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How to Play Reverse Hex

Ryan B. Hayward[1], Bjarne Toft[2], and Philip Henderson[3]

[1] Dept. Comp. Sci., University of Alberta, hayward@ualberta.ca
[2] IMADA, Syddansk Universitet, btoft@imada.sdu.dk
[3] Dept. Comp. Sci., University of Alberta, ph@cs.ualberta.ca
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Abstract. We present new results on how to play Reverse Hex, also known as Rex, or Misère Hex, on $n\times n$ boards. We give new proofs — and strengthened versions — of Lagarias and Sleator's theorem (for $n\times n$ boards, each player can prolong the game until the board is full, so the first/second player can always win if n is even/odd) and Evans's theorem (for even n, opening in the acute corner wins). Also, for even $n \geq 4$, we find another first-player winning opening (adjacent to the acute corner, on the first player's side), and for odd $n \geq 3$ and for each first-player opening, we find second-player winning replies. Finally, in response to comments by Martin Gardner, for each $n \leq 5$ we give a simple winning strategy for the $n\times n$ board.
Key words: Hex; Reverse Hex; Rex; Misère Hex

1 Background

Reverse Hex, also known as Rex or as Misère Hex, is the variant of Hex in which the player who connects her two sides loses.

Rex uses a Hex board, i.e., an $m\times n$ array of hexagons. There are two players: each is assigned her own color — say black or white — and two opposing sides of the board. Players alternate moves; white moves first. On each move, a player colors any empty cell. The loser is the player who forms a *connecting chain* — a set of cells of her color connecting her two opposing sides. See Figure 1.

A Hex board with every cell colored has a connecting chain for exactly one player, so — like Hex — Rex cannot end in a draw. (For more on Hex, including a proof of this result, see the survey by the first author and Jack van Rijswijck [9] or the books by Cameron Browne [1, 2]). Thus either the first player or the second player wins, i.e., has a winning strategy.

In the July 1957 *Mathematical Games* column in *Scientific American*, Martin Gardner introduced Hex and Rex to his readers. He

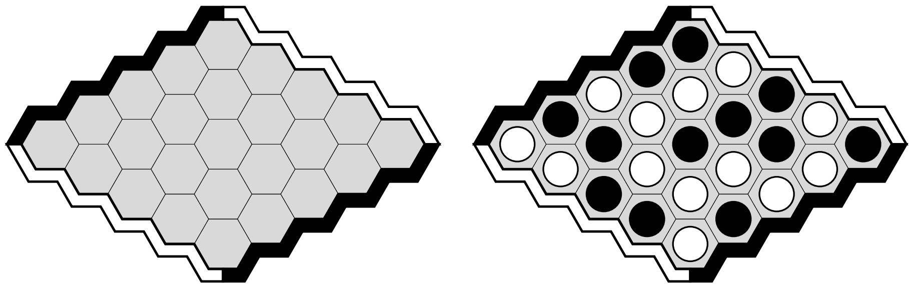

Fig. 1. An empty 5×5 board and a Reverse Hex game lost by white.

reported a result by Robert Winder: for Rex on the $n \times n$ board, the first player wins if n is even, and the second player wins if n is odd [4–8]. Winder's proof was apparently never published, but in 1974 Ronald Evans proved that for even n the acute corner is a winning opening move for the first player [3], and in 1999 Jeffrey Lagarias and Daniel Sleator proved that each player can avoid creating a connecting chain until every cell is colored, which implies Winder's result [10].

In 1988 Martin Gardner commented further: "*... it is easy to see the win on the 3×3* [board], "*but the 4×4 is so complex that a winning line of play for the first player remains unknown. David Silverman reported in a letter that he had found an unusual pairing strategy for a second-player win on the 5×5.* [6](p.183),[8](p.91).

In this paper we say more about how to play Rex. We give new proofs and strengthened versions of Lagarias and Sleator's theorem and Evans's theorem. Also, for even $n \geq 4$ we find another first-player winning opening (adjacent to the acute corner, on the first player's side), and for odd $n \geq 3$ and for each first-player opening we find a winning second-player reply. Finally, in response to Gardner's comments we give simple winning strategies for board sizes up to 5×5.

2 Notation, holes, and color-symmetry

Throughout the paper, X is an arbitrary player and Y is X's opponent. We describe a game state (P, X) by giving the position P and the player to move X.

