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A Hex Handicap Strategy

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Abstract

We give an $\lceil \frac{n+1}{6} \rceil$-cell handicap strategy for the game of Hex on an $n \times n$ board: the first player is guaranteed victory if she is allowed to colour $\lceil \frac{n+1}{6} \rceil$ cells on her first move. Our strategy exploits a new kind of inferior Hex cell.

1 Introduction

Hex was invented independently by Piet Hein in 1942 [9] and John Nash in 1948 [10]. The game is played by two players, Black and White, on a board with hexagonal cells. The players alternate turns, colouring any single uncoloured cell[1] with their colour. The winner is the player who creates a path of her colour connecting her two opposing board sides. Figure 1 shows a 5×5 board at the start and end of a game won by White.

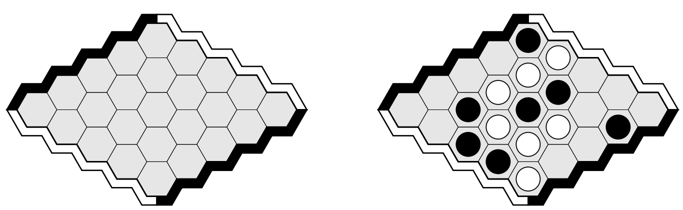

Figure 1: An empty 5×5 Hex board (left) and a completed game position (right).

Hein and Nash observed that Hex cannot end in a draw [9, 10]: exactly one player has a winning path if all cells are coloured [1]. Also, an extra coloured cell is never disadvantageous for the player with that colour [10]. For $n \times n$ boards, Nash showed the existence of a first-player winning strategy [10]; however, his proof reveals nothing about the moves of such a strategy. For 7×7 and smaller boards, computer search can find all winning first moves [7]. For 8×8 and 9×9 boards, Jing Yang found by human search that moving to the centre cell is a winning first move [13].

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[1]Or by indicating cell ownership in some other way, e.g. by placing a coloured stone on the cell.

For $m \times n$ boards with $m \neq n$ (hereafter *irregular boards*), the player whose opposing board sides are closer together can win, even as the second player, via a pairing strategy due to Claude Shannon [5]. Figure 2 shows this strategy for the 5×4 Hex board. By contrast, Stefan Reisch showed that solving arbitrary $n \times n$ Hex positions is PSPACE-complete [11].

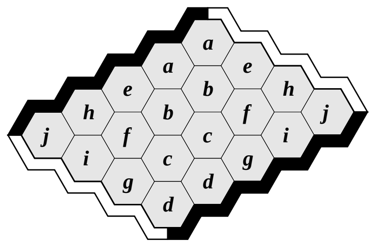

Figure 2: Shannon's pairing strategy on a 4×5 Hex board.

The problem of efficiently, say in polynomial time, identifying a winning first move on an empty $n \times n$ Hex board has been unsolved for roughly 60 years. As a step towards solving this problem we ask the following:

> *Starting from the* n×n *Hex board, what is the least number* h(n) *of cells the first player needs to colour in order to reach a winning position with a known and polytime strategy, and what should these initial cells be?*

For n up to 9, explicit winning strategies are known for opening on the centre board cell. Notice that $h(n) \geq 1$, since the second player can win if the first player colours no cells. Also, by colouring cells as in Figure 3, notice that $h(n) \leq \frac{n}{2}$; this can be improved slightly by omitting cells closest to the sides. Before now, little else was known about $h(n)$.

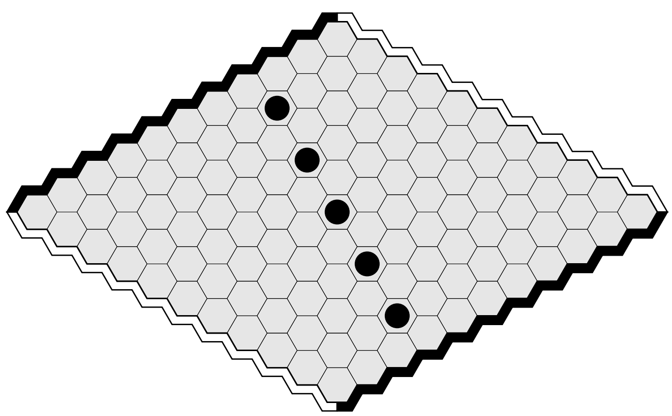

Figure 3: A winning Black handicap position.

