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Bichromatic P_4-composition schemes for perfect orderability

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Abstract

A P_4 is an induced path with four vertices. A bichromatic P_4 composition scheme is as follows: (1) start with two graphs with vertex sets of different colour, say black • and white ○, (2) select a set of allowable four-vertex bichromatic sequences, for example {••••, ○○○○, •○○•, ○••○}, (3) add edges between the graphs so that in the composed graph each P_4 is coloured with an allowable sequence. Answering a question of Chvátal, we determine all such schemes which preserve perfect orderability.

Keywords: Graph composition; P_4; P_4-structure; Perfect order; Perfectly orderable graph

1. Introduction

Given that determining the chromatic number of a graph is computationally infeasible [11], restricted approaches are often considered for vertex colouring. One such approach is due to Chvátal [1], who conceived of a class of graphs for which a certain efficient algorithm always yields an optimal colouring. Precisely, he defined as *perfectly orderable* those graphs for which there exists a *perfect order*, namely, an ordering of the vertices and an ordering of colours such that for every (vertex induced) subgraph, the so-called "greedy" or "sequential" colouring algorithm (proceed through the vertices in vertex order, assigning to each vertex the first colour in colour order which has not been assigned to any neighbour) yields an optimal colouring. Although

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it is well motivated, the definition of perfect order is unwieldy, referring to a graph property which must hold for all induced subgraphs. A more useful property, verifiable in polynomial time, is the following characterization, also established by Chvátal [1]:

> a linear order $<$ of the vertices of a graph is a perfect order if and only if there is no induced path $(abcd)$ with $a < b$ and $d < c$.

Since vertex colouring is easy for perfectly orderable graphs once a perfect order is found, it is of interest to answer the question “which graphs are perfectly orderable?”. Middendorf and Pfeiffer [12] showed that there is probably no good (namely, polynomial time) answer to this question by showing that recognizing perfectly orderable graphs is NP-complete. However, many partial answers have been established by researchers who have shown that various classes of graphs are perfectly orderable. Some such classes are discussed in the chapter on perfectly orderable graphs by Hoàng in [13] ; see for example [5–7,10]. In this paper, we give a partial answer of a different form, namely by considering certain composition schemes which preserve perfect orderability.

The composition schemes we consider are: start with two graphs, and add edges between the graphs to form a new graph. We are interested in imposing conditions which preserve perfect orderability: the composed graph should be perfectly orderable if the starting graphs are. A P_4 in a graph is an induced four-vertex path.[2] In light of Chvátal’s characterization, it is reasonable to consider composition schemes defined in terms of P_4’s. If we distinguish the starting graphs by giving the two vertex sets different colours, the colour patterns on the composed graph’s P_4’s might determine whether perfect orderability is preserved.

By a *bichromatic graph*, we mean a graph together with a two-colouring of the vertices (namely, a partition of the vertex set into two colour classes); in this paper, the two colours used will always be black $\bullet$ and white $\circ$. The *pattern* of a bichromatic P_4 is the sequence of vertex colours along the path. For example, the pattern of a bichromatic P_4 $(wxyz)$ in which w, x, z are coloured black and y is coloured white is $\bullet\bullet\circ\bullet$. The *pattern set* of a bichromatic graph is the set of patterns of the P_4’s of the bichromatic graph. Since the reversal of a P_4 is a P_4, pattern sets are closed under the property of reversing patterns. We append ‘$*$’ to a pattern to indicate either the pattern or its reverse; for example, $\{\bullet\bullet\circ\bullet^*\}$ is the same as $\{\bullet\bullet\circ\bullet, \bullet\circ\bullet\bullet\}$. A *bichromatic P_4 composition scheme* (or simply a *P_4 composition scheme*) is a two colour (one for each starting graph) composition scheme in which the pattern set of each composed graph is a subset of a specified pattern set. P_4 composition schemes are identified by the corresponding pattern set. An example of P_4 composition is shown in Fig. 1. These notions were introduced by Chvátal, who asked “*which P_4 composition schemes preserve perfect orderability*?” and who gave the following partial answer [2]: *the P_4 composition scheme with pattern set* $\{\bullet\bullet\bullet\bullet, \bullet\bullet\circ\bullet^*, \bullet\circ\bullet\circ^*, \bullet\circ\circ\bullet, \bullet\circ\circ\circ^*, \circ\circ\circ\circ\}$ *preserves perfect orderability*.

[2] This differs slightly from the usual definition of a P_4 as a *set* of four vertices inducing a path in a graph. In this paper, we want to know not only which vertices are in the path, but also which edges.

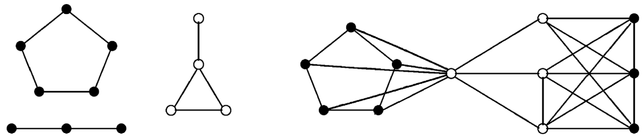


Fig. 1. An example of P_4 composition. With scheme {● ● ●●, ○ ○ ○○, ● ○ ○●, ○ ● ●○}, the bichromatic graph on the right can be composed from the two monochromatic starting graphs on the left.

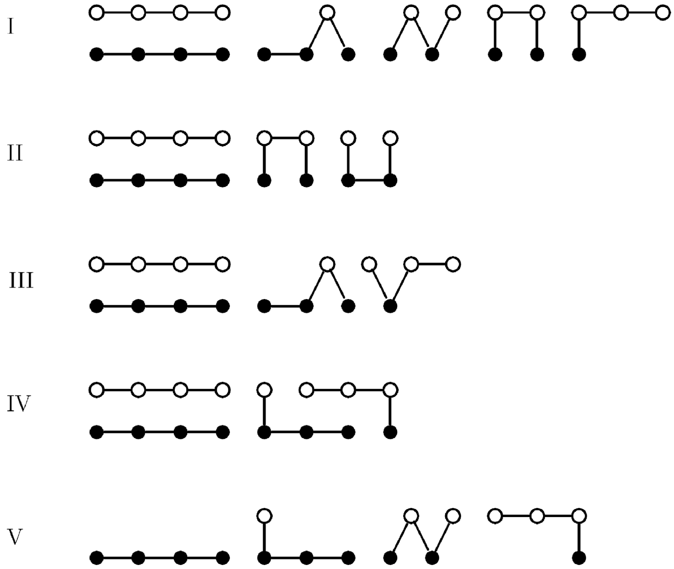


Fig. 2. The five maximal P_4 composition schemes which preserve perfect orderability.

The main result of this paper is a complete answer to Chvátal's question, namely Theorem 1; the five schemes are illustrated in Fig. 2.

Theorem 1. *A P_4 composition scheme preserves perfect orderability if and only if its associated pattern set or its colour exchange equivalent is a subset of one of the following*:

(I) {● ● ●●, ● ● ○●*, ● ○ ●○*, ● ○ ○●, ● ○ ○○*, ○ ○ ○○},
(II) {● ● ●●, ● ○ ○●, ○ ● ●○, ○ ○ ○○},
(III) {● ● ●●, ● ● ○●*, ○ ● ○○*, ○ ○ ○○},
(IV) {● ● ●●, ● ● ●○*, ● ○ ○○*, ○ ○ ○○},
(V) {● ● ●●, ● ● ●○*, ● ○ ●○*, ● ○ ○○*}.

An idea which is useful in proving the above theorem is the natural correspondence between a pattern set and two associated graph classes. For each pattern set $K = \mathrm{I}, \ldots, \mathrm{V}$, let $\mathcal{G}_K$ be the class of graphs with some non-monochromatic two-colouring, such that the pattern set of the resulting bichromatic graph is a subset of K, and let $\mathcal{H}_K$ be those graphs of $\mathcal{G}_K$ for which every non-trivial (namely with at least two vertices) induced subgraph is also in $\mathcal{G}_K$. For example, the rightmost graph in Fig. 1 is in $\mathcal{G}_{\mathrm{II}}$ but not $\mathcal{H}_{\mathrm{II}}$, since the induced cycle with five vertices is not in $\mathcal{G}_{\mathrm{II}}$. In the process of proving our main theorem, we will characterize some of $\mathcal{G}_{\mathrm{I}}, \ldots, \mathcal{G}_{\mathrm{V}}$ and $\mathcal{H}_{\mathrm{I}}, \ldots, \mathcal{H}_{\mathrm{V}}$.

The rest of the paper is divided into four sections. In Section 2, we present background material on compositions related to perfectly orderable graphs. In Section 3, we establish one direction of the theorem, namely that each of the described schemes preserves perfect orderability. In Section 4, we establish the other direction of the theorem, namely that no other scheme preserves perfect orderability. In Section 5, we present some concluding remarks.

2. Basic results

In this section, we present several well-known results on perfect orderability and introduce a class of graphs fundamental to our later results.

One result is a modification of Chvátal's characterization in which the notion of linear vertex order is replaced with the notion of *edge orientation*, namely an ordering $<$ of the vertices of the edges, in which each edge (u, v) is oriented either $u < v$ or $v < u$. Every linear vertex order implies an edge orientation, and every edge orientation which is *acyclic*, namely with no directed cycle, can be extended (via topological sorting) to a linear vertex order. An oriented P_4 $(abcd)$ is *bad* if $a < b$ and $d < c$. Thus Chvátal's characterization implies that

> a graph is perfectly orderable if and only if the graph has an acyclic edge orientation with no bad P_4.

