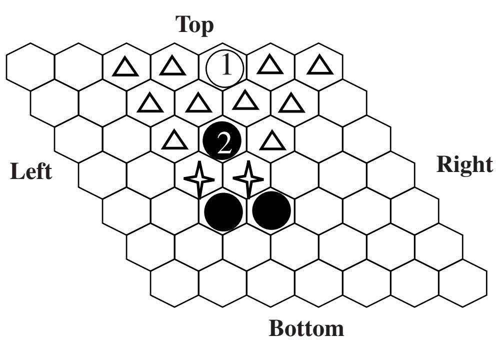

Fig. 34. If White plays at ①, LocalPattern 9, whose influence region is marked by △s, would ensure a connection between ❷ and Top.

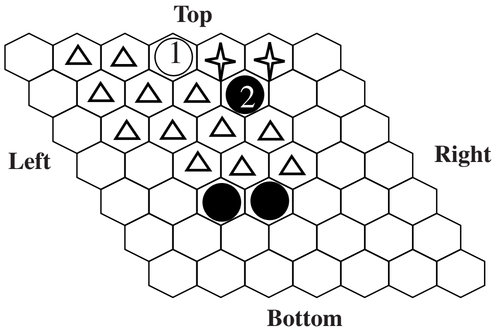

Fig. 35. The influence region distinguished by △s is explained in LocalPattern 38. This is the last case on the further descriptions of LocalPattern 19. From Fig. 27 to Fig. 35, all of White's defense moves in the influence region of LocalPattern 19 are covered.

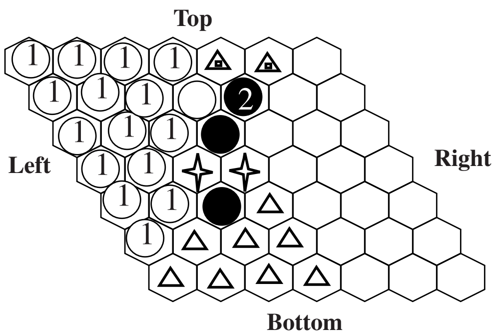

Fig. 36. This is the first figure to describe LocalPattern 11 one layer further. If White plays at any of ①s, Black can play at ❷ to win the game. The influence region marked by △s is illustrated in LocalPattern 5.

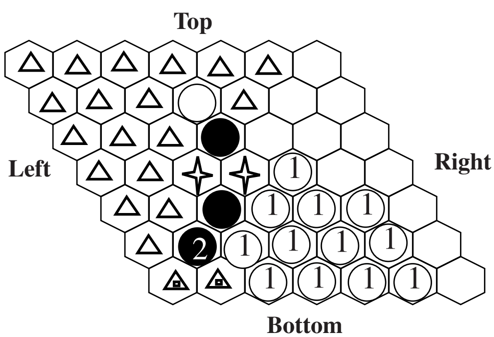

Fig. 37. If White plays at any of ①s, LocalPattern 52, whose influence region is marked by △s, will ensure ❷ a connection to Top.

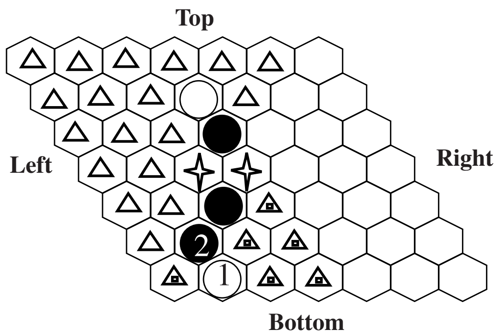

Fig. 38. Similar to the above case, Black is ensured a connection between ❷ and Top.

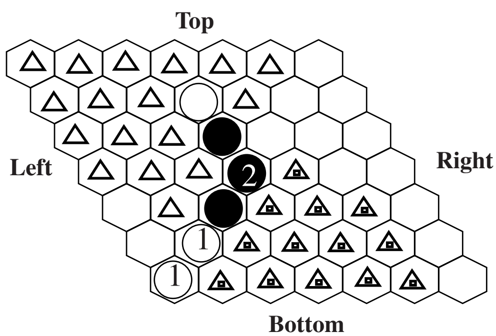

Fig. 39. Black's ❷ divides the board into two independent local regions. LocalPattern 57, marked by △s, ensures a connection to Top; LocalPattern 8, marked by ⚠s, guarantees a connection to Bottom.

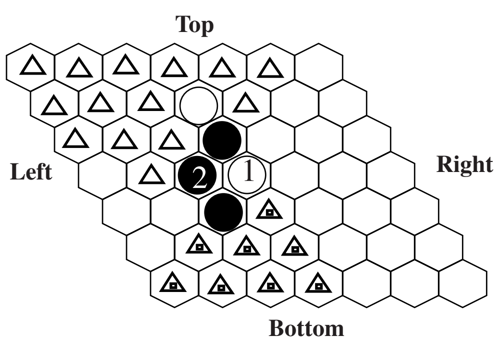

Fig. 40. Distinguished by △s, LocalPattern 36 would ensure a connection between ❷ and Top.

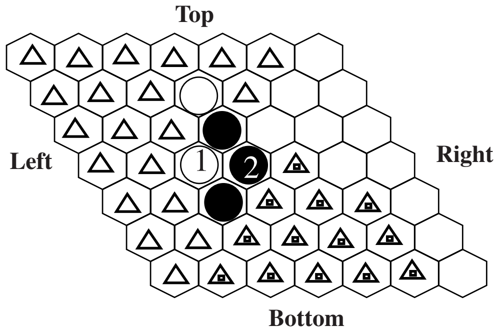

Fig. 41. If White plays at ①, Black can play at ❷ to win the game. Local regions marked by △s and △s are explained in LocalPattern 47 and LocalPattern 8.

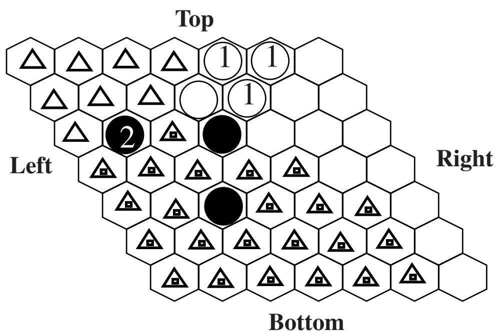

Fig. 42. The influence regions marked by △s and △s are discussed in LocalPattern 5 and LocalPattern 10, which would ensure a connection to Top and Bottom, respectively. This is the last case on the further discussion of LocalPattern 11. From Fig. 36 to Fig. 42, all of White's defense moves in the influence region of LocalPattern 11 are covered.