In this paper, we show that $h(n) \leq \lceil \frac{n+1}{6} \rceil$ for all positive n. Our strategy combines Shannon's $(n-1) \times n$ pairing strategy with the exploitation of a new kind of inferior cell: we colour cells in the second row in a way that allows us to negate opponent moves to the first row. The resulting $\lceil \frac{n+1}{6} \rceil$-cell handicap strategy is both explicit and efficient.

In §2 we review previous Hex inferior cell analysis, in §4 we introduce a new type of inferior cell, and in §5 we present our handicap strategy.

2 Inferior Cell Analysis

Hex has a large branching factor, so it is important for players to find some partial order indicating comparative cell value. If one cell is provably inferior to another, in the sense that (a move to) the former loses if the latter loses, then the former can be replaced by the latter on the list of moves to consider. At one end of this order are cells that are provably useless.

Following observations of Beck et al. [1] and Schensted and Titus [12], Hayward and Jack van Rijswijck defined one form of useless cell: with respect to a particular Hex position, a cell c (coloured or not) is *live* if there is some *completion* of the position (namely, a colouring of all uncoloured cells) in which changing c's colour changes the winner in the completion; a cell is *dead* if it is not live [8]. See Figure 4.

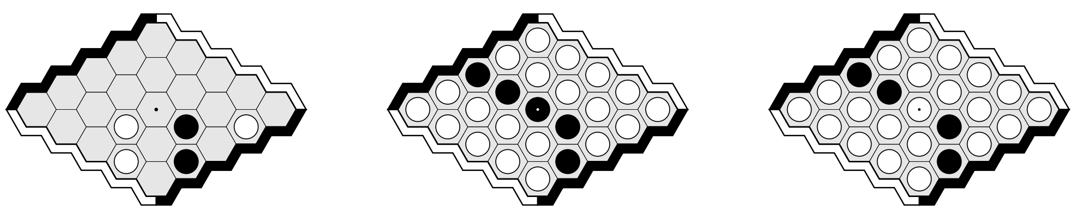

Figure 4: A live cell (left). In the completion (middle), changing the cell's colour (right) changes the winner.

Any move to a dead cell is useless, since there is no completion in which the cell's colour matters. It follows that a dead cell can be assigned an arbitrary colour (uncoloured, black, white) without changing the position's winner.[2]

Live cells can be identified by considering connecting sets [8]. In a Hex position, a *connector* for a player is a set of uncoloured cells that when coloured connects the player's two opposing board sides. An uncoloured cell is *live* if it is on some minimal connector; otherwise it is *dead*. A coloured cell is *live* if it is live in the position obtained by uncolouring the cell; otherwise it is dead. See Figure 5.

Some dead cells can be recognized by matching a pattern of neighbouring cells. For example, for each pattern of Figure 6, the uncoloured cell is dead [7]. By using only these five patterns and by representing each side of the board as a row of coloured cells, one can identify all the dead cells in Figure 5. Just as moves to dead cells are useless, so are cells to moves which can be immediately killed. A cell c is *vulnerable* for player P if her opponent $\overline{P}$

[2]More generally, this holds for any set of uncoloured dead cells, and for any single coloured dead cell. However, it does not always hold for a larger set of coloured dead cells, since changing the colour of one such cell might cause other dead cells of the same colour to become live.