To simplify our arguments, we allow positions in which particular cells are excluded from play. We call such excluded cells – which neither player is allowed to color – *holes*. Thus, to describe a position, we specify for each cell whether it is black, white, a hole, or empty (uncolored). See Figure 2.

Punctured Rex (PRex) is Rex played on a position which can have holes. A game of PRex can end with no connecting chain, in which case it is a draw.

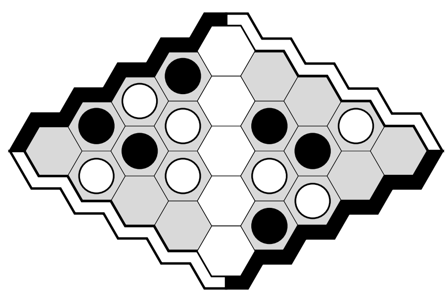

Fig. 2. A position with holes on the short diagonal. Neither player can win.

Some PRex results which hold for an empty board hold more generally for positions which look the same for the two players. A position is *color-symmetric* if the position is topologically identical for each player, i.e., if changing the color of each side and each colored cell and then either reflecting the board through the long diagonal, or reflecting the board through the short diagonal, yields a position that is identical to the original position; the associated diagonal is the *symmetry axis*. A color-symmetric position can have holes. Notice that every cell on the symmetry axis of a color-symmetric position must be either empty or a hole. See Figure 3.

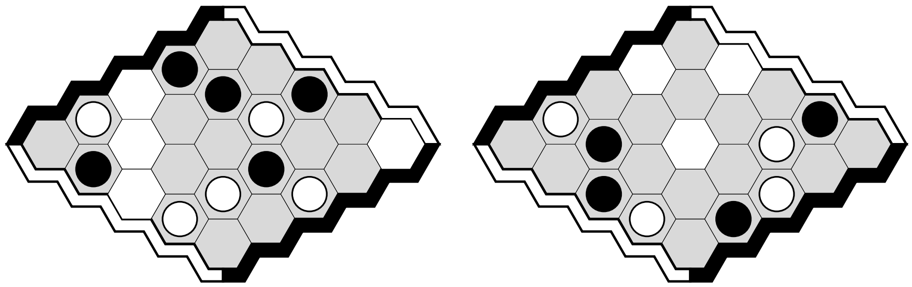

Fig. 3. Left, a color-symmetric position with symmetry axis the long diagonal. Right, a color-symmetric position with symmetry axis the short diagonal.

3 TRex

In reasoning about PRex it is helpful to consider the variant we call Terminated PRex (TPRex), in which the game is terminated just before the board is filled. The rules for TPRex are as for PRex, with one change: *a player moves only if there remain at least two empty cells; if there remains at most one empty cell (and neither player has lost) then the game ends in a draw.* See Figure 4.

The following lemma — inspired by an argument used by Lagarias and Sleator — shows that for TPRex, (un)coloring a cell changes a winning state into (at worst) a non-losing state.

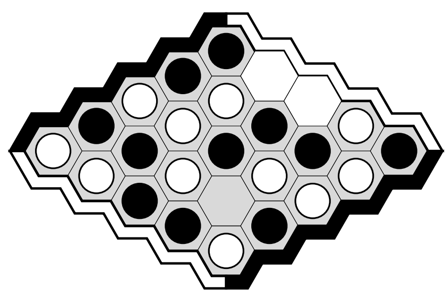

Fig. 4. A TPRex game that ended in a draw.

Lemma 1. *For TPRex from a position P with an empty cell c, for $Z = X$ or Y, if X has a winning TPRex strategy for state $(P + Y(c), Z)$, then X has a non-losing TPRex strategy for state (P, Z).*

Equivalently: if Y has a winning TPRex strategy for state (P, Z), then Y has a non-losing TPRex strategy for state $(P + Y(c), Z)$.

Proof. To play from the state (P, Z), X imagines that c is Y-colored and follows the strategy from $(P + Y(c), Z)$. If Y ever plays in the imaginary Y cell, creating a position Q, then X imagines that some new empty cell d is Y-colored and follows the strategy from $(Q + Y(d), Z)$. Eventually X reaches a point in her strategy where Y creates a connecting chain, which is either actual (i.e., contains no imaginary Y cell) or imaginary (i.e., contains exactly one imaginary cell). In each case this chain forms a cutset separating X's two sides. The chain contains at most one imaginary cell, so from this point X can avoid losing by never playing in the imaginary cell; this is always possible, since in TPRex a player moves only if there exists at least two empty cells.