This holds even for acyclic orientations which leave some edges unoriented, as long as at least one edge of each P_4 is oriented so that the P_4 cannot be bad.

Another result concerns properties which imply perfect orderability. A vertex in a graph is *no-mid* (respectively *no-end*) if it is not a middle (respectively end) vertex of any P_4. Hoàng and Khouzam [8] observed that

> a graph with every proper subgraph perfectly orderable and with a no-mid or no-end vertex is perfectly orderable.

This result also follows from Chvátal's result that scheme I preserves perfect orderability: colouring any no-mid vertex black and all other vertices white, or any no-end vertex white and all other vertices black, yields a pattern set which is a subset of {○○○○, ●○○○*} or {●●●●, ●●○●*}. A graph is *split* if its vertex set can be partitioned into two sets which induce respectively a clique and an independent set. A vertex in a split graph is no-mid if in the independent set and no-end if in the clique, so

> split graphs are perfectly orderable.

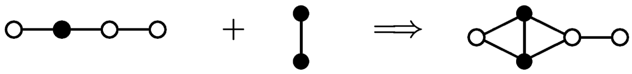


Fig. 3. Substitution.

Yet another result concerns a graph composition scheme. *Substitution* is the following composition process: given a vertex v in a graph G and a graph H, form a new graph by replacing v with H, making every neighbour of v in G adjacent to every vertex of H. This composition process is illustrated in Fig. 3. A corresponding notion is that of a *homogeneous set* of a graph, namely a proper non-trivial[3] vertex subset, such that every vertex which is not in the set is adjacent to all or none of the vertices in the set. For example, in the rightmost graph in Fig. 3, the black vertices form a homogeneous set. A graph has a homogeneous set if and only the graph can be obtained by substituting a non-trivial graph into a non-trivial graph. For any graph formed by the substitution of one perfectly orderable graph into another, the vertex order which is consistent with the two perfect orders is a perfect order of the new graph, so

substitution preserves perfect orderability.

Equivalently, if every proper induced subgraph of a graph with a homogeneous set is perfectly orderable, then the graph is perfectly orderable. For example, a graph is *split-substitute* (respectively *recursively split-substitute*) if it can be created from a split graph by repeatedly substituting any graph (respectively any recursively split-substitute graph) for any vertex, so

recursively split-substitute graphs are perfectly orderable.

On the other hand, split-substitute graphs are not necessarily perfectly orderable, since these graphs may have homogeneous sets which do not induce perfectly orderable graphs. A graph is *prime* if it has no homogeneous set. The prime graph obtained by replacing every maximal homogeneous set with a single vertex is the *characteristic graph* of a graph. It is easy to see that a graph is split-substitute if and only if its characteristic graph is split, and recursively split-substitute if and only if every prime induced subgraph is split.

We conclude this section by establishing a forbidden induced subgraph characterization of recursively split-substitute graphs. The characterization will be used later to show that this class of graphs is exactly $\mathscr{H}_{\mathrm{II}}$. C_k and P_k denote the induced cycle and path, respectively, with k vertices and $\bar{X}$ denotes the complement of X. D_6, also known as the domino, is any graph isomorphic to the graph with vertex set $\{1,\ldots,6\}$ and edge set $\{12,16,23,25,34,45,56\}$. H_6, also known as the hat, is any graph isomorphic to the graph with vertex set $\{1,\ldots,6\}$ and edge set $\{12,23,25,34,45,56\}$.

[3] A set is non-trivial if it has at least two elements. A graph is non-trivial if its vertex set is non-trivial.

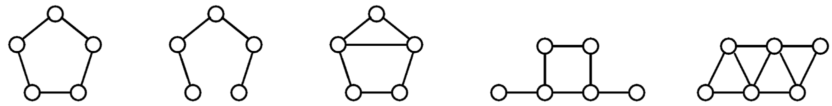


Fig. 4. The forbidden recursively split-substitute graphs: C_5, P_5, $\bar{P}_5$, H_6, $\bar{H}_6$.

A graph is called *chordal* if it contains no $C_{k\geqslant 4}$, and recursively chordal-substitute if it can be created from a chordal graph by repeatedly substituting a recursively chordal-substitute graph for any vertex. (Equivalently, a graph is recursively chordal-substitute if and only if every prime induced subgraph is chordal.) Földes and Hammer [4] showed that

> a graph is split if and only if it contains no C_4, $\bar{C}_4$, C_5.

It follows that a graph is split if and only if both the graph and its complement are chordal, and similarly

> a graph is recursively split-substitute if and only if both the graph and its complement are recursively chordal-substitute.

Hoàng and Reed [9] showed that (in our terminology)

> a graph is recursively chordal-substitute if and only if it contains no $C_{k\geqslant 5}$, no $\bar{P}_5$, no H_6, and no D_6.

Since $C_{k\geqslant 6}$ and D_6 contain P_5, the previous two results imply our desired characterization, illustrated in Fig. 4.

Theorem 2. *A graph is recursively split-substitute if and only if it contains no* C_5, P_5, $\bar{P}_5$, H_6, *or* $\bar{H}_6$.

3. The schemes preserve perfect orderability

The proof of Theorem 1 consists of two parts, namely showing that schemes I, ..., V preserve perfect orderability, and showing that there are no other schemes. We present the former part in this section and the latter part in the next section. Several different proof forms will be employed here, including explicit construction of perfect orders, reduction, and structural analysis of some of the graph classes $\mathscr{G}_{\mathrm{I}}, \ldots, \mathscr{G}_{\mathrm{V}}, \mathscr{H}_{\mathrm{I}}, \ldots, \mathscr{H}_{\mathrm{V}}$. The rest of this section is broken into five subsections, namely one for each scheme.

3.1. *Scheme I:* {● ● ●●, ● ● ○●*, ● ○ ●○*, ● ○ ○●, ● ○ ○○*, ○ ○ ○○}

In initiating the study of P_4 composition schemes, Chvátal was motivated by a desire to study schemes whose rules of composition are suggested by the structure of certain graph classes. For example, by colouring adjacent vertices different colours, a bipartite graph can be two coloured so that its pattern set is a subset of $\{\bullet\circ\bullet\circ^*\}$. The associated scheme B generalizes the process of creating bipartite graphs (since the set of non-trivial graphs which can be two-coloured to have exactly this pattern set strictly includes all non-trivial bipartite graphs). Thus, it is not surprising that the proof that bipartite graphs are perfectly orderable (order all vertices of one colour before all vertices of the other colour) easily extends to a proof that B preserves perfect orderability (consistent with perfect orders of the starting graphs, order all vertices of one graph before all vertices of the other graph). This proof also works for schemes whose pattern set is a superset of B; Chvátal observed that scheme I is the largest scheme for which this particular proof works.

Theorem 3 (Chvátal [2]). *Scheme* I *preserves perfect orderability*.

Proof. Combine perfect orders of the monochromatic starting graphs into a vertex order of the composed graph by ordering all white vertices before all black vertices. This order is a perfect order, since every bichromatic P_4 $(abcd)$ has either $\circ b$ and $\bullet a$ or $\circ c$ and $\bullet d$ and so $b < a$ or $c < d$. □

Recall that $\mathscr{G}_{\mathrm{I}}$ is the class of bichromatic graphs whose pattern set is I, and $\mathscr{H}_{\mathrm{I}}$ is all graphs of $\mathscr{G}_{\mathrm{I}}$ all of whose non-trivial induced subgraphs are also in $\mathscr{G}_{\mathrm{I}}$. The preceding comments imply that $\mathscr{G}_{\mathrm{I}}$ contains all bipartite-substitute graphs and $\mathscr{H}_{\mathrm{I}}$ contains all recursively bipartite-substitute graphs. We shall see later that these containments are strict.

3.2. *Scheme II:* $\{\bullet\bullet\bullet\bullet, \bullet\circ\circ\bullet, \circ\bullet\bullet\circ, \circ\circ\circ\circ\}$

Chvátal's proof that scheme I preserves perfect orderability explicitly constructs a perfect order of the composed graph from the perfect orders of the starting graphs. We use a different proof technique to show that scheme II preserves perfect orderability, namely we characterize the graph class $\mathscr{G}_{\mathrm{II}}$.

Theorem 4. *For any non-trivial graph G*, *the following are equivalent*:

(1) *G is in* $\mathscr{G}_{\mathrm{II}}$, *namely has a two-colouring with pattern set a subset of* $\{\bullet\bullet\bullet\bullet, \bullet\circ\circ\bullet, \circ\bullet\bullet\circ, \circ\circ\circ\circ\}$,
(2) *G is in* $\mathscr{G}_{\mathrm{II}-}$, *namely has a two-colouring with pattern set a subset of* $\{\bullet\bullet\bullet\bullet, \bullet\circ\circ\bullet, \circ\circ\circ\circ\}$,
(3) *G is split-substitute*.

In proving structural theorems based on P_4 composition, results on the connectedness of P_4's are useful. Our starting point is the observation (due to Seinsche [14]) that

for each non-trivial graph with no P_4, the graph or its complement is disconnected.