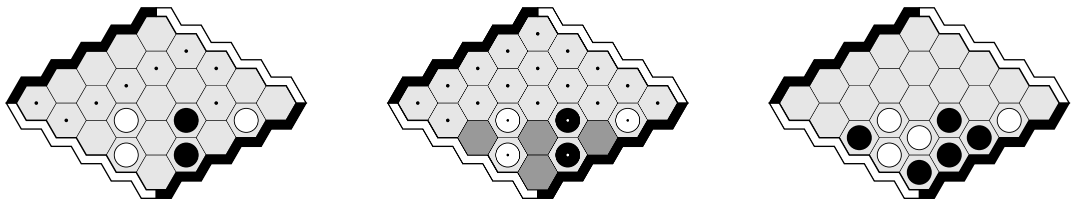

Figure 5: A minimal Black connector (left, dotted), all live/dead cells (middle, dotted/shaded), and an equivalent state (right) obtained by arbitrarily colouring dead cells.

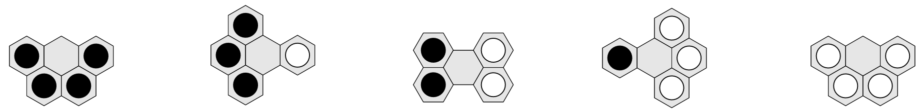

Figure 6: Dead patterns.

has a reply that makes c dead; this reply is c's *killer*. For example, in each pattern in Figure 7, the dotted cell is White-vulnerable to the shaded cell.

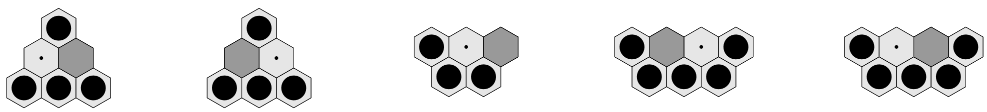

Figure 7: White-vulnerable patterns.

A set S of cells is *captured* by P if she has a second-player strategy to make all cells in S dead or her colour. We may assume that each player P follows a capturing strategy for each of their captured sets, so P-colouring the cells of a P-captured set does not alter a position's winner. See Figure 8.

With respect to a partition U_1, U_2 of the cells of a position B, for each $j = 1, 2$, let α_j be a player P 2nd-player strategy for a game continuation in which each player is restricted to colouring cells of U_j. Then $\alpha_1 \oplus \alpha_2$ is P's 2nd-player strategy for B defined by combining α_1 and α_2 in the obvious way:

> For each $j = 1, 2$, in response to the $\overline{P}$-colouring of a cell of U_j, find the state in α_j defined by the $\overline{P}$-coloured cells of U_j and colour the cell specified by α_j; if the specified cell is already P-coloured or if no cell is specified (for example if $\overline{P}$ has just coloured the last uncoloured cell in U_j) and if there is some uncoloured cell, then colour any uncoloured cell.

Notice that this sum operator $\oplus$ is essentially the combinatorial game theory sum operator modified so that a player never passes. For a player P and set of cells C of a position B, let $B + P(C)$ be the state obtained from B by P-colouring all cells of C.

Figure 8: Black-captured patterns (left). Black-colouring a Black-captured set does not alter the winner (right).

Theorem 1. *Let U be the uncoloured cells of a position B such that a proper subset of C of U is captured by P. Further let α_1 be a P-winning 2nd-player for $B + P(C)$ and let α_2 be a P-capturing strategy for C. Then $\alpha_1 \oplus \alpha_2$ is a P-winning strategy for B.*

Proof. Consider any terminal position T obtained by following $\alpha_1 \oplus \alpha_2$. To be continuted. □

Extra cells of a player P's colour are never disadvantageous, so a P-move to a cell c that yields P-captured cell set S is at least as good for P as any move to a cell of S. Such a cell c is said to *P-capture-dominate* all cells in set S. When considering moves, a player P can ignore all cells that are dead, captured by either player, P-vulnerable, or P-capture-dominated; if she has a winning move and there are other cells available, then she has a winning move to one of the other cells. See [4, 6, 7, 8] for more on inferior cell analysis.

