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The Story of Perfectly Orderable Graphs

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Abstract. We give the story behind one examplar of the work of VašekChvátal, namely his conception of the class of perfectly orderable graphs.

Key words. Perfect order, Perfectly orderablegraph, Chvátal

I cannot hope in only a few pages to survey all, or most, or even a significant portion, of the many scholarly achievements of Vašek Chvátal. Instead, I present here the story behind one exemplar of his work, the story of perfectly orderable graphs.

The main part of this story takes place in 1981 around the time of Chvátal's 35th birthday; the origins go back further. In the 1960s, the development of the silicon-chip computer had many scientists thinking about two fundamental issues: measuring algorithm efficiency, and measuring problem hardness. Around the time of his 25th birthday, Chvátal was working on the latter issue. In particular, he was thinking about Jack Edmonds' 1965 paper on the matching polytope[1] of a graph [12]. Ever since Dantzig developed his efficient simplex algorithm [10], there has been interest in knowing when the fractional relaxation of an integer polytope is equal to the polytope; Edmonds had just shown that for any matching polytope, adding the "blossom inequality"[2] cutting planes to the fractional relaxation yields the original polytope.

Chvátal realized that the "hardness" of an integer program could be measured in terms of a cutting plane metric defined on the program's associated integer polytope. In his paper "*On a hierarchy of Edmonds polytopes*" [7] Chvátal showed that any integer polytope can be obtained from its fractional relaxation by iterating the following step a finite number of times; the cutting planes described in this step are now called *Chvátal–Gomory cutting planes*.

> *To the system of integer inequalites* $A\overline{x} \leq \overline{b}$ *defining the current polytope, add all inequalities of the form* $(\overline{y}^T A)\overline{x} \leq \lfloor \overline{y}^T \overline{b} \rfloor$, *where* $\overline{y}^T$ *is a non-negative row vector such that* $(\overline{y}^T A)$ *is integral.*

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[1] A matching in a graph is a set of pairwise nonadjacent edges. The matching polytope of a graph, so-called because it is the convex hull of the characteristic edge vectors of the graph's matchings, is the integer polytope defined by $A\overline{x} \leq \overline{1}, \overline{x} \geq \overline{0}$, where each row of A corresponds to the set of edges incident with a particular vertex.

[2] The blossom inequalities are as follows: for every set of edges between an odd-sized set of vertices and all remaining vertices, the sum over the set of the edge weights is at least 1.

Chvátal's polytope hardness metric is simply the number of iterations required before the fractional relaxation equals the original polytope; this number is now called the *Chvátal rank* of the polytope. For example, the Chvátal rank of a polytope whose fractional relaxation has only integral vertices is 0. As another example, since blossom inequalities are Chvátal-Gomory cuts, the Chvátal rank of a matching polytope is at most 1.

As Chvátal was writing his polytope hierarchy paper, the graph theory world was buzzing over the latest news on perfect graphs[3]: László Lovász had proved Claude Berge's "weak" perfect graph conjecture [18, 19]. Since the early 1960s Berge had been promoting the study of perfect graphs via his "weak" and "strong" conjectures[4] [1–3]; Lovász's result was seen as a breakthrough towards resolving the strong conjecture[5]. Chvátal and Lovász were among the invitees to the seminar on hypergraphs organized by Berge in the summer of 1972 at Ohio State University; around this time, Chvátal starting working on perfect graphs.

Chvátal soon discovered an unexpected relationship between perfect graphs and "easy" polytopes. He had long been interested in the problem of determining the stability number[6] of a graph. The usual integer program formulation of this problem is

$$\max \overline{1}^T \overline{x} \quad \text{such that} \quad A\overline{x} \le \overline{1}\,, \;\; \overline{x} \ge \overline{0}\,,$$

where A is the clique-vertex matrix[7] of the graph. This formulation led Chvátal to wonder when, for an arbitrary 0–1 matrix A, the polytope of the system $A\overline{x} \le \overline{1}$, $\overline{x} \ge \overline{0}$ is "easy", namely has Chvátal rank 0. In his paper "*On certain polytopes associated with graphs*" [8] he answered this question: for a 0–1 matrix A in a certain standard form[8], A has Chvátal rank 0 if and only if A is the clique-vertex matrix of a perfect graph.

Since linear programs[9] can be solved in polynomial time, it follows that, for perfect graphs with a polynomially bounded number of maximal cliques, the stability number can be found in polynomial time. A few years after Chvátal's graph polytope paper appeared, Grötschel, Lovász, and Schrijver found polynomial time ellipsoid-like algorithms for determining the chromatic number of arbitrary perfect graphs [14, 15].