Theorem 4 follows from Lemma 5. $\mathscr{F}$ is the set of minimal forbidden graphs of Theorem 2, namely $\mathscr{F} = \{C_5, P_5, \bar{P}_5, H_6, \bar{H}_6\}$.

Lemma 5. *Every graph G which is prime and not split-substitute has some graph F of $\mathscr{F}$ as an induced subgraph, and for any such F, the vertices of the graph can be labelled $v_1, \ldots, v_n$ so that*

(1) *F is induced by $\{v_1, \ldots, v_t\}$ (so $t = 5$ or 6),*
(2) *for each $k > t$, some set $\{v_a, v_b, v_c, v_k\}$ with $a < b < c < k$ induces a P_4.*

Throughout, this paper *sees* and *misses* indicate adjacency and non-adjacency, respectively. A vertex is *universal*, *partial*, or *null* on a set of vertices if it sees, respectively, all, some but not all, or none of the vertices in the set; these properties are extended from a vertex to a set if all vertices in the set have the same property. For example, if the vertices of the rightmost graph in Fig. 3 are labelled left to right as $\{abb'cd\}$, then $\{bb'\}$ is $\{ac\}$-universal and $\{d\}$-null.

Proof of Lemma 5. (1) holds by Theorem 2, so we need only show that (2) holds. If $n = t$, we are done, so suppose $n > t$. Argue by induction. Suppose that for some $t \leqslant u < n$, the vertices of $V_u = \{v_1, \ldots, v_u\}$ have been labelled so that (2) holds restricted to V_u. Since V_u is not homogeneous in G, some vertex z not in V_u is V_u-partial. Let j be the smallest index such that z is $V_j = \{v_1, \ldots, v_j\}$-partial. Thus $2 \leqslant j$, and z is not $(V_j - v_j)$-partial. If $j \leqslant t$ then z is F-partial, and a routine case analysis shows that z is in a P_4 with three vertices of F and so can be labelled v_{t+1}. If $j > t$ then by inductive assumption there are indices $a < b < c < j$ such that $Q = \{v_a, v_b, v_c, v_j\}$ induces a P_4. Since z is V_j-partial but not $(V_j - v_j)$-partial, z is Q-partial but not $(Q - v_j)$-partial, so z sees either exactly one or exactly three vertices of Q, and in each case z is in a P_4 with three vertices of Q and again can be labelled v_{t+1}. Thus, the labelling can always be extended, so the lemma holds. □

Proof of Theorem 4. Any split-substitute graph can be two-coloured so that its pattern set is a subset of $\Pi^- = \{\bullet\bullet\bullet\bullet, \bullet\circ\circ\bullet, \circ\circ\circ\circ\}$: in the original split graph, colour the clique white and the independent set black; for each substitution, colour the vertices being added with the colour of the vertex being replaced. Thus (3) implies (2). Since Π^- is a subset of Π, (2) implies (1). It remains to show that (1) implies (3).

Argue the contrapositive: assuming that a graph is not split-substitute, show that is has no two-colouring with pattern set a subset of Π. Consider a bichromatic graph G which is a smallest counterexample.

First observe that G has no homogeneous set H. Assume the contrary: then for every h in H, $G - H + h$ is not split-substitute, and there is at least one h in H whose colour is different from the colour of some vertex in $G - H$ (otherwise is monochromatic), and $G - H + h$ is a smaller counterexample than G, contradiction.

Next, by Theorem 2, G contains some graph from $\mathscr{F}$, so we can label vertices $v_1,\ldots,v_n$ as in Lemma 5. A case analysis shows that no graph in $\mathscr{F}$ can be two-coloured with patterns only from II, so $v_1,\ldots,v_t$ are all the same colour, say black. Assume that $v_1,\ldots,v_r$ are all black, for some $t \leqslant r < n$. Since there are indices $a < b < c < r+1$ which induce a P_4, and since II does not contain any pattern with exactly three black vertices, v_{r+1} is black. Thus, all vertices of G are black, so G is monochromatic, contradiction. Thus, there is no counterexample, so (1) implies (3) and the proof is complete. □

Corollary 6. *Scheme* II *preserves perfect orderability*.

Corollary 7. $\mathscr{H}_{\mathrm{II}}$ *is the class of recursively split-substitute graphs*.

The first corollary follows from Theorem 4 because split graphs are perfectly orderable and substitution preserves perfect orderability. The second follows from Theorem 4 because $\mathscr{H}_{\mathrm{II}}$ consists of those graphs of $\mathscr{G}_{\mathrm{II}}$ which have all non-trivial proper induced subgraphs also in $\mathscr{G}_{\mathrm{II}}$.

3.3. Scheme III: {● ● ●●, ● ● ○●*, ○ ● ○○*, ○ ○ ○○}

A P_4 in a bichromatic graph has pattern set $\{abcd^*\}$ if and only if the complement of the P_4 has pattern set $\{bdac^*\}$, so a bichromatic graph has pattern set III if and only if the complement of the graph with the same two-colouring has pattern set IV. Also, a vertex set is homogeneous in a graph if and only if it is homogeneous in the graph's complement. Furthermore, a vertex is no-end in a graph if and only if it is no-mid in the graph's complement. It follows from these three statements that the lemma and theorem below are equivalent to the corresponding results in Section 3.4. Since we will prove in Section 3.4 all results stated in Section 3.4, we do not need to prove any of the results stated below.

A graph is *no-end-substitute* (respectively *no-mid-substitute*) if it can be created by substitution, starting from a graph with some no-end (no-mid) vertex; an equivalent definition is that a graph is no-end-substitute (no-mid-substitute) if and only if its characteristic graph has a no-end (no-mid) vertex.

We use □ to indicate an unspecified colour, namely either colour ● or colour ○. We represent a coloured P_4 by listing the pattern and the path together in the same relative order; furthermore, whenever we list a four vertex path followed by its colour sequence, the path is understood to be a P_4 unless otherwise stated.

Lemma 8. *If a bichromatic graph* (Fig. 5) *whose pattern set is a subset of* III *contains* (● ○ ● ● *bxac*) *and* (○□□○ *xzwy*), *then*

(1) *at least one of* w,z *is white and* $\{xzwy\}$ *is* $\{c\}$*-null and* $\{ab\}$*-universal and*
(2) *the graph has a homogeneous set*.

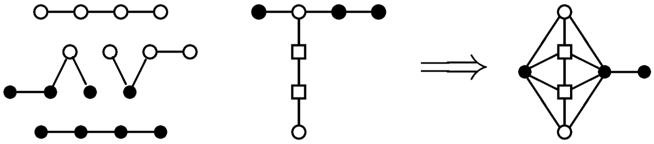


Fig. 5. An illustration of Lemma 8.

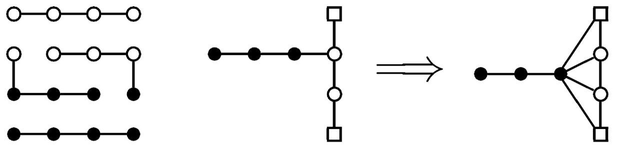


Fig. 6. An illustration of Lemma 11.

Theorem 9. *For any non-trivial graph G, the following are equivalent:*

(1) *G is in* $\mathscr{G}_{\mathrm{III}}$, *namely has a two-colouring with pattern set a subset of* {● ● ●●, ● ● ○●*, ○ ● ○○*, ○ ○ ○○}
(2) *G is in* $\mathscr{G}_{\mathrm{III}-}$, *namely has a two-colouring with pattern set a subset of* {●●●●, ● ● ○●*, ○ ○ ○○}
(3) *G is no-end-substitute.*

Corollary 10. *Scheme* III *preserves perfect orderability.*

3.4. *Scheme IV:* {● ● ●●, ● ● ●○*, ● ○ ○○*, ○ ○ ○○}

As in Section 3.2, the results here will be obtained by characterizing the associated bichromatic graphs, in this case the graphs of $\mathscr{G}_{\mathrm{IV}}$. The (long and straightforward) proof of the following lemma (Fig. 6) is postponed to the end of this subsection.

Lemma 11. *If a bichromatic graph with pattern set a subset of* IV *contains* (● ● ● ○ *abcx*) *and* (□ ○ ○□ *wxyz*), *then*

(1) *at least one of w,z is white and* {*wxyz*} *is* {*c*}*-universal and* {*ab*}*-null and*
(2) *the graph has a homogeneous set.*

Theorem 12. *For any non-trivial graph G, the following are equivalent:*

(1) *G is in* $\mathscr{G}_{\mathrm{IV}}$, *namely has a two-colouring with pattern set a subset of* {● ● ●●, ● ● ●○*, ● ○ ○○*, ○ ○ ○○}
(2) *G is in* $\mathscr{G}_{\mathrm{IV}-}$, *namely has a two-colouring with pattern set a subset of* {●●●●, ● ○ ○○*, ○ ○ ○○}
(3) *G is no-mid-substitute.*

Proof of Theorem 12. Any no-mid-substitute graph can be two-coloured so that its pattern set is a subset of $\mathrm{IV}^- = \{$● ● ●●, ● ○ ○○*, ○ ○ ○○$\}$: in the original graph, colour some no-mid vertex black and all other vertices white; for each substitution, colour the vertices being added with the colour of the vertex being replaced. Thus (3) implies (2). Since IV^- is a subset of IV, (2) implies (1). It remains to show that (1) implies (3).

