A Note on Domination in Hex

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Abstract. We establish some move domination results for the game of Hex. One corollary is that for a player P, any opening move to a cell in the second-row which is adjacent to two first-row cells is at least as good as an opening move to either of the two latter cells.

For a player P, a *side cell* is a cell which touches one of P's two borders, a *side pair* (a_1, a_2) consists of two side cells which touch the same border, and a *side triangle* (x_1, x_2, t) consists of a side pair (x_1, x_2) together with a third cell (the *tip*) adjacent to the two side cells.

Lemma 1. *Let (x_1, x_2, t) be a Black side triangle, let B_0 be a game state with x_1 and x_2 unoccupied and a black piece on t, let B_1 be the state obtained from B_0 by adding a black piece on x_1, and let B_2 be the state obtained from B_1 by adding a black piece on x_2. Then Black has a winning strategy for any one of B_0, B_1, B_2 if and only if P has a winning strategy for all of B_0, B_1, B_2.*

Proof. Hex is regular (adding a piece for a player is never disadvantageous for the player), so for $j = 0$ and 1, a Black winning strategy for B_j implies a Black winning strategy for B_{j+1}. Thus to prove the lemma it sufficies to show that the existence of a Black winning strategy S_2 for B_2 implies the existence of a Black winning strategy S_0 for B_0.

Let S' be the strategy obtained from any such S_2 by ignoring any Black or White moves to either x_1 or x_2. Let East and West be the two Black sides, and let (x_1, x_2, t) be a western triangle. Now it is easy to verify that S' is winning strategy on the board obtained by removing x_1, x_2, where Black wins by completing a chain which reaches from the East to either the West or t. Combining S' with the strategy which forms the connection from t through x_1, x_2 to the West yields the desired strategy S_0. □

Corollary 1. *Let (x_1, x_2, t) be a Black side triangle, let B_0 be a game state with x_1 and x_2 unoccupied and a black piece on t, and let B^* be the state obtained from B by moving the black piece on t to x_1. Then Black has a winning strategy for B_0 if Black has a winning strategy for B^*.*

Proof. Since Hex is regular, a Black winning strategy for B^* implies a Black winning strategy for the state B^+ obtained from B^* by adding a black piece at t. By relabelling x_1, x_2 if necessary, B^+ is equal to the state B_1 described in the lemma. Since Black has a winning strategy for B_1, it follows by the lemma that Black has a winning strategy for B_0. □

References

1. A. Beck, Games, in *Excursions into Mathematics*, A. Beck, M. Bleicher and D. Crowe, editors, Worth, New York, 1969, pp. 317-387
2. A. Beck, Appendix 2000, in *Excursions into Mathematics: The Millenium Edition*, A. Beck, M. Bleicher and D. Crowe, editors, A.K. Peters, Natick, 2000.

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