Theorem 1. *For TPRex from a color-symmetric position P with at least two empty cells, the player to move has a non-losing strategy.*

Proof. Consider the state $S = (P, X)$. We want to show that X has a non-losing strategy for S. Argue by contradiction: assume Y wins S. Thus, for any empty cell c in P, Y wins $(P+X(c), Y)$. By Lemma 1, Y does not lose (P, Y). But P is color-symmetric, so by exchanging colors X does not lose $(P, X) = S$. But now Y wins S and X does not lose S, contradiction.

Theorem 2. *For TPRex from a color-symmetric position P with at least two empty cells, the player not to move has a non-losing strategy.*

Proof. Consider the state $S = (P, X)$. We want to show that Y does not lose S. Argue by contradiction: assume X wins S. Thus, for some empty cell c in P, X wins $S' = (P + X(c), Y)$.

Now observe that any winning X strategy for S' is a winning X strategy for $S'' = (P + Y(c), Y)$, since changing a cell's color from X to Y is never disadvantageous for X. Thus X wins S''.

By Lemma 1, X does not lose (P, Y). P is color-symmetric, so by color exchange and reflection Y does not lose $(P, X) = S$. But X wins S, contradiction.

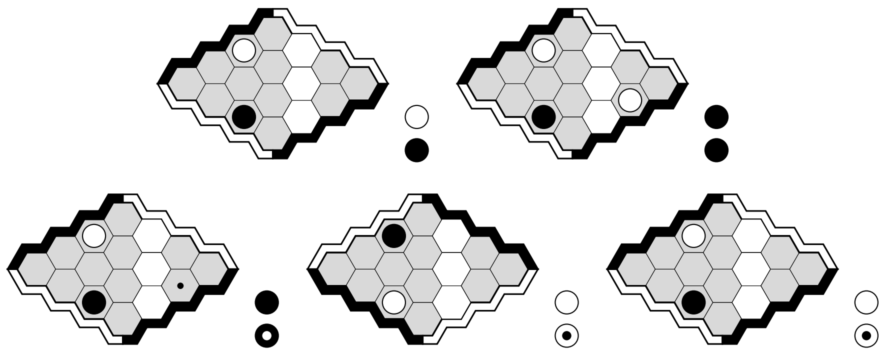

Fig. 5. Proof of Theorem 1. Off-board circles show player to move (top) and player to win (bottom, if solid) or not to lose (bottom, if dotted). Assume color-symmetric position, $X = W$ (white) to play, $Y = B$ (black) to win (1). After W moves, B to play, B to win (2). Lemma 1: uncolor W cell, B to play, B not to lose (3). Exchange colors: W to play, W not to lose (4). Reflect board: position now same as original, but with contradicting outcome (5).

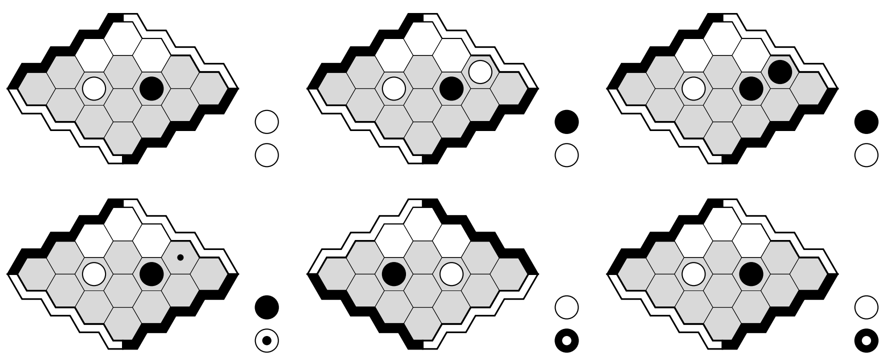

Fig. 6. Proof of Theorem 2. Assume color-symmetric position, $X = W$ (white) to play, W to win (1). After W moves, B to play, W to win (2). Change last move to B, still B to play, W to win. (3). Lemma 1: uncolor B cell, B to play, W not to lose (4). Exchange colors: W to play, B not to lose (5). Reflect: position same as original, contradicting outcome (6).