For any graph G in $\mathscr{G}_{\mathrm{IV}}$, let G' be any induced bichromatic subgraph obtained by replacing every maximal homogeneous set with any one vertex from that set. Thus, G' is prime and isomorphic to the characteristic graph of G. If the pattern set of G' is a subset of II, then by Theorem 2 G' is a split graph, so G is split-substitute and therefore no-mid-substitute and we are done; if not, then by exchanging colours if necessary, we may assume that G' has some (●●●○ *abcd*), so d is no-mid by Lemma 11, so G is no-mid-substitute, and we are done. □

Corollary 13. *Scheme* IV *preserves perfect orderability*.

Corollary 13 follows from Theorem 12 and the observation of Hoàng and Khouzam mentioned in Section 2.

Proof of Lemma 11. The proof of (1) is a case analysis of all possible bichromatic graphs induced by the two P_4's. The pattern set implies that at least one of z, w is white, so there are three cases to consider.

Case 1: ○z and ○w. In this case the seven vertices are distinct, so

a misses *z* (otherwise *c* misses *z* [else (○●○● *xcza*)] so *b* misses *z* [else (○●●○ *xcbz*)] so *a* misses *w* [else (○ ○ ● ○ *xwaz*)] so *b* misses *w* [else (○ ○ ● ● *xwba*)] so *c* sees *w* [else (○ ○ ● ● *wxcb*)] so *c* misses *y* [else (○ ○ ● ○ *zycw*)] so *y* misses *b* [else (● ● ○ ○ *cbyz*)] so (○ ○ ● ● *yxcb*), contradiction). Also
a misses *w* (otherwise *a* misses *y* [else (○○●○ *zyaw*)] so *b* misses *y* [else (●●○○ *abyx*)] so *y* sees *c* [else (● ● ○ ○ *bcxy*)] so *w* misses *c* [else (○ ● ○ ● *ycwa*)] so (● ○ ○ ● *cxwa*), contradiction). Thus
b misses *w* [else (● ● ○ ○ *abwx*)] and
w sees *c* [else (● ● ○ ○ *bcxw*)] so
y sees *c* (otherwise *y* misses *b* [else (○●●○ *wcby*)], so (●●○○ *bcxy*), contradiction). Thus
y misses *a* [else (○ ● ○ ● *wcya*)] so
y misses *b* [else (○ ○ ● ● *xyba*)] so
b misses *z* [else (● ● ○ ○ *abzy*)] so
z sees *c* [else (● ● ○ ○ *bcyz*)]

so (1) holds in this case.

Case 2: ●z and ○w. This colouring and the P_4 adjacencies imply that the seven vertices are distinct (otherwise, $z = b$ or $z = a$: in the former case, w misses a [else (● ● ○ ○ *zawx*)], so w sees c [else (● ● ○ ○ *zcxw*)], so a sees y [else (○ ○ ● ● *xyza*)], so *c* sees *y* [else (○ ● ● ○ *wczy*)], so (● ○ ● ○ *aycw*), contradiction; in the latter case, *w* misses *b* [else (● ● ○ ○ *zbwx*)], so *w* sees *c* [else (● ● ○ ○ *bcxw*)], so *y* sees *c* [else (● ○ ○ ● *zyxc*)], so (● ○ ● ○ *zycw*), contradiction). Thus

y sees *c* (otherwise *y* misses *a* [else (●○○● *ayxc*)] so *b* misses *y* [else (○○●● *xyba*)] so (● ● ○ ○ *bcxy*), contradiction); similarly

w sees *c* so

z sees *c* [else (○ ● ○ ● *wcyz*)] so

y misses *a* (otherwise *w* sees *a* [else (○●○● *wcya*)], so *z* misses *a* [else (●●○○ *zawx*)], so (○ ● ○ ● *wayz*), contradiction) so

w misses *a* [else (○ ● ○ ● *ycwa*)] so

z misses *a* [else (○ ○ ● ● *xyza*)] so

b misses *y* [else (○ ○ ● ● *xyba*)] so similarly

b misses *w* so

b misses *z* [else (○ ○ ● ● *xyzb*)]

so (1) holds in this case.

Case 3: ○ *z* and ● *w*. This colouring and the P_4 adjacencies imply that the seven vertices are distinct (otherwise $w=c$, so *y* sees *b* [else (○ ○ ● ● *yxwb*)], so *y* sees *a* [else (● ● ○ ○ *abyx*)], so (● ○ ○ ● *wxya*), contradiction). Now

b misses *w* (otherwise *b* misses *z* [else (○●●○ *xwbz*)], so *b* misses *y* [else (○○●● *zybw*)], so (○ ○ ● ● *yxwb*), contradiction) and similarly

a misses *w* so

b misses *y* [else (● ○ ○ ● *byxw*)] and similarly

a misses *y* so

y sees *c* [else (● ● ○ ○ *bcxy*)] so

w sees *c* [else (● ● ○ ● *bcxw*)] so

z sees *c* [else (● ● ○ ○ *wcyz*)] so

z misses *a* [else (○ ● ○ ● *xcza*)] so

z misses *b* [else (● ● ○ ○ *abzy*)]

so (1) holds in this case, and so in all cases.

It remains to show that (2) holds. Let *G* be a graph as described in the lemma, let *D* be the set of vertices *d* such that (*abcd*) is a P_4, and let D^* be any subset of *D* which contains $\{wxyz\}$ and which is minimal with respect to the following property: every vertex in $D-D^*$ is D^*-universal or D^*-null. Thus, either $D^*=D$ or D^* is a minimal homogeneous set of $G[D]$ which contains $\{wxyz\}$. We shall show that D^* is a homogeneous set of *G*.

Argue by contradiction. Let *q* be a D^*-partial vertex of $G-D^*$. Then *q* is in $G-\{abc\}-D$, since *q* is not in $\{abc\}$ by the definition of *D* and not in $D-D^*$ by the definition of D^*.

We first claim that *q* misses all white vertices of D^*. Again, argue by contradiction. Since $G[D^*]$ contains at least one white vertex and (by the definition of D^*) is connected, $G[D^*]$ contains adjacent vertices dd' such that d is white and q sees d but not d'. First suppose that d' is white. Then q sees c [otherwise, either (●●○□ $bcdq$) or both (□●●○ $qbcd'$) and (□●○○ $bqdd'$), and the latter implies (○●●○ $qbcd'$) or (●●○○ $bqdd'$), contradiction] q misses a [else (●○●○ $aqcd'$) or (●●○○ $aqdd'$)], q misses b [else (●●○○ $abqd$) or (●●○○ $bqdd'$)], and q is in D, contradiction. Next suppose that d' is black. Then q misses a [else (●□○● $aqdd'$)], q misses b [else (●□○● $bqdd'$)], q sees c [else (●●○□ $bcdq$)], and again q is in D, contradiction. Thus the first claim holds.

We next claim that q is partial on some P_4 (○○○□o) of D^*. Argue as follows. By the minimality property of D^* and the fact that D^* contains $\{wxyz\}$, there is a labelling $\{v_1\ldots\}$ of the vertices of D^* so that (○○○□ $v_1v_2v_3v_4$) is a P_4, and for each $k \geqslant 3$, v_k is $(\{v_1\ldots v_{k-1}\})$-partial. Call a vertex *extreme* with respect to a vertex subset S if it is S-universal or S-null. Since q is D^*-partial, for any such labelling there is a smallest index t with $1 \leqslant t \leqslant |D^*|$ such that q is $\{v_1\ldots v_t\}$-partial but $\{v_1\ldots v_{t-1}\}$-extreme. Consider a labelling which minimizes t.

Observe that for $6 \leqslant k \leqslant t$, v_k is $\{v_1\ldots v_{k-2}$-extreme [otherwise, for the largest such k^* such that v_{k^*} is $\{v_1\ldots v_{k^*}\}$-partial and the smallest j^* such that v_{k^*} is $\{v_1\ldots v_{j^*}\}$-partial, the vertex sequence obtained from $(v_1\ldots v_t)$ by omitting vertices v_r with $j^*+1 \leqslant r \leqslant k^*-1$ contradicts the minimality of t].

Now it follows that $t \leqslant 4$. Argue by contradiction: suppose that $t \geqslant 5$. Then v_5 is not in a P_4 with any three vertices of $\{v_1v_2v_3v_4\}$ [otherwise, since no P_4 in scheme IV has exactly two white vertices, this P_4 would have at least three white vertices and omitting v_4 from $(v_1\ldots v_t)$ contradicts the minimality of t]. Since v_5 is $\{v_1\ldots v_4\}$-partial, it follows by a simple case analysis that v_5 sees v_2v_3 and misses v_1v_4. It follows that $t=5$ [otherwise v_6 sees v_5 and misses $v_1\ldots v_4$ or misses v_5 and sees $v_1\ldots v_4$, and $(v_1v_2v_5v_6)$ or $(v_1v_6v_3v_5)$ can replace $(v_1v_2v_3v_4)$, contradicting the minimality of t]. But then, since q is $\{v_1\ldots v_{t-1}\}$-extreme, either q sees v_5 and misses $v_1\ldots v_4$ or q misses v_5 and sees $v_1\ldots v_4$. By the previous claim, q misses all white vertices of D^*, so the latter case doe not occur (v_2 is white), and in the former case v_5 is black, but then (○○●□ $v_1v_2v_5q$), contradiction.