3 Strategy Sum

4 Permanently Inferior Cells

Before describing a new kind of inferior Hex cell, we first generalize the notion of vulnerability. A cell c is *vulnerable-by-capture* for player P if her opponent $\overline{P}$ has a reply that captures a set S which when $\overline{P}$-coloured makes c dead; the reply is the *killer*, and the subsequent capturing strategy is the *killing strategy*. See Figure 9. S can be empty, so vulnerable-by-capture generalizes vulnerable. $\overline{P}$-colouring a $\overline{P}$-captured set does not change a position's winner [8] so, if in a Hex position a player P has a winning move, then she has a winning move that is not at a P-vulnerable-by-capture cell.

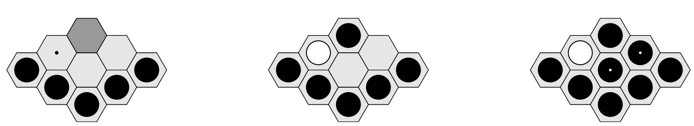

Figure 9: A White-vulnerable-by-capture cell (left, dot). If White plays there, Black can reply (next) and capture cells which, if Black-coloured (next), kill the White cell.

We now introduce a new kind of inferior cell. Let P be a player with opponent $\overline{P}$, and consider a Hex position with a set C of uncoloured cells, a cell c_1 in C such that each cell in

$C \setminus \{c_1\}$ is P-vulnerable-by-capture to a killing strategy using only cells in C, and such that some c_2 in C different from c_1 is $\overline{P}$-vulnerable to c_1. Then we say that c_2 is *permanently inferior* for P. See Figure 10.

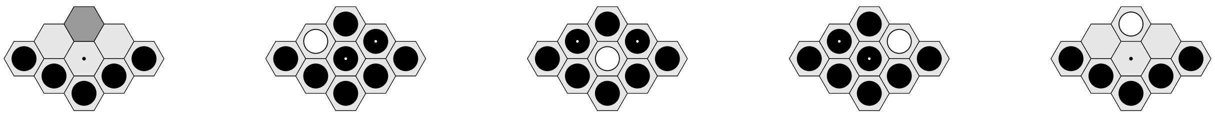

Figure 10: A White-permanently inferior pattern (left): the dotted cell is White-permanently inferior. The three unshaded cells are each White-vulnerable-by-capture to the shaded cell (next, next, next); also, White-colouring the shaded cell kills the dotted cell (right).

Theorem 2. *Let c_2 be a P-permanently inferior cell in a Hex position S, and let T be the position obtained from S by $\overline{P}$-colouring c_2. Then the winner of S is the winner of T.*

Proof. Extra $\overline{P}$-coloured cells are never $\overline{P}$-disadvantageous, so if $\overline{P}$ wins S then $\overline{P}$ wins T and we are done. Suppose then that P wins S. Thus P has a winning strategy for S. Furthermore, she has such a strategy in which she never plays at a dead, $\overline{P}$-captured, or P-vulnerable-by-capture cell.

Let c_1 and C be as in the definition of permanently inferior. Notice that the uncoloured cells of $C \setminus \{c_1\}$ remain P-inferior in any continuation from S in which P has not coloured any cell of $C \setminus \{c_1\}$: they will be either dead, $\overline{P}$-captured, or still P-vulnerable-by-capture. It follows that if P ever plays in C, then her first such move is to c_1, at which point c_2 is killed unless it is already $\overline{P}$-coloured. Thus P never plays at c_2, and her winning strategy also applies to position T. □

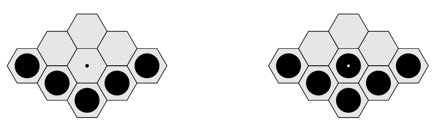

Figure 11: Black-colouring a White-permanently inferior cell does not alter the winner.

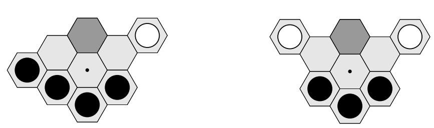

Figure 12: Two more White permanently inferior patterns.

