Meyniel Weakly Triangulated Graphs and a Theorem of Dirac*

RYAN B. HAYWARD†

Abstract. We generalize a theorem due to Dirac and show that every Meyniel weakly triangulated graph has some vertex which is not the middle vertex of any P_5. Our main tool is a separating set notion known as a handle.

Key Words. weakly triangulated graph, Meyniel graph, handle

1. Introduction. C_k and P_k represent the induced cycle and path with k vertices, and $\overline{C}_k$ and $\overline{P}_k$ their complements. The first of the following two theorems is a well-known result of Dirac from 1961; the second is the main result of this paper.

THEOREM 1.1. [D] *Every graph with no C_k for $k \geq 4$ has some vertex which is not the middle vertex of a P_3. Moreover, if the graph is not a clique then it has two non-adjacent such vertices.* □

THEOREM 1.2. *Every graph with no C_k for $k \geq 5$ and no $\overline{P}_5$ has some vertex which is not the middle vertex of a P_5. Moreover, if the graph is not a clique then it has two non-adjacent such vertices.* □

The classes of graphs referred to in Theorems 1.1 and 1.2 are called *triangulated* and *Meyniel weakly triangulated* respectively, the latter being the intersection of *Meyniel graphs* [M] (those in which every odd cycle other than a triangle has at least two chords) with *weakly triangulated graphs* [H1] (those with no C_k or $\overline{C}_k$ for $k \geq 5$).

We prove our theorem by focusing on handles, nontrivial connected subgraphs which satisfy certain separation and adjacency conditions. We first establish new results for handles in arbitrary graphs and then apply these results to Meyniel weakly triangulated graphs.

2. Handles. Roughly, a handle is a non-trivial connected subgraph highly connected to and minimally separated by its neighborhood. Handles were introduced recently as a structural concept useful for graph decomposition [H2]. In this section we extend results on handles by showing how to find a $2K_2$-free handle in an arbitrary graph. ($2K_2$ is the graph with four vertices and two non-adjacent edges.)

Our notation is standard. We use the terms *sees* and *misses* for "is adjacent to" and "is not adjacent to" respectively. The *neighborhood* $N(S)$ of a vertex subset S of

* This research was supported by NSERC Operating Grant OGP-0137764.

† Department of Mathematics and Computer Science, University of Lethbridge, 4401 University Drive, Lethbridge AB Canada T1K 3M4, e-mail `hayward@cs.uleth.ca`

a graph G is the set of vertices of $G - S$ that see at least one vertex of S. A vertex set is *non-trivial* if it contains at least two vertices. We call vertex subsets X and Y of a graph G *separable* if $G[X]$ and $G[Y]$ are connected and $G[X \cup Y]$ is disconnected, and *minimally separated components* if in addition $N(X) = N(Y)$.

A *handle* of a graph G is a non-trivial vertex subset H such that

- $G[H]$ is connected,
- some component J other than H of $G - N(H)$ satisfies $N(J) = N(H)$, and
- every vertex of $N(H)$ sees at least one vertex of every edge of $G[H]$.

Any J as described above is a *co-handle* of H.

Handles are somewhat like tree leaves, in that each can be isolated by a sequence of separating operations. The following algorithms make this analogy precise, by showing how to go from separable sets to minimally separated components to handles to $2K_2$-free handles.

The correctness of Algorithm 2.1 is straightforward, while that of Algorithm 2.2 follows from Theorem 2.1 in [H2].

ALGORITHM 2.1.
Input: *A graph G and separable sets X_0, Y_0.*
Output: *Minimally separated components $X \supseteq X_0$, $Y \supseteq Y_0$.*

1. $Y \leftarrow$ *the component of* $G - N(X_0)$ *containing* Y_0
2. $X \leftarrow$ *the component of* $G - N(Y)$ *containing* X_0 □

ALGORITHM 2.2.
Input: *A graph G and minimally separated components X, Y with X non-trivial.*
Output: *A handle $H \subseteq X$ with co-handle $J \supseteq Y$.*

1. *while some vertex s in $N(X)$ misses both vertices of some edge e of X do*
   (a) $Y \leftarrow$ *the component of* $G - N(e)$ *containing* s
   (b) $X \leftarrow$ *the component of* $G - N(Y)$ *containing* e
2. $H, J \leftarrow X, Y$ □

ALGORITHM 2.3.
Input: *A handle H of a graph G.*
Output: *A $2K_2$-free handle $H^* \subseteq H$ of G with co-handle J^*.*