Thus $t \leqslant 4$, so the second claim holds and q is partial on some (○○○□ $v_1v_2v_3v_4$) of D^*. This and the first claim implies that q misses $v_1\ldots v_3$, so q sees v_4, so v_4 is black, so (○○●□ $v_2v_3v_4q$). This contradiction completes the proof of (2), which completes the proof of Lemma 11. □

3.5. Scheme V: {●●●●, ●●●○*, ●○●○*, ●○○○*}

As in Section 3.1, the desired result will be obtained by explicitly constructing a perfect order of the composed graph. We begin with a structural lemma.

Lemma 14. *Let G be a bichromatic graph with pattern set a subset of set* V.
Then, if (●●●○ $abcd$) *then*

(1) *if* ○e *sees* d, *then* e *sees* c *and is not* $\{ab\}$-*partial*,
(2) *there is no* (●○○○ $cdef$),
(3) *if* (●○●○ $cdef$) *then* (●●●○ $cbef$),

(4) *if* ($\circ\circ\circ\bullet$ $fedg$) *then*
   (a) $G[\{a,b,\ldots,g\}]$ *has edge set* $\{ab,bc,cd,ce,cf,cg,de,ef,dg\}$,
   (b) *if* $\bullet x$ *sees* g *but not* c, *then* x *sees* $bdefg$ *and misses* ac (*and so* x *and* c *have the same neighbourhood in* $G[\{a\ldots g\}]$), *and*
   (c) *there is no* $\bullet\bullet\bullet\bullet$ $cgxy$.

Similarly, *if* ($\circ\circ\circ\bullet$ $abcd$) *then*

(5) *if* $\bullet e$ *sees* d, *then* e *sees* c *and is not* $\{ab\}$-*partial*,
(6) *there is no* ($\circ\bullet\bullet\bullet$ $cdef$),
(7) *if* ($\circ\bullet\circ\bullet$ $cdef$) *then* ($\circ\circ\circ\bullet$ $cbef$),
(8) *there is no* ($\bullet\bullet\bullet\circ$ $fedg$).

Proof. Consider the first statement: suppose ($\bullet\bullet\bullet\circ$ $abcd$). If e sees a then e sees b [else ($\bullet\bullet\circ\circ$ $baed$)] and e sees c [else ($\bullet\circ\circ\bullet$ $aedc$)]; if e misses a then e misses b [else ($\bullet\bullet\circ\circ$ $abed$)] and e sees c [else ($\bullet\bullet\circ\circ$ $bcde$)]. Thus (1) holds. If $\circ e$ sees d, then e sees c by (1), so $cdef$ is not a P_4, so (2) holds. If ($\bullet\circ\bullet\circ$ $cdef$) then e sees b [else ($\bullet\bullet\circ\bullet$ $bcde$)] and f misses b [else ($\circ\bullet\bullet\circ$ $fbcd$)], so (3) holds.

Consider (4): suppose ($\circ\circ\circ\bullet$ $fedg$). By (1), e sees c and either both or neither of ab. In fact, the latter holds (otherwise, g sees a [else ($\bullet\circ\circ\bullet$ $gdea$)]; and ($\circ\bullet\bullet\circ$ $fagd$) if f sees a whereas ($\circ\circ\bullet\bullet$ $feag$) if f misses a, contradiction). Now g misses a [else ($\circ\circ\bullet\bullet$ $edga$)], g misses b [else ($\circ\circ\bullet\bullet$ $edgb$)], g sees c [else ($\bullet\circ\bullet\bullet$ $gdcb$)], f sees c [else ($\circ\circ\bullet\bullet$ $fecg$)], f misses a [else ($\bullet\circ\bullet\bullet$ $afcg$)], and f misses b [else ($\bullet\bullet\circ\circ$ $abfe$)], so (4.a) holds. If $\bullet$ x sees g and not c then x sees d (by (1) of the second statement with ($\circ\circ\circ\bullet$ $fedg$) in place of ($\circ\circ\circ\bullet$ $abcd$)), so x sees b [else ($\bullet\circ\bullet\bullet$ $xdcb$)], x sees e [else ($\bullet\bullet\circ\circ$ $bxde$)], x sees f [else ($\circ\circ\bullet\bullet$ $fexg$)], x misses a [else ($\bullet\bullet\circ\bullet$ $axdc$)], so (4.b) holds. If ($\bullet\bullet\bullet\bullet$ $cgxy$) then either ($\bullet\circ\bullet\bullet$ $yecg$) (if y sees e) or ($\bullet\bullet\circ\bullet$ $yxec$) (if y misses e), contradiction. Thus (4.c) holds, so the first statement of the lemma holds.

For the second statement, the proofs of (5)–(7) are the same as the proofs of (1)–(3) with the colours exchanged. The proof of (8) is the same as the proof of (4) with the colours exchanged, together with the observation that the colour exchange of the graph described in (4a) cannot occur, as such a graph has ($\circ\circ\circ\circ$ $abcg$). □

Theorem 15. *Scheme* V *preserves perfect orderability*.

Proof. It suffices to show that the following algorithm is correct.

Algorithm V. *Input*: a bichromatic graph G whose pattern set is a subset of V, together with perfect orders of $G[B]$ and $G[W]$ (where B and W are the black and white vertices). *Output*: a perfect order of G.

(1) For every ($\bullet\bullet\bullet\circ$ $abcd$), orient $c<d$.
(2) For every ($\circ\circ\circ\bullet$ $abcd$), orient $c<d$.
(3) For every ($\bullet\circ\bullet\circ$ $abcd$) with neither ab nor cd oriented by (1) or (2), orient $c<d$.

(4) For every pair (● ● ● ○ $abce$) (○ ○ ○ ● $gfed$), orient $c < d$.
(5) For every (● ● ● ● $abcd$) with no edge oriented by (4), orient $c < d$ if and only if $c < d$ in the given perfect order of $G[B]$.
(6) Extend the edge orientation to any total vertex order, and return this order.

With respect to an orientation of some of the edges of a graph, call a P_4 ($abcd$) *good* if at least one of $b < a$, $c < d$ holds (namely, if at least one end edge is oriented "in"). Since a good P_4 cannot be bad (even if additional edges are oriented), the correctness of Algorithm V follows from these claims:

- no edge receives two opposite orientations
- at the end of Step 5, every P_4 is good
- at the end of Step 5, the orientation is acyclic.

Consider the first claim. The only steps which orient bichromatic edges are 1,2,3; of these, only 2 orients ○$x < y$●, in which case neither 1 (by Lemma 14.2) nor 3 (by its definition) orients ●$y < x$○. Thus the claim holds for bichromatic edges. The only steps which orient monochromatic (black) edges are 4,5. If ●$x <$ ● y by 4, then there is no (● ● ● ● $vwyx$) (by Lemma 14), so it cannot be that $y < x$ by 4 or 5; if ●$x <$ ● y by 5, then it cannot be that $y < x$ by 5, since all orientations from 5 are from one perfect order. Thus the first claim holds.

Consider the second claim. Consider all P_4's after Steps 1–5 have been applied. By 1 and 2 every P_4 ● ● ● ○ or ○ ○ ○● is good. Also, for any (● ● ● ● $uvwx$), if $u < v$ then (by Lemma 14) neither $u < v$ nor $x < w$ is oriented by 4, so $u < v$ is oriented by 5, so neither 4 nor 5 orients $x < w$, so $uvwx$ is good. Finally, consider any (● ○ ● ○ $uvwx$). If $u < v$ by 1 then (by Lemma 14) $w < x$ by 1 as well; similarly, if $x < w$ by 2 then $v < u$ by 2; in each case, the P_4 is good. If $v < u$ by 2 or $w < x$ by 1, the P_4 is good. If neither wx nor uv is oriented by 1 or 2, then $w < x$ by 3; thus in all of these cases the P_4 is good. Thus the second claim holds.

In justifying the third claim, we use the following lemma.

Lemma 16. *Let G be a bichromatic graph in $\mathcal{G}_{\mathrm{V}}$ whose edges have been oriented according to Steps 1–5 of Algorithm V.*

(1) *If* ●x ● y ○ z, $x < y$, $y < z$, $x \nless z$, *then* (● ● ● ● $vwxy$) *for some* v, w *and* z *is* $\{vwx\}$*-null.*
(2) *There are no* ○x ● y ○ z *with* $x < y$ *and* $y < z$,
(3) *If* (●●●○ $abce$) *and* (○○○● $gfed$) *and* ($d = v_1, v_2, \ldots, v_k$) *is a sequence of* ● *vertices, such that for* $1 < j \leqslant k$ $v_{j-1} < v_j$, $v_j \nless v_1$, *and* $c \nless v_j$, *then for* $1 < j \leqslant k$
  (●) v_j *sees* e *and* c *
  (●) $v_j \nless c$. **

Proof. First prove (1). Let x, y, z be as stated. Since ●$x <$ ● y there is some (● ● ● ● $vwxy$). It suffices to show that z misses x, since then z misses v (else (● ○ ● ● $vzyx$)) and z misses w (else (● ● ○ ● $vwzy$)). Argue by contradiction: suppose z sees x.