4 Who wins Rex

Using our TPRex results, we can strengthen Lagarias and Sleator's theorem — and thereby Winder's theorem — by finding the winner not only from the start of the game, but from any color-symmetric position, including those with holes.

Theorem 3. *For PRex from a color-symmetric position P with k empty cells, each player has a strategy which avoids creating a connecting chain as long as there remain at least two empty cells. Thus, if k is even (resp. odd), then the first (resp. second) player has a non-losing strategy; furthermore, if P has no holes, this player has a winning strategy.*

Proof. The first statement follows by having each non-winning player use a non-losing TPRex strategy guaranteed by the previous theorems.

Let X be the first (resp. second) player if k is even (resp. odd). Before each move by X, the number of empty cells is at least two. Thus X can use any non-losing TPRex strategy for the associated PRex game, so X has a non-losing PRex strategy on P. But if P has no holes, then no draw is possible, so X's non-losing PRex strategy must be a winning Rex strategy.

5 Winning openings on even boards

In this section we strengthen Evans's theorem by showing that both the acute corner and a neighboring cell are winning Rex openings on even boards. Furthermore, our theorem applies not only to the empty board, but to any color-symmetric position, including those with holes; this is useful in the proof of Corollary 1. The next two lemmas — inspired by similar arguments by Evans which he used in more restricted situations — show that for PRex, coloring a cell for the last player (without changing the turn) does not change the outcome (assuming perfect play).

Lemma 2. *For PRex from a position P with an even number $k \geq 2$ of empty cells, if X wins (P, Y) then, for each empty cell c, X wins $(P + X(c), Y)$.*

Proof. Assume X wins the state $S = (P, Y)$ in PRex with strategy W. Before each Y-move there are at least two empty cells. Observe therefore that, for each empty cell d in a state (Q, X) reachable from S when Y plays against X's strategy W, the position $Q + X(d)$ has no connecting X-chain: otherwise, Y can win from Q by never coloring d.

To win from the state $S' = (P + X(c), Y)$, X imagines that c is empty and follows this strategy W' adapted from W: whenever W requires X to color the current imagined-empty cell, X instead colors any other empty cell, and then sets this other cell to be the new imagined-empty cell. In play from S the number of empty cells before each Y-move is positive and even; so in play from S', before each Y-move (resp. X-move) the number of empty cells is positive and odd (resp. even); so it is always possible for X to color an empty cell. By the observation above, an X-move never creates a connecting X-chain.

Let Q' be a position reached from $(P + X(c), Y)$ by X following strategy W', and let Q be the corresponding position reached from (P, Y) by X following strategy W. To finish the proof, we claim that Q' has a Y-connecting chain if Q' has no empty cells.

Argue as follows. For all pairs Q and Q', Q has one more empty cell than Q'. Thus Y's available move choices in Q' are a strict subset of Y's available move choices in Q, so the set of Y-colored cells in Q' is the same as in Q. Thus if Q' has no empty cells then Q has exactly one empty cell and so a Y-connecting chain (since W wins for X from (P, Y)), so Q' has a Y-connecting chain. So the claim holds, and in PRex W' wins for X from $(P + X(c), Y)$.

Lemma 3. *For PRex from a position P with an odd number $k \geq 3$ of empty cells, if X wins (P, X) then, for each empty cell c, X wins $(P + X(c), X)$.*

Proof. Assume X wins (P, X). Then, for some empty cell b in P, X wins $(P + X(b), Y)$. By Lemma 2, for each empty cell c of $P + X(b)$, X wins $(P + X(b) + X(c), Y)$. Thus X wins $(P + X(c), X)$ by playing b and $(P + X(b), X)$ by playing c.

For a board, a *corner wedge* comprises an acute corner cell and the five other cells nearest that corner. See Figure 7, where the corner wedge is shaded. Notice that an $n \times n$ board has a corner wedge if and only if $n \geq 3$. A corner wedge is *empty* if all cells in it are empty.

With respect to a player X, a cell is *peripheral* if it is on X's own side, not an obtuse corner, and either an acute corner or adjacent to an acute corner. See Figure 8, where the white-peripheral cells are all marked.