1. *while H has some $2K_2$ whose edges are p, q do*
   (a) $Y \leftarrow$ *the component of* $H - N(p)$ *containing* q
   (b) $X \leftarrow$ *the component of* $H - N(Y)$ *containing* p
   (c) $H, J \leftarrow$ *Algorithm 2.2*$(G[H], X, Y)$
2. $H^*, J^* \leftarrow H, J$ □

THEOREM 2.4. *Algorithm 2.3 is correct.*

Proof. Consider G, H, p, q, Y, X just before the execution of step 1(c) of the algorithm. Observe that performing steps 1(a) and 1(b) is equivalent to calling Algorithm 2.1 with input $(G[H], p, q)$. By the correctness of this algorithm it follows that output sets X and Y are minimally separated components of $G[H]$ and so $(G[H], X, Y)$ is a valid input for Algorithm 2.2 in step 1(c). Now consider the handle H returned at the completion of step 1(c); in order to distinguish between this and the input handle H, we label the new handle H_1, and its co-handle J_1. Since H_1 is a handle of $G[H]$, H_1 is non-trivial and $G[H_1]$ has some edge; since the co-handle J_1 contains the input set Y which contains the edge q, $G[J_1]$ also contains an edge. Since H is a handle, every vertex of $N_G(H)$ sees at least one vertex of every edge of $G[H]$, and so

$$\begin{aligned} N_G(H_1) &= N_{G[H]}(H_1) \cup N_G(H) \\ \text{and} \quad N_G(J_1) &= N_{G[H]}(J_1) \cup N_G(H)\ . \end{aligned}$$

In other words, H_1 and J_1 are handle and co-handle respectively not only of the graph $G[H]$ but also of the graph G. Now the proof follows from the fact that H_1 does not contain the edge p and so is a proper subset of the original handle H; thus the algorithm terminates and when it does the final handle contains no $2K_2$. □

The following summarizes some consequences of the preceding algorithms.

THEOREM 2.5.

- *A graph has a handle if and only if it contains* $\overline{P}_3$.
- *A graph has two separable handles if and only if it contains* $2K_2$.
- *Every handle contains a* $2K_2$*-free handle.* □

Proof. The first part, proved in [H2], follows by observing that any handle together with its co-handle induces a graph which contains $\overline{P}_3$, and that a $\overline{P}_3$ yields a valid input for Algorithm 2.1 which in turn yields a valid input for Algorithm 2.2.

To show the second part, first observe that any two separable handles together induce a graph which contains $2K_2$. Next, observe that any $2K_2$ is valid input for Algorithm 2.1, and that the output is two non-trivial minimally separated components; relabel these sets A and B. Executing Algorithm 2.2 first with input A,B and then with input B,A yields handles $H_A \subseteq A$ and $H_B \subseteq B$. Since A and B are separable so are H_A and H_B.

The third part is an immediate corollary of Theorem 2.4. □

3. The main result. In this section we prove Theorem 1.2 by considering $2K_2$-free handles in Meyniel weakly triangulated graphs. We use a characterization of handles in Meyniel weakly triangulated graphs from [H2]. We need a bit more terminology.

With respect to a set S of vertices in a graph G, a vertex x of $G-S$ is said to be *S-universal*, *S-partial*, or *S-null* if it sees respectively all, some but not all, or none of the vertices of S; a set X of $G-S$ is *S-universal* or *S-null* if every vertex in X is respectively S-universal or S-null.

THEOREM 3.1. [H2] *Let H be a handle of a Meyniel weakly triangulated graph G=(V,E), let S=$N(H)$, let H_0 be the vertices of H that are S-universal, and let R= V−H−S.*

- *Suppose H_0 is empty. Then there is a partition (H_1, H_2) of H, and a partition (S_1,S_2) or (S_0,S_1,S_2) of S such that*
  1. *$G[H]$ is bipartite, with parts H_1 and H_2,*
  2. *H_k is S_k-universal and S_{3-k}-null, for $k=1,2$,*
  3. *S_1 is S_2-universal,*
  4. *S_0 is ($H \cup S$−S_0)-universal,*
  5. *each vertex of R = V−($H \cup S$) is S_1-null or S_2-null, and*
  6. *each vertex of R partial on a component of $\overline{G}[S_0]$ is ($S_1 \cup S_2$)-null,*
  7. *there are vertices r_1 and r_2 in R such that r_k is ($r_{3-k} \cup S_k$)-universal and S_{3-k}-null, for $k=1,2$.*
- *Suppose H_0 is non-empty. Let $H_1 = H - H_0$. Then*
  8. *$G[H_1]$ is an independent set, and*
  9. *every vertex of H_1 is S'-null or S'-universal, for each component S' of $\overline{G}[S]$,*
  10. *some vertex of R is S-universal.*[1]

□

We now prove our main result. For fixed k, a vertex v is *mid-P_k* if there is some P_k $(p_1, p_2, \ldots, p_k)$ such that $v = p_{\lceil k/2 \rceil}$.