5 Handicap Strategy

We now describe Black's $\lceil \frac{n+1}{6} \rceil$-handicap strategy for $n \times n$ Hex. Orient the board so that the two bottom borders meet at an obtuse angle. The row of cells along the bottom Black border is the *first row*; the column of cells along the bottom White border is *column 1*. The *handicap cells* are the cells in the second row in column $n-3$, and in columns $2 + 6 \times j$ for each j in $\{0,\ldots,\lfloor \frac{n}{6} \rfloor - 1\}$. The *primary cells* are the first row cells adjacent to a handicap cell. The *inferior regions* are the sets of four cells in the first two rows that, assuming all handicap cells and primary cells are Black, match the permanently inferior pattern in Figure 10. See Figures 13 and 14.

Black's Strategy *Colour the* $\lceil \frac{n+1}{6} \rceil$ *handicap cells. Then, in response to each White move,*

1. *if White colours a first row cell that is killed by Black-colouring all primary cells, then colour any cell;*

2. *if White colours a vulnerable-by-capture first or second row cell, then colour any killer;*

3. *if White colours any other cell, then colour its pair in the* $(n-1) \times n$ *Shannon strategy on the board obtained by ignoring the first row.*

Theorem 3. *Black's strategy is a winning* $\lceil \frac{n+1}{6} \rceil$*-handicap strategy for* $n \times n$ *Hex.*

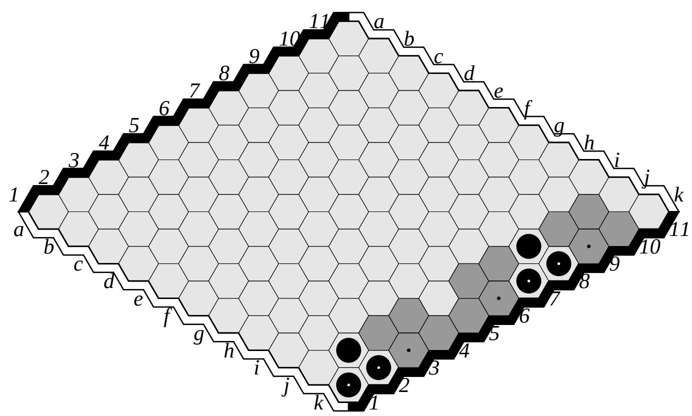

Figure 13: Our handicap strategy. The first row is labelled k. Handicap cells are solid Black. Primary pairs are dotted Black. Inferior regions are shaded.

Proof. It suffices to show that every White first row cell is dead when the game ends. This holds for each primary cell, since these cells form a Black-captured set: each such cell is White-vulnerable to a Black reply at any neighbouring primary cell.

It also holds for each White first row cell c of an inferior region R, as follows. Suppose firstly that c is the first cell in R to be coloured White; then, since the primary cells are

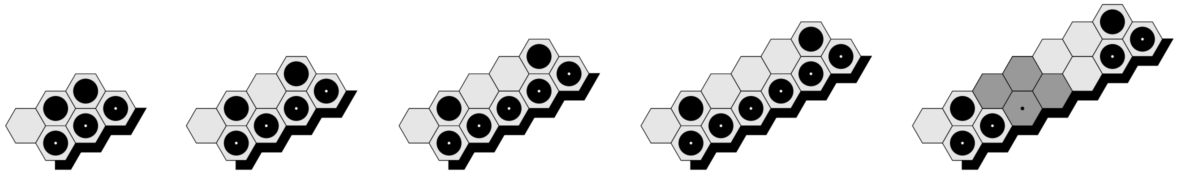

Figure 14: Strategy when the gap between the last two handicap cells is less than five.