This lemma will be useful in the next theorem.

Lemma 4. *For a player X and a PRex position P, let P' be a position obtained by uncoloring an even number of X-colored cells. For $Z = X$ or Y, if X wins (P, Z) then X wins (P', Z).*

Proof. Modify a winning strategy for P by adding a pairing strategy on the set of cells U that were uncolored: X plays in U only if Y has just played there, or if there are no other empty cells (in which case each remaining Y-move must be in U). Then, for every terminal state in a game played from P', the set of Y-colored cells is a superset of the set of Y-colored cells in a corresponding game played from P. Y forms a connecting chain in the latter game, and so also in the former.

Theorem 4. *Consider PRex from a color-symmetric $n \times n$ position P such that the long diagonal is a symmetry axis, $n \geq 3$ and a corner wedge is empty, and the total number of empty cells is even. Let c be a cell that is in an empty wedge and X-peripheral. Then X does not lose $(P + X(c), Y)$.*

Proof. Without loss of generality, assume $X = W$ (white) and $Y = B$ (black). The proof is illustrated in Figure 7 and uses board labels shown there. Cell c is either $a1$ or $b1$; assume the latter. Let position $P_1 = P + W(b1)$. Argue by contradiction: assume B wins (P_1, B).

Let $P_2 = P_1 + B(b2)$. By Lemma 3, B wins (P_2, B). Let P_3 be either $P_2 + B(a2) + W(a3)$ or $P_2 + W(a2) + B(a3)$, and let $P_4 = P_2 + B(a2) + B(a3)$. We claim that B wins (P_4, B).

To prove the claim, consider a winning strategy S for (P_2, B), and assume B follows S on (P_4, B). B makes the move recommended by S, unless that move is $a2$ or $a3$, in which case B imagines that she makes the recommended move (the cell is already black in P_3), then imagines that W immediately replies with the other of $a2, a3$ — as

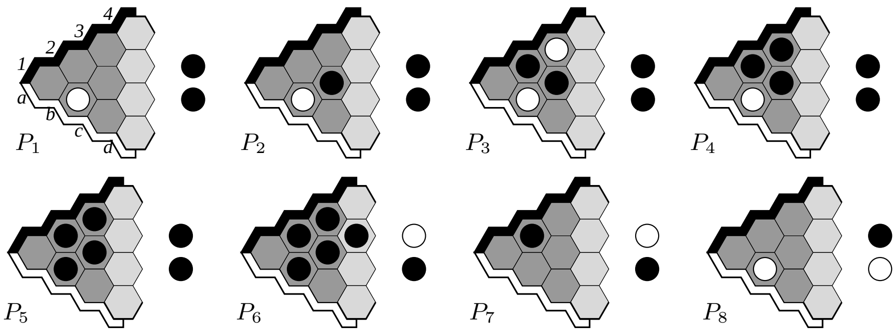

Fig. 7. Proof of Theorem 4. Off-board circles show player to move/win (above/below). P_1: W opens *b1* (proof similar if W opens *a1*), assume B to move and win. P_2: by Lemma 3, B-color *b2*, B wins. P_3: W can play a pairing strategy on $\{a2, a3\}$, B wins. P_4: paired W-cell cannot be on minimal W-connecting chain, so recolor cell, B wins. P_5: original W-cell cannot be on minimal W-connecting chain, so recolor cell, B wins. P_6: B makes winning move. P_7: by Lemma 4, uncolor four B-cells, B wins. P_8: swap colors, reflect board, same position as start, contradicting outcome.

in P_3 — and then makes the next move recommended by S after the two moves just imagined.

Since S wins for B, a white chain eventually appears in the imaginary game. The chain might contain a white cell d at $a2$ or $a3$ as in P_3 that is black in the real game, starting from (P_4, B); however, in such a case there is also a white chain not using d, since d is effectively (representing the black border as a line of black cells) surrounded by four consecutive black cells, meaning that the two white neighbors of d on the chain must themselves be adjacent. Thus there is always a white chain in the game starting from (P_4, B), so B wins (P_4, B) and the claim holds.

Let P_5 be the position obtained from P_4 by changing the color of $b1$ to B. Notice that B wins (P_5, B): at the end of any game that continues from P_4, the only W-neighbor of b_1 is a_1 or c_1, which are both on W's side, so any W-connecting chain that contains b_1 is still a W-connecting chain if b_1 is removed.