Proof of Theorem 1.2. Let G be a Meyniel weakly triangulated graph. If G is $2K_2$-free then no vertex of G is the middle of a P_5 and the theorem holds. Suppose then that G has some $2K_2$. By Theorem 2.5 G has separable handles X and Y, and each contains a $2K_2$-free handle, so G has separable $2K_2$-free handles X' and Y'. If each of these contains some not mid-P_5 vertex, then the theorem holds. Suppose then that for one of these sets, say X', every vertex is mid-P_5 in G. Relabel X' as H, and let S, H_0 and R be as described in Theorem 3.1.

[1] Also, there is some handle in which an additional property holds when H_0 is non-empty, namely
11. every P_4 of $G[H]$ has both middle vertices in H_0.
The proof follows by applying an algorithm similar to Algorithm 2.3 to a $2K_2$-free handle in a Meyniel weakly triangulated graph, using any P_4 which violates the conclusion to find a suitable starting set.

First suppose that H_0 is empty. Let h_1 be any vertex of H_1. Since every vertex of H is mid-P_5, G has some P_5 (a, b, h_1, d, e). Theorem 3.1 and $G[H]$ being $2K_2$-free (and so P_5-free) imply that b and d are in S_1 and a and e in R, and that furthermore (a, b, s_2, d, e) is a P_5 for every vertex s_2 in S_2. It follows that every vertex in S_2 is mid-P_5 in G. Similarly, by considering any vertex h_2 in H_2 it follows that every vertex in S_1 is mid-P_5 in G.

Let G' be the graph obtained from G by removing all edges of $G[H]$. Using Theorem 3.1, it is routine to verify that G' is Meyniel weakly triangulated. Observe that every vertex of $H \cup S_1 \cup S_2$ is mid-P_5 in G'. By induction on the number of edges, we may assume that the theorem holds for G', so (since G' is not a clique) G' contains two non-adjacent vertices v_1 and v_2 that are each not mid-P_5 in G'. By the conclusion of the preceding paragraph, these vertices are in $V - H - S_1 - S_2 = S_0 \cup R$. Now by the following observation v_1 and v_2 are not mid-P_5 in G as well as G', and the theorem holds for this case.

OBSERVATION 3.2. *Let G' be a graph obtained by removing a set of edges T from a graph G, let v be a vertex that is not mid-P_5 in graph G', and let v be universal or null in G on each edge of T. Then v is not mid-P_5 in G.*

Next suppose that H_0 is non-empty. Let H_1 be $H - H_0$. Argue as in the previous case. Let h be any vertex of H; every vertex of H is mid-P_5, so G has some P_5 (a, b, h, d, e). Theorem 3.1 and H being $2K_2$-free imply that b and d are in S and a and e in R. Furthermore, let s be any vertex of S that misses some vertex h of H; since H_0 is S-universal, $h = h_1$ is in H_1. The vertex h_1 is mid-P_5 in G, so there is some P_5 (u, v, h_1, x, y) with v and x in S and u and y in R. Let S' be the component of $\overline{G}[S]$ containing v and x. By (9) of Theorem 3.1 h_1 is universal on S', so s is in $S - S'$, so s sees v and x, and misses u (otherwise $\{s, h_1, u, x, v\}$ is a $\overline{P}_5$ of G, contradiction) and similarly misses y. Thus (u, v, s, x, y) is a P_5 and s is mid-P_5 in G.

Again, let G' be the graph obtained by removing all edges of $G[H]$; again, using Theorem 3.1 it is routine to verify that G' is Meyniel weakly triangulated. By inductive assumption G' has two non-adjacent vertices v_1 and v_2 that are not mid-P_5; by the arguments above each of these vertices is either in R, or in S and H-universal. By Observation 3.2 v_1 and v_2 are not mid-P_5 in G, and the theorem holds. □

4. Related results. Results with a form similar to Dirac's theorem and our generalization include the following. (D_6, sometimes called the *domino*, is the graph obtained by adding an edge between the end vertices of a P_4 in a C_6.)