Then z sees both of vw (if z sees exactly one then ($\bullet\bullet\circ\bullet$ $vwzy$) or ($\bullet\bullet\circ\bullet$ $wvzy$); if z sees neither then ($\bullet\bullet\bullet\circ$ $vwxz$) and $x<z$). Since $y<z$, for some st either ($\bullet\bullet\bullet\circ$ $styz$) or ($\bullet\circ\bullet\circ$ $styz$); in each case s is $\{vw\}$-null (else ($\bullet\bullet\circ\bullet$ sv/wzy)) and s misses x (else ($\bullet\bullet\circ\bullet$ $sxzv$)).

Now suppose $\bullet$ t. Then t is $\{vw\}$-universal (else ($\bullet\bullet\circ\bullet$ $tyzv/w$)), t sees x (else ($\bullet\bullet\circ\bullet$ $tvzx$)), and ($\bullet\bullet\bullet\circ$ $stxz$) implies $x<z$, contradiction. Thus $\circ$ t and there is no ($\bullet\bullet\bullet\circ$ $abyz$). In particular, this implies that $y<z$ is a consequence of Step 3, so st is not oriented by Step 1 or 2. Since t sees x (else $\bullet\circ\bullet\bullet$ $styx$), there is no ($\circ\circ\circ\bullet$ $pqzx$) (else ($\circ\circ\circ\bullet$ $zqts$) by Lemma 14 and $t<s$ by Step 2). Now either ($\bullet\bullet\bullet\circ$ $abxz$) for some ab and $x<z$ by Step 1, or neither st nor xz are oriented by Step 1 or 2, so $x<z$ by Step 3 because of ($\bullet\circ\bullet\circ$ $stxz$), contradiction. Thus (1) holds.

Next consider (2). Argue by contradiction: let x,y,z be as stated. Since $x<y$, ($\circ\circ\circ\bullet$ $vwxy$) for some vw, so by Lemma 14 there is no ($\bullet\bullet\bullet\circ$ $pqyz$), so $y<z$ by Step 3, so ($\bullet\circ\bullet\circ$ $styz$) with st not oriented by Step 1 or 2. Now z sees x (else z misses v [($\circ\circ\bullet\circ$ $vzyx$)], z sees w [($\circ\circ\bullet\circ$ $wxyz$)], so ($\circ\circ\circ\bullet$ $vwzy$), so $z<y$, contradiction). Thus $x\neq t$, so the vertices $vwxystz$ are distinct. Now v misses t (else v sees z [($\circ\circ\bullet\circ$ $vtyz$)], v misses s [($\bullet\circ\circ\bullet$ $svzy$)], so ($\circ\circ\circ\bullet$ $zvts$), so ts is oriented by Step 2, contradiction). Similarly, w misses t. Now x misses t (else ($\circ\circ\circ\circ$ $vwxt$)), but then ($\circ\circ\bullet\circ$ $wxyt$), contradiction. Thus (2) holds.

Finally consider (3). By Lemma 14, each of $gfed$ sees c but neither of ab. To prove *, we first show that v_j sees e. Consider a smallest counterexample: suppose that v_j misses e, but that v_i sees e for all $i<j$.

Then v_j misses $d=v_1$ (by Lemma 14 with ($\circ\circ\circ\bullet$ $gfed$) and $\bullet v_j$), and v_j sees v_{j-1} (since $v_{j-1}<v_j$ by the hypothesis of (3)). Thus there is a smallest index p with $1<p<j$ such that v_j sees v_p. Notice that v_j misses f (else ($\bullet\circ\circ\bullet$ v_jfed)), so v_j misses g (else $c<v_j$ by Step 4 because of ($\bullet\bullet\bullet\circ$ $abcg$) and ($\circ\circ\circ\bullet$ $efgv_j$)). Also, v_p sees d [else ($\bullet\bullet\circ\bullet$ v_jv_ped)] and v_p sees f [else ($\bullet\bullet\circ\circ$ v_jv_pef)], so v_p sees g [else ($\bullet\bullet\circ\circ$ v_jv_pfg)]. Since $v_{p-1}<v_p$, ($\bullet\bullet\bullet\bullet$ $wxv_{p-1}v_p$) for some wx, and e sees both wx (if e sees exactly one, then ($\bullet\bullet\circ\bullet$ $wxev_p$) or ($\bullet\bullet\circ\bullet$ $xwev_p$); if e sees neither and $p=2$ then ($\bullet\bullet\bullet\circ$ $wxde$) and $\circ f$ contradicts Lemma 14; if e sees neither and $p\geqslant 3$ then ($\bullet\bullet\bullet\circ$ $wxv_{p-1}e$) and ($\circ\circ\circ\bullet$ $gfed$) implies $v_{p-1}<d=v_1$ by Step 4, contradiction). So v_j sees w [else ($\bullet\bullet\circ\bullet$ v_jv_pew)], and, since v_j misses v_{p-1} by the choice of p, ($\bullet\bullet\circ\bullet$ v_jwev_{p-1}), contradiction. Thus v_j sees e as claimed.

Hence, v_j sees c (otherwise v_j sees b [else ($\bullet\circ\bullet\bullet$ v_jecb)], v_j misses a [else ($\bullet\bullet\circ\bullet$ av_jec)], and $v_j<d$ by Step 4 with ($\bullet\bullet\bullet\circ$ abv_je) and ($\circ\circ\circ\bullet$ $gfed$), contradiction), so * holds.

Prove ** by contradiction: suppose $v_j<c$. Then ($\bullet\bullet\bullet\bullet$ wxv_jc) for some wx. Thus, x sees e (otherwise, w misses e [else ($\bullet\bullet\circ\bullet$ $xwec$)] and $v_j<d$ by Step 4 with ($\bullet\bullet\bullet\circ$ wxv_je) and ($\circ\circ\circ\bullet$ $gfed$), contradiction). So w sees e [else ($\bullet\circ\bullet\bullet$ $cexw$)], so w sees b [else ($\bullet\bullet\circ\bullet$ $bcew$)], so w misses a [else ($\bullet\bullet\circ\bullet$ $awec$)], so ($\bullet\bullet\bullet\circ$ $abwe$) and ($\circ\circ\circ\bullet$ $gfed$), so w sees all of $defg$ by Lemma 14. Thus v_j misses a [else ($\bullet\bullet\circ\bullet$ av_jew)] and v_j sees b [else ($\bullet\bullet\circ\bullet$ $bwev_j$)], so ($\bullet\bullet\bullet\circ$ abv_je) and ($\circ\circ\circ\bullet$ $gfed$), so $v_j<d=v_1$ by Step 4. This contradiction completes the proof of ** and (3), which concludes the proof of the Lemma 16. □

We now prove the last claim, namely that the orientation produced by Steps 1–5 is acyclic. There are no white di-cycles, since no ∘∘ edges are oriented. To show that there are no black di-cycles, it is sufficient to show that no black di-cycle contains at least one edge oriented by Step 4, since black edges oriented by Step 5 are oriented consistent with one perfect order and so are acyclic. So it is sufficient to show that for any sequence of black vertices $v_0, \ldots, v_k$ such that $v_0 < v_1$ by Step 4 and $v_{j-1} < v_j$ for $1 \leqslant j < k$ we have $v_k \nless v_0$. But if there is a counterexample, then in a smallest counterexample $v_0 \nless v_j$ for $1 < j \leqslant k$ and $v_j \nless v_1$ for $1 < j \leqslant k$, so $c = v_0 \ngtr v_k$ by Lemma 16 (with $c = v_0$), contradiction. Thus there are no monochromatic di-cycles.

To show there are no bichromatic di-cycles, argue by contradiction. Consider any shortest such di-cycle $D=(v_0, \ldots, v_k)$, where $v_j < v_{j+1}$ for $0 \leqslant j < k$ and $v_k < v_0$. Since no white edges are oriented, there are no consecutive white vertices in this cycle. Since there is at least one vertex of each colour, the sequence • ∘ • must occur. Label three such vertices yza respectively. By Lemma 16 the predecessor x of y in D must be black. (It is possible that $x = a$; this does not affect the following argument.) Thus •x •y∘z, $x < y$, $y < z$, and (since D is shortest) $x \nless z$, so by Lemma 16 (• • • • $vwxy$) for some vw, where z sees y but none of vwx. Now $z < a$ implies that (∘ ∘ ∘ • $stza$) for some st, and (• • • ∘ $wxyz$) and (∘ ∘ ∘ • $stza$) implies $y < a$ by Step 4, contradicting the assumption that D is a shortest di-cycle. Thus the last claim holds, so Algorithm V is correct, and the proof of Theorem 15 is complete. □

4. Necessity: no other schemes

In this section, we show that the list of schemes of Theorem 1 is complete, namely we show that any P_4 composition scheme whose pattern set or its colour exchange equivalent is not a reversal-closed subset of the five sets listed in the theorem does not preserve perfect orderability. This is established by the following two lemmas. The composition schemes of Lemma 17 and the graphs which establish that they do not preserve perfect orderability are illustrated in Figs. 7 and 8.