Now let P_6 be the position obtained from P_5 after B makes some winning move. Thus B wins (P_6, W).

Let $P_7 = P + B(a2)$, obtained from P_6 by uncoloring four B-cells. By Lemma 4, B wins (P_7, W).

Let P_8 be obtained from P_7 by color exchange and reflection through the long diagonal. Thus W wins (P_8, B).

But $P_8 = P_1$, so the initial assumption that B wins (P_1, B) implies that W wins $(P_8, B) = (P_1, B)$, a contradiction. This concludes the proof for $c = b1$; a similar proof holds for $c = a1$.

Corollary 1. *Consider Rex on an empty $n \times n$ board with n even. For $n \geq 2$, player X wins by opening in an X-peripheral cell.*

Proof. For $n = 2$, see Figure 12. For $n \geq 4$, use Theorem 4.

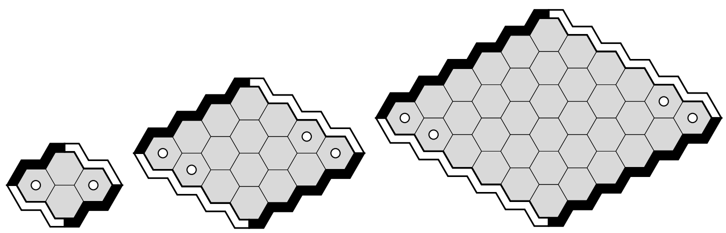

Fig. 8. By Corollary 1, some winning first-player (white) Rex openings.

6 Winning initial replies on odd boards

The following color-symmetric result allows us to find winning replies for openings on odd boards.

Theorem 5. *For PRex from a color-symmetric $n \times n$ position P with an odd number of empty cells, assume X next colors cell c.*

- *If c is not on some symmetry axis of P, then Y does not lose by coloring that axis's reflection of c.*
- *If the long diagonal is a symmetry axis for P, c is a cell on the long diagonal, $n \geq 3$, and at least one corner wedge is empty in $P + X(c)$, then Y does not lose by coloring a cell that is in an empty wedge and Y-peripheral.*

Proof. For the first case, the result follows by Theorem 3, since Y's move restores the position's color-symmetry. For the second case, consider the position P' obtained from P by making cell c a hole, and let b be a cell of P' that is in an empty wedge and Y-peripheral. By Theorem 4, Y does not lose $(P' + Y(b), X)$. Thus Y does not lose $(P + X(c) + Y(b), X)$.

Corollary 2. *For Rex on an empty $n \times n$ board with odd $n \geq 3$, assume the first player colors cell c.*

- *If some diagonal does not contain c, then the second player wins by coloring that diagonal's reflection of c.*
- *If c is on the long diagonal and an empty wedge remains, then the second player Y wins by coloring a cell that is Y-peripheral and in an empty wedge.*
- *If c is the unique center of the board and $n = 3$, then the second player Y wins by coloring a cell that is Y-peripheral.*

Proof. For $n = 3$ with c the unique center the proof is simple: see Figure 13 and use Theorem 7. For all other cases, use Theorem 5.

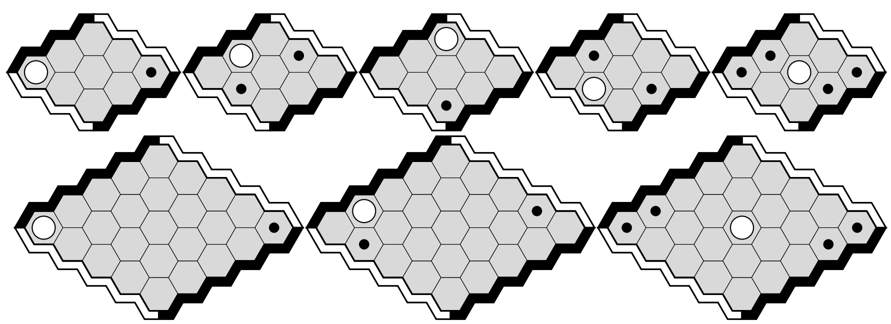

Fig. 9. By Corollary 2, some winning second player (black) Rex replies.