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3 A graph is perfect if, for every induced subgraph, its chromatic number equals its clique size.

4 The weak conjecture: a graph is perfect if and only if its complement is perfect. The strong conjecture: a graph is perfect if and only if neither the graph nor its complement contains an induced odd cycle with at least five vertices.

5 The strong conjecture was finally confirmed in 2002 by Chudnovsky et al. [5].

6 The stability number of a graph is the size of a smallest set of pairwise non-adjacent vertices.

7 The clique-vertex matrix of a graph is the 0–1 matrix whose rows index the maximal cliques and whose columns index the vertices.

8 The matrix is in standard form if no row dominates another. A row dominates another if for each index, its component is at least as large as the corresponding component of the other.

9 See Chvátal's text *Linear Programming* [6] for a primer on linear programming.

Many of these results were fresh in Chvátal's mind in the summer of 1981 when, at the invitation of Claudio Lucchesi, he was in the midst of presenting a (presumably 5 day) series of two-hour talks on combinatorial optimization at the University of Campinas. Campinas is not far from São Paulo and so, lured by the siren song of the big city, Chvátal soon established a daily routine: 10:00 to noon, lecture; noon to 2:00, lunch and espresso; 14:00 to 16:00, bus to São Paulo; 22:00 to midnight, bus back to Campinas.

On one of the return trips, while staring out the darkened bus at the moonlit Brazilian landscape, Chvátal found himself thinking about graph colouring. While graph colouring is NP-complete in general [17], it can be solved in polynomial time for many classes of graphs. In one of the Campinas lectures Chvátal had reviewed why colouring is especially easy for three well known classes of perfect graphs.

Consider for example chordal graphs[10]. By a theorem of Dirac [11], every chordal graph has a simplicial vertex[11]; as observed by Fulkerson and Gross [13], it follows that chordal graphs are characterized by the existence of a linear vertex order in which each vertex is simplicial with respect to the vertices up to that point in the order. Now it follows easily that applying a certain greedy colouring algorithm[12] to such a vertex order yields a minimum colouring of any chordal graph.

Comparability graphs[13] and complements of chordal graphs also have this property: it is easy to find a linear vertex order for which the greedy algorithm yields a minimum colouring.

As the bus rolled towards Campinas, Chvátal wondered: which graphs allow such a vertex order? It did not take long to see that the answer is "all graphs", so this question was not so interesting.

Chvátal next considered the corresponding "induced subgraph" version of the question: which graphs allow a *perfect order*, namely a linear vertex order so that, for every vertex induced subgraph, the greedy colouring of the subgraph (using the corresponding induced linear order) yields a minimum colouring?

This question was more interesting. Chvátal realized that, with respect to a linear vertex order, one "obstruction" to a perfect order—an ordered induced subgraph that can prevent the greedy algorithm from yielding a minimum colouring—is a "bad P_4", namely an induced path with four vertices (say $abcd$) such that each end vertex precedes its neighbouring path vertex in the order (so a precedes b, and d precedes c). He suspected that this was the only obstruction and, when he next had access to daylight, pencil, and paper, he proved it. Thus Chvátal had a definition (a graph is *perfectly orderable* if some linear vertex order is a perfect order) and a theorem (a graph is perfectly orderable if and only if some linear vertex order induces no bad P_4).

[10] A graph is chordal if every cycle with four or more vertices has a chord. Equivalently, a graph is chordal if it has no induced cycle with four or more vertices.

[11] A vertex is simplicial if its neighbours induce a clique.

[12] The algorithm is as follows. Use positive integers as colours; assign the first vertex colour 1; assign each subsequent vertex the smallest colour not yet assigned to any neighbour.

[13] A graph is a comparability graph if it admits a transitive edge orientation.

The following week, in Rio de Janeiro at the invitation of Jayme Szwarcfiter, it was on one of Rio's famous beaches that Chvátal finished writing "*Perfectly Ordered Graphs*" [9]. Three years later the paper appeared in the volume *Topics on Perfect Graphs* [4] edited by Chvátal and Berge.

Of course, the story of perfectly orderable graphs continues. Since Chvátal's seminal paper appeared in 1984, almost 100 papers on perfectly orderable graphs have been published. For a recent survey, see the chapter by Chính Hoáng [16] in the volume *Perfect Graphs* edited by Jorge Ramírez-Alfonsín and Bruce Reed [20].

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