Lemma 17. *The pattern set of a P_4 composition scheme which preserves perfect orderability is not a superset of any of the following*:

(VI) {• • ∘∘*}
(VII) {• • •∘*, • • ∘•*}
(VIII) {• • •∘*, • ∘ ∘•}
(IX) {• • ∘•*, ∘ • •∘}
(X) {• • ∘•*, • ∘ •∘*, ∘ • ∘∘*}
(XI) {• ∘ •∘*, • ∘ ∘•, ∘ • •∘}
(XII) {• • ••, • • •∘*, • ∘ •∘*, • ∘ ∘∘*, ∘ ∘ ∘∘}

Lemma 18. *Every reversal-closed subset, or its colour exchange equivalent, of the set of all patterns is either a subset of (at least) one of sets* I, …, V, *or a superset of (at least) one of sets* VI, …, XII.

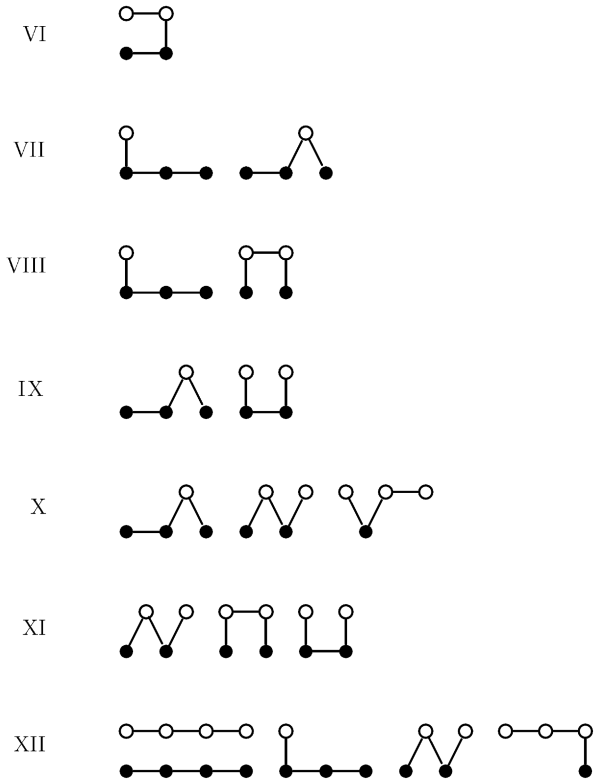


Fig. 7. Composition schemes which destroy perfect orderability.

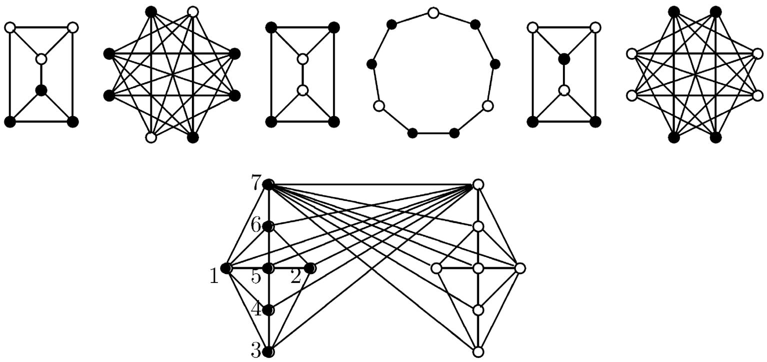


Fig. 8. Composition counterexamples.

Proof of Lemma 17. To prove that a pattern set is not a subset of any pattern set which preserves perfect orderability, it is sufficient to present a bichromatic graph with that pattern set, such that the graph is not perfectly orderable but the monochromatic induced subgraphs are.

Consider the top left graph in Fig. 8. This graph is the complement of an induced cycle with six vertices, and so has six vertex sets which induce P_4. Checking each set reveals that the graph's pattern set is $\{\bullet\,\bullet\,\circ\circ^*\}$, namely set VI. Using Chvátal's characterization, it is routine to verify that the graph is not perfectly orderable: orienting any white triangle edge forces an orientation of a black triangle edge, in turn forcing an orientation of a white triangle edge, and so on, eventually forcing a cyclic orientation of each triangle. On the other hand, the graph's monochromatic subgraphs are perfectly orderable: each such subgraph is P_4-free, so any vertex order is a perfect order. Thus, set VI is not a subset of any P_4 composition scheme which preserves perfect orderability.

Considering in similar fashion the remaining graphs of the top row of Fig. 8, it is routine to verify that none of sets VII, ..., XI is a subset of a P_4 composition scheme which preserves perfect orderability.

Finally, consider the bottom graph in Fig. 8. Let G, B, W represent respectively the graph and its black and white vertex sets. Observe that $G[W]$ is isomorphic to $G[B]$. Let $B = \{b_1, b_2, \ldots, b_7\}$ and $W = \{w_1, w_2, \ldots, w_7\}$ so that for each j, b_j and w_j correspond to the vertex with label j in Fig. 8. In G, the P_4's which include neither b_7 nor w_7 have pattern $\bullet\bullet\bullet\bullet$ or $\circ\circ\circ\circ$, the P_4's which include exactly one of b_7 and w_7 have pattern $\bullet\bullet\bullet\circ^*$ or $\bullet\circ\circ\circ^*$, and the P_4's which include both b_7 and w_7 have pattern $\bullet\circ\bullet\circ^*$. Thus G's pattern set is $\{\bullet\bullet\bullet\bullet, \bullet\bullet\bullet\circ^*, \bullet\circ\bullet\circ^*, \bullet\circ\circ\circ^*, \circ\circ\circ\circ\}$, namely set XII. It is routine to verify that $b_1 < b_2 < b_3 < \cdots < b_7$ is a perfect order of $G[B]$ and so that both monochromatic subgraphs are perfectly orderable. Thus to show that set XII is not a subset of any P_4 composition scheme which preserves perfect orderability, it remains only to show G is not perfectly orderable.

Argue by contradiction: consider any perfect order $<$ of G. Suppose that $b_7 < b_1$: this forces $b_3 < b_2$ by Chvátal's characterization, since $(b_7 b_1 b_3 b_2)$ is a P_4, which in turn forces $b_6 < b_7$. On the other hand, suppose that $b_7 < b_6$: this forces $b_2 < b_3$ which in turn forces $b_1 < b_7$. Thus in all cases, $b_1 < b_7$ or $b_6 < b_7$ (or both), and similarly $w_1 < w_7$ or $w_6 < w_7$ (or both). Pick x in $\{1, 6\}$ so that $b_x < b_7$ and pick y in $\{1, 6\}$ so that $w_y < w_7$. For any $b_{j \neq 7}$ in B, $w_7 < b_j$ [since $b_j < w_7$ would force $w_6 < w_2$ in turn forcing $w_3 < w_4$ in turn forcing $w_5 < w_6$ in turn forcing $w_7 < b_j$, contradiction]. Similarly, for any $w_{k \neq 7}$ in W, $b_7 < w_k$. But now $w_7 < b_x < b_7 < w_y < w_7$, a contradiction since $<$ is a linear order and therefore acyclic. Thus, G is not perfectly orderable and the proof of the theorem is complete. □

Proof of Lemma 18. Let $\mathscr{S}$ denote the set $\{\mathrm{I}, \mathrm{I}', \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{V}, \mathrm{V}'\}$ and let $\mathscr{T}$ denote the set $\{\mathrm{VI}, \mathrm{VII}, \mathrm{VII}', \mathrm{VIII}, \mathrm{VIII}', \mathrm{IX}, \mathrm{IX}', \mathrm{X}, \mathrm{XI}, \mathrm{XII}\}$. We wish to show that every pattern set is either a subset of some element of $\mathscr{S}$ or a superset of some element of $\mathscr{T}$. To simplify notation, we represent patterns with numbers as follows:

0	1	2	3	4	5	6	7	8	9
$\bullet\bullet\bullet\bullet$	$\circ\circ\circ\circ$	$\bullet\bullet\bullet\circ^*$	$\bullet\circ\circ\circ^*$	$\bullet\bullet\circ\bullet^*$	$\circ\bullet\circ\circ^*$	$\bullet\circ\circ\bullet$	$\circ\bullet\bullet\circ$	$\bullet\circ\bullet\circ^*$	$\bullet\bullet\circ\circ^*$

	0	1	2	3	4	5	6	7	8	9
I = {013468}			VII			VII′		VIII′		VI
I′ = {012578}				VII′	VII		VIII			VI
II = {0167}			VIII	VIII′	IX	IX′			XI	VI
III = {0145}			VII	VII′			IX′	IX	X	VI
IV = {0123}					VII	VII′	VIII	VIII′	XII	VI
V = {0238}		XII			VII	VII′	VIII	VIII′		VI
V′ = {1238}	XII				VII	VII′	VIII	VIII′		VI

Fig. 9. Maximality of I, …, V′. For example, the row IV column 5 entry indicates that adding pattern set 5 = ○ ● ○○* to IV yields a superset of VII′.

	0	1	2	3	4	5	6	7	8	9
VI = {9}										I
VII = {24}			III		V					
VII′ = {35}				III		V				
VIII = {26}			II				V			
VIII′ = {37}				II				V		
IX = {47}					II			III		
IX′ = {56}						II	III			
X = {458}					I′	I			III	
XI = {678}							I′	I	II	
XII = {01238}	V′	V	I	I′					IV	

Fig. 10. Minimality of VI, …, XII. For example, the row VIII column 6 entry indicates that removing pattern set 6 = ● ○ ○● from VIII yields a subset of IV.