7 Complete strategies on small boards

To this point, we have given winning initial moves or replies on $n \times n$ boards. In this section, we give complete winning strategies for small boards — up to 5×5. The key idea is to use pairing strategies.

For a player X and a position, a *pair-completion* is a partition of a subset of the empty cells into pairs such that, for every set S

containing one cell from each pair, X-coloring S yields a connecting chain for X. Figure 10 shows a white pair-completion.

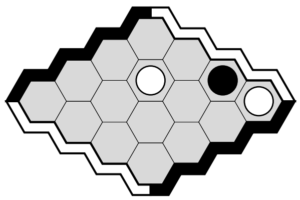

Fig. 10. A position with a white pair-completion.

For a player X and a position, a *pre-pair-completion* is a partition of a subset of the empty cells into pairs and a singleton such that, for every set S containing the singleton and one cell from each pair, X-coloring S yields a connecting chain for X. Figure 11 shows a white pre-pair-completion. The next theorems show how (pre-)pair-completions yield winning PRex strategies.

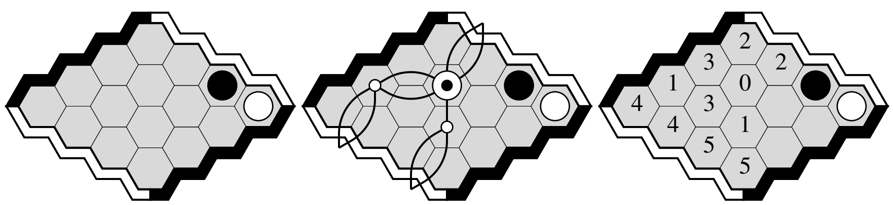

Fig. 11. A position with a white pre-pair-completion. The singleton is labelled 0.

Theorem 6. *For PRex from a position P with an X-pre-pair-completion, X loses (P, X) (resp. (P, Y)) if the number of empty cells in P is odd (resp. even).*

Proof. With respect to a pre-pair-completion α of the current position, a pair of the completion is *empty* if both its cells are empty. Using α, Y plays as follows; notice that exactly one of (1-3) applies at any one time: (1) whenever X colors a cell of an empty pair, Y colors the other cell; (2) whenever X colors the singleton, or a cell of a non-empty pair, Y colors a cell of an empty pair; (3) whenever neither (1) nor (2) applies, Y colors any cell not in α.

By induction on the number of empty cells, it follows that after each Y-move each pair either is empty, or has one cell of each color, or has one empty cell and one Y-colored cell (in which case there is only one such pair and the singleton is X-colored), and so the number of empty cells not in α is even. Thus Y can always move as prescribed, and after each Y-move either both players have played outside α, or both players have played inside α, resulting in a new X-pre-pair-completion with exactly two fewer cells than α.

Figures 12–15 show pre-pair-completions that give explicit winning Rex strategies on board sizes up to 5×5.

Consider for example Figure 12. The first player (white) moves first and follows the strategy from Theorem 6. Rule (3) applies, so white colors the cell not in the pre-partition-completion. Next, if black colors the singleton, (2) applies and white colors a cell from the empty pair; if black colors a cell from the empty pair, (1) applies and white colors the other cell in the pair. Next, black colors the last cell and loses.

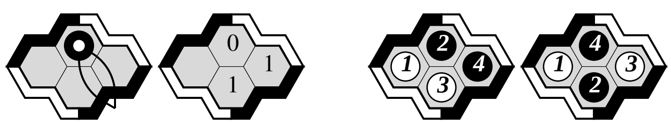

Fig. 12. Via Theorem 6's strategy, the first player (white) wins 2×2 Rex with this black pre-pair-completion. The last two diagrams show possible game sequences.

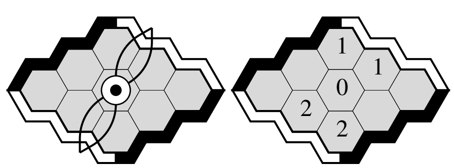

Fig. 13. The second player (black) wins 3×3 Rex.

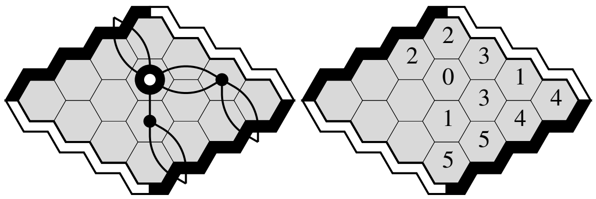

Fig. 14. The first player (white) wins 4×4 Rex.