THEOREM 4.1. [HK] *Every graph with no C_k for $k \geq 5$, no $\overline{P}_5$ and no D_6 has some vertex not mid-P_4. Moreover, if the graph is not a clique then it has two*

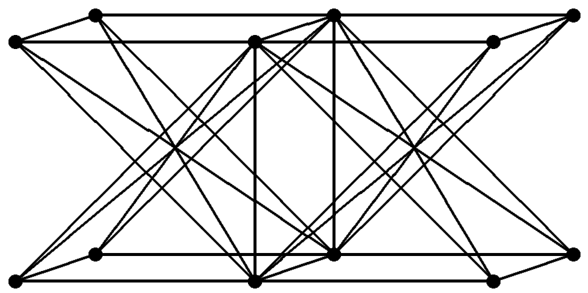

FIG. 1. *A minimal graph for the property "every vertex is the middle of a P_5".*

non-adjacent such vertices. □

THEOREM 4.2. [HMOP] *Every graph with no C_k for $k \geq 5$, no $\overline{P}_5$ and no P_6 has some vertex not end-P_5.* □

It is natural to ask to what extent results of this form might be strengthened:

> *for fixed k, what are the sets of graphs $\mathcal{M}_k$ and $\mathcal{E}_k$ for which every induced subgraph has some vertex not mid-P_k and not end-P_k?*

One way to describe $\mathcal{M}_k$ and $\mathcal{E}_k$ is to specify the corresponding sets $f(\mathcal{M}_k)$ and $f(\mathcal{E}_k)$ of minimal forbidden graphs. A graph is in $f(\mathcal{M}_k)$ if and only if every vertex is mid-P_k and every proper induced subgraph has some vertex not mid-P_k. The set $f(\mathcal{E}_k)$ is defined analogously. An element of $f(\mathcal{M}_5)$ is shown in Figure 1. The sets $f(\mathcal{M}_k)$ and $f(\mathcal{E}_k)$ are easily determined for $k \leq 3$, and unknown for larger k. (K_t is the complete graph with t vertices.)

$$
\begin{aligned}
f(\mathcal{M}_1) = f(\mathcal{E}_1) &= \{K_1\} \\
f(\mathcal{M}_2) = f(\mathcal{E}_2) &= \{K_2\} \\
f(\mathcal{M}_3) &= \{C_k \mid k \geq 4\} \\
f(\mathcal{E}_3) &= \{C_k \mid k \geq 4\} \cup \{P_4\}
\end{aligned}
$$

Dirac's theorem cannot be strengthened, since the set of graphs excluded in the hypothesis is exactly $f(\mathcal{M}_3)$. On the other hand, each of Theorems 1.2, 4.1 and 4.2 can be strengthened, since for $k \geq 4$ neither $f(\mathcal{M}_k)$ nor $f(\mathcal{E}_k)$ contains $\overline{P}_5$.

There are many other forms of generalization of Dirac's theorem. One involves replacing the property "not mid-P_3" of the conclusion with a list of possible properties, such as "not mid-P_4, or not end-P_4"; see [HK] and [HMOP]. Another involves replacing the vertex property with an edge property, for example "not the middle edge of a P_4"; see [CR] and [ES].

Acknowledgements. I thank Derek Corneil and the Department of Computer Science at the University of Toronto for hospitality extended during the writing of this paper.

REFERENCES

[CR] V. Chvátal and I. Rusu, A note on graphs with no long holes, to appear in *Proc. 1993 Workshop on Perfect Graphs, Princeton NJ.*

[D] G.A. Dirac, On rigid circuit graphs, *Abh. Math. Sem. Univ. Hamburg* **25** (1961) 71-76.

[ES] E. Eschen and R. Sritharan, A characterization of some graph classes with no long holes, *J. Comb. Theory* **B** 65 (1995) 156-162.

[H1] R.B. Hayward, Weakly triangulated graphs, *J. Comb. Theory, Series B* **39** (1985) 200-209.

[H2] R.B. Hayward, On P_5-free, weakly triangulated, and perfectly orderable graphs, University of Lethbridge DMCS-TR-CS-02-94.

[HK] C.T. Hoàng & N. Khouzam, On brittle graphs, *J. Graph Theory* **12** (1988) 391-404.

[HMOP] C.T. Hoàng, F. Maffray, S. Olariu & M. Preissman, A charming class of perfectly orderable graphs, *Discrete Math.* **102** (1992) 67-74.
On rigid circuit graphs, *Abh. Math. Sem. Univ. Hamburg* **25**, 71-76.

[M] H. Meyniel, On the perfect graph conjecture, *Discrete Math.* **16** (1976) 334-342.

Meyniel Weakly Triangulated Graphs
and a Theorem of Dirac

Ryan B. Hayward
hayward@cs.uleth.ca

November 1995

Report TR-CS-03-95