Fig. 9 establishes that any proper superset of an element of $\mathscr{S}$ is a superset of some element of $\mathscr{T}$.[4] Similarly, Fig. 10 establishes that any proper subset of an element of $\mathscr{T}$ is a subset of some element of $\mathscr{S}$. Thus to complete the proof, it suffices to consider any pattern set Z which is not a superset of any element of $\mathscr{T}$, and show that Z is a subset of some element of $\mathscr{S}$. Begin by observing that Z does not contain 9 (otherwise Z is a superset of XII in $\mathscr{T}$).

[4] At this point in the proof we have established that the pattern sets I, …, V and their colour exchange equivalents are maximal with respect to preserving perfect orderability. The rest of the proof establishes that there are no other maximal sets.

First suppose that 4 is in Z. Then 2 is not in Z (otherwise Z contains VII) and 7 is not in Z (otherwise Z contains IX). If 5 is not in Z then Z is a subset of I and we are done; if 5 is in Z then 3 is not in Z (otherwise Z contains VII′), 6 is not in Z (otherwise Z contains IX′), 8 is not in Z (otherwise Z contains X), and Z is a subset of III. So we are done if 4 is in Z. Since 5 is the colour exchange of 4, we are also done if 5 is in Z.

Suppose then that neither 4 nor 5 is in Z. If 2 is in Z then 6 is not in Z (otherwise Z contains VIII); now if 3 is not in Z then Z is a subset of I′, whereas if 3 is in Z then 7 is not in Z (otherwise Z contains VIII′) and Z is a subset of XII. Since Z does not contain XII, Z is a proper subset of XII, and so by the earlier argument using Fig. 10, Z is a subset of some element of $\mathscr{S}$. So we are done unless 2 is not in Z, and by a symmetric argument (since 3 is the colour exchange of 2) 3 also is not in Z. Then at least one of 6,7,8 is not in Z (otherwise Z contains XI), and Z is a subset of one of I,I′,II. □

5. Conclusions

We have shown that schemes I–V are the only maximal P_4 composition schemes which preserve perfect orderability. For the most part, our method has been to describe the structure of graphs which admit two-colourings with various pattern sets. Recall that $\mathscr{G}_{\mathrm{I}}, \ldots, \mathscr{G}_{\mathrm{V}}$ are the classes of non-trivial graphs which can be two-coloured (with at least one vertex of each colour) using pattern sets of sets I, …, V. In Theorems 4, 9 and 12 we have shown that

- $\mathscr{G}_{\mathrm{II}}$ consists of all non-trivial split-substitute graphs,
- $\mathscr{G}_{\mathrm{III}}$ consists of all non-trivial no-end-substitute graphs,
- $\mathscr{G}_{\mathrm{IV}}$ consists of all non-trivial no-mid-substitute graphs;

these theorems also imply that

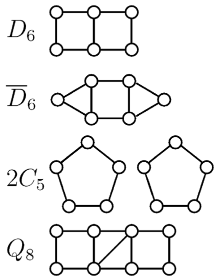
- $\mathscr{G}_{\mathrm{II}} \subset \mathscr{G}_{\mathrm{III}}, \mathscr{G}_{\mathrm{II}} \subset \mathscr{G}_{\mathrm{IV}}$, and $(\mathscr{G}_{\mathrm{III}} \cup \mathscr{G}_{\mathrm{IV}}) \subset \mathscr{G}_{\mathrm{I}}$.[5]

It would be interesting to find similar structural characterizations for $\mathscr{G}_{\mathrm{I}}, \mathscr{G}_{\mathrm{V}}$ and to establish whether $\mathscr{G}_{\mathrm{V}} \subset \mathscr{G}_{\mathrm{I}}$. Fig. 11 shows that no other inclusions among the five classes are possible, and that not all perfectly orderable graphs are in $\mathscr{G}_{\mathrm{I}}$ (since Q_8 is perfectly orderable). The graphs D_6 (the domino), $\bar{D}_6$, $2C_5$, and Q_8 are shown in Fig. 12, which also shows the set containment hierarchy among the classes $\mathscr{G}_{\mathrm{I}}, \ldots, \mathscr{G}_{\mathrm{V}}$.

Recall that $\mathscr{H}_{\mathrm{I}} \ldots \mathscr{H}_{\mathrm{V}}$ are the subsets of $\mathscr{G}_{\mathrm{I}} \ldots \mathscr{G}_{\mathrm{V}}$ which satisfy the hereditary property (namely, every non-trivial induced subgraph of a graph in $\mathscr{H}_K$ is also in $\mathscr{H}_K$). Thus

- $\mathscr{H}_{\mathrm{II}}$ consists of all non-trivial recursively split-substitute graphs,
- $\mathscr{H}_{\mathrm{III}}$ consists of all non-trivial recursively no-end-substitute graphs,
- $\mathscr{H}_{\mathrm{IV}}$ consists of all non-trivial recursively no-mid-substitute graphs.

[5] The first two inclusions follow from the fact that each vertex in a split graph is no-end or no-mid. The last inclusion follows from $\mathscr{G}_{\mathrm{III}} = \mathscr{G}_{\mathrm{III}-}, \mathscr{G}_{\mathrm{IV}} = \mathscr{G}_{\mathrm{IV}-}, \mathrm{III}^{-} \subset I$, and $\mathrm{IV}^{-} \subset \mathrm{I}$.



	$\mathcal{G}_I$	$\mathcal{G}_{II}$	$\mathcal{G}_{III}$	$\mathcal{G}_{IV}$	$\mathcal{G}_V$
D_6	√	×	×	√	√
$\overline{D}_6$	√	×	√	×	√
$2C_5$	√	√	√	√	×
Q_8	×	×	×	×	×

Fig. 11. Some separating examples for inclusions among $\mathscr{G}_{\mathrm{I}},\ldots,\mathscr{G}_{\mathrm{V}}$. For example, the first row indicates that D_6 is in $\mathscr{G}_{\mathrm{I}},\mathscr{G}_{\mathrm{IV}},\mathscr{G}_{\mathrm{V}}$ but not $\mathscr{G}_{\mathrm{II}},\mathscr{G}_{\mathrm{III}}$.

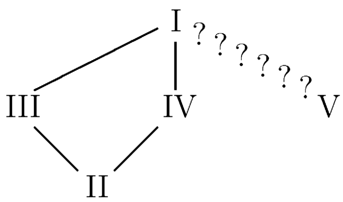


Fig. 12. Inclusions among graph classes $\mathscr{G}_{\mathrm{I}},\ldots,\mathscr{G}_{\mathrm{V}}$.

Hoáng and Reed [9] called a graph *strongly brittle* if every induced subgraph has some no-end vertex. Since substituting a graph with a no-end vertex into a graph with a no-end vertex yields a graph with a no-end vertex, every graph in $\mathscr{H}_{\mathrm{III}}$ is strongly brittle. On the other hand, it follows easily from the definition of $\mathscr{H}_{\mathrm{III}}$ that every strongly brittle graph is in $\mathscr{H}_{\mathrm{III}}$. Thus $\mathscr{H}_{\mathrm{III}}$ is exactly the set of all non-trivial strongly brittle graphs; similarly, $\mathscr{H}_{\mathrm{IV}}$ is the set of all complements of strongly brittle graphs.

It would be interesting to find similar characterizations for $\mathscr{H}_{\mathrm{I}}$, $\mathscr{H}_{\mathrm{V}}$ and to find forbidden induced subgraph characterizations for $\mathscr{H}_{\mathrm{I}}$, $\mathscr{H}_{\mathrm{III}}$, $\mathscr{H}_{\mathrm{IV}}$, $\mathscr{H}_{\mathrm{V}}$.

The characterizations we have established have algorithmic consequences, as we now explain. The tasks of computing a graph's characteristic graph, checking whether a graph is split, and checking whether a vertex is no-mid (or no-end) can all be performed in polynomial time. Also, a graph is strongly brittle if and only if the vertices can be linearly ordered so that each vertex is no-end in the subgraph induced by that and all previous vertices [9]. It follows that recognizing whether a graph is in $\mathscr{G}_{\mathrm{II}}$, $\mathscr{G}_{\mathrm{III}}$, $\mathscr{G}_{\mathrm{V}}$, $\mathscr{H}_{\mathrm{III}}$, or $\mathscr{H}_{\mathrm{IV}}$ takes polynomial time. Also, it follows from our forbidden induced subgraph characterization of $\mathscr{H}_{\mathrm{II}}$ that these graphs can be recognized in polynomial time. It would be interesting to determine whether graphs in classes $\mathscr{G}_{\mathrm{I}}$, $\mathscr{G}_{\mathrm{V}}$, $\mathscr{H}_{\mathrm{I}}$, $\mathscr{H}_{\mathrm{V}}$ can be recognized in polynomial time.

It would also be interesting to consider P_4 composition schemes which preserve other properties. For example, Chvátal et al. [3] find all maximal P_4 composition schemes which preserve perfection. (A graph is *perfect* if for every induced subgraph, the chromatic number is the same as the size of a largest clique.)

For any of these open problems, general results on the P_4-structure of graphs may be useful; for a recent survey, see the chapter by Hougardy in [13].

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