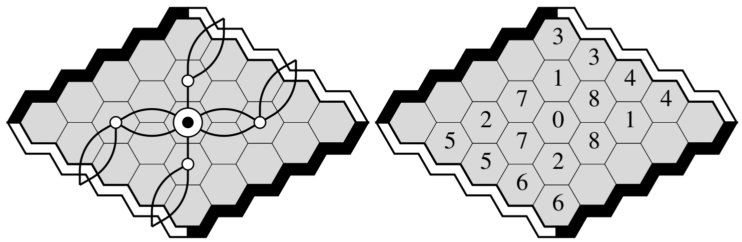

Fig. 15. The second player (black) wins 5×5 Rex.

To conclude, observe that if X has a pair-completion, then X loses regardless of whose move it is.

Theorem 7. *For PRex from a position P with an X-pair-completion, X loses (P, X) and (P, Y).*

Proof. Using the pair-completion β, Y plays as follows: if X colors one cell of an empty β-pair, Y colors the other; otherwise, if possible color a cell not in β; otherwise (every empty cell is one of an empty β-pair) color any cell, and then apply Theorem 6 to the resulting X-pre-pair-completion (as the resulting position has an odd number of empty cells).

Using Theorems 6 and 7, it is easy to find the Rex win/loss value of all openings on the 2×2 board, all replies on the 3×3 board, and — with one exception — all openings on the 4×4 board. For example, by Theorem 6, any cell that is not in the pre-pair-completion of Figure 14 or Figure 16 is a winning 4×4 opening. See Figures 17–19.

As Gardner reported and apparently Shannon observed[5], for every $m \times (m+1)$ board there is, in our terminology, a pair-completion for the player with longer sides. It follows that there is such a pair-completion for any $m \times n$ board with $m < n$: consider any $m \times (m+1)$

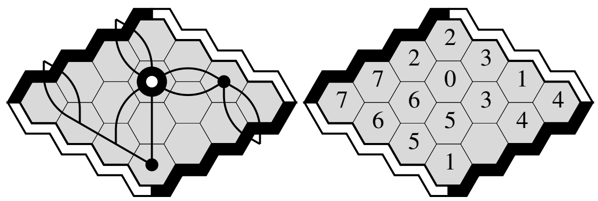

Fig. 16. Another pre-pair-completion. By Theorem 6, the unused cell is a winning opening move for 4×4 Rex.

sub-board. Thus by Theorem 7, regardless of who plays first, Rex on a $m \times n$ board with $m < n$ is a win for the player with shorter sides.

Finally, we leave the reader with two questions: For what values of $n \geq 6$ are there (pre-)pairing connection strategies for the $n \times n$ board? And what is the complexity of solving an arbitrary Rex position?

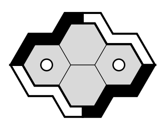

Fig. 17. All 2×2 winning openings.

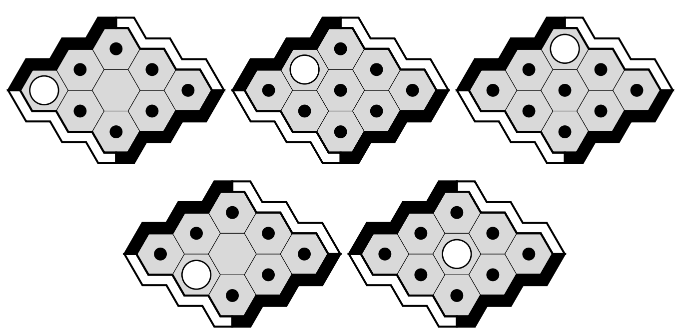

Fig. 18. All 3×3 winning initial replies.

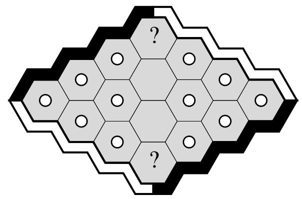

Fig. 19. 4×4 openings. 12 winning (first-player white) openings (dots), 2 losing openings (empty), and 2 unresolved openings (‘ ? ’) left as a challenge for the reader.

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