Black-captured, c is White-vulnerable-by-capture as in Figure 10. Suppose secondly that the first cell in R to be coloured White is different from c and one of the three White-vulnerable-by-capture cells; then, since the primary cells are Black-captured and Black is following the killing strategy as in Figure 10, c will be dead when the game ends. Suppose finally that the first cell in R to be coloured White is the single non-White-inferior cell of R; thus, since the primary cells are Black-captured, the White-permanently inferior cell w will be dead. We are done if c is w, so assume c is not w, in which case it is adjacent to w; let x be other first row neighbour of c. We are done if x is coloured since, x is either Black or is White and will be dead by game end, so in each case c will be dead. But if x is uncoloured, c is White-vulnerable-by-capture to a Black replay at x. See Figure 15.

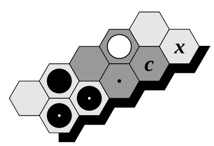

Figure 15: The dotted cell is dead, so c is White-vulnerable to a Black reply at x.

Lastly, the property holds for any cell c not in a primary pair or an inferior region (as in column n, or in the last pattern of Figure 14), as the neighbouring first row inferior region cell x will be either Black, or White and dead, or uncoloured; in these three cases c is respectively dead, dead, and White-vulnerable to a Black reply at x. □

Figure 16 shows a game following our strategy. `Black 1` and `Black 2` are the handicap moves. `Black 4` through `Black 64` follow the Shannon strategy; `White 35` is at the non-White-inferior cell of an inferior region, so `Black 36` follows the Shannon strategy. The primary cells *k7,k8* are captured by `Black 2`, so `White 65` is at a vulnerable-by-capture cell of an inferior region, and the reply is the killer `Black 66`. `White 67` is then vulnerable-by-capture to the killer `Black 68`. Similarly, the primary cells *k1,k2* are captured by `Black 1`, so the reply to the vulnerable-by-capture `White 91` is the killer `Black 92`, and the reply to the vulnerable-by-capture `White 93` is the killer `Black 94`. `White 97` will be dead because of Black-captured primary cells and `White 35`, so `Black 98` could have been played anywhere. `White 99` is at a primary cell, so the reply is the neighbouring killer `Black 100`. This move also connects Black's two sides, ending the game.

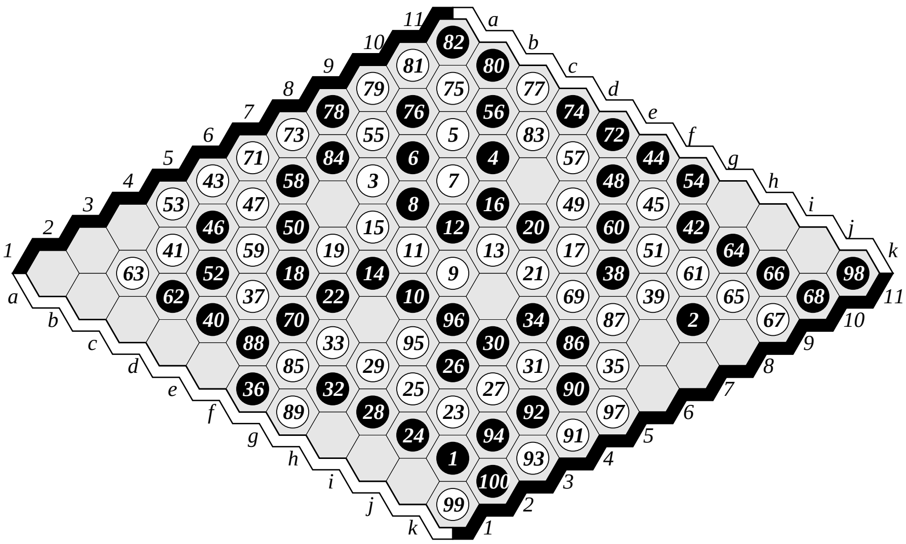

Figure 16: A game following our strategy.

As far as we know, this is the first two-stone handicap strategy for 11×11 Hex. The late Claude Berge, who was a Hex enthusiast [2, 3], would often give beginners three handicap stones on 11×11 Hex boards, suggesting that he did not expect them to find a winning strategy requiring fewer than four handicap stones. We would like to think that our result would have surprised him.

5.1 Acknowledgements

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