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A Lower Bound for the Optimal Crossing-Free Hamiltonian Cycle Problem*

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Abstract. Consider a drawing in the plane of K_n, the complete graph on n vertices. If all edges are restricted to be straight line segments, the drawing is called rectilinear. Consider a Hamiltonian cycle in a drawing of K_n. If no pair of the edges of the cycle cross, it is called a crossing-free Hamiltonian cycle (cfhc). Let $\Phi(n)$ represent the maximum number of cfhc's of any drawing of K_n, and $\bar{\Phi}(n)$ the maximum number of cfhc's of any rectilinear drawing of K_n. The problem of determining $\Phi(n)$ and $\bar{\Phi}(n)$, and determining which drawings have this many cfhc's, is known as the optimal cfhc problem. We present a brief survey of recent work on this problem, and then, employing a recursive counting argument based on computer enumeration, we establish a substantially improved lower bound for $\Phi(n)$ and $\bar{\Phi}(n)$. In particular, it is shown that $\bar{\Phi}(n)$ is at least $k \times 3.2684^n$. We conjecture that both $\Phi(n)$ and $\bar{\Phi}(n)$ are at most $c \times 4.5^n$.

1. A Survey of the Optimal Crossing-Free Hamilton Cycle Problem

Let K_n be the complete graph on n vertices. All drawings in this paper are assumed to be drawn in the plane. If all the edges of a drawing of a graph are restricted to be straight line segments, the drawing is said to be *rectilinear.* By a *crossing* of a drawing we mean a pair of edges which intersect in the drawing. A *Hamiltonian cycle* of a graph is a cycle that visits each vertex of the graph exactly once. Consider a particular Hamiltonian cycle in a drawing of K_n. If the cycle includes no crossings, it is called a *crossing-free Hamiltonian cycle,* or a *cfhc* for short. Let $\Phi(n)$ (and respectively $\bar{\Phi}(n)$) represent the maximum number of cfhc's of any drawing (respectively rectilinear drawing) of K_n. The *optimal cfhc problem* is to determine $\Phi(n)$ and $\bar{\Phi}(n)$, and to determine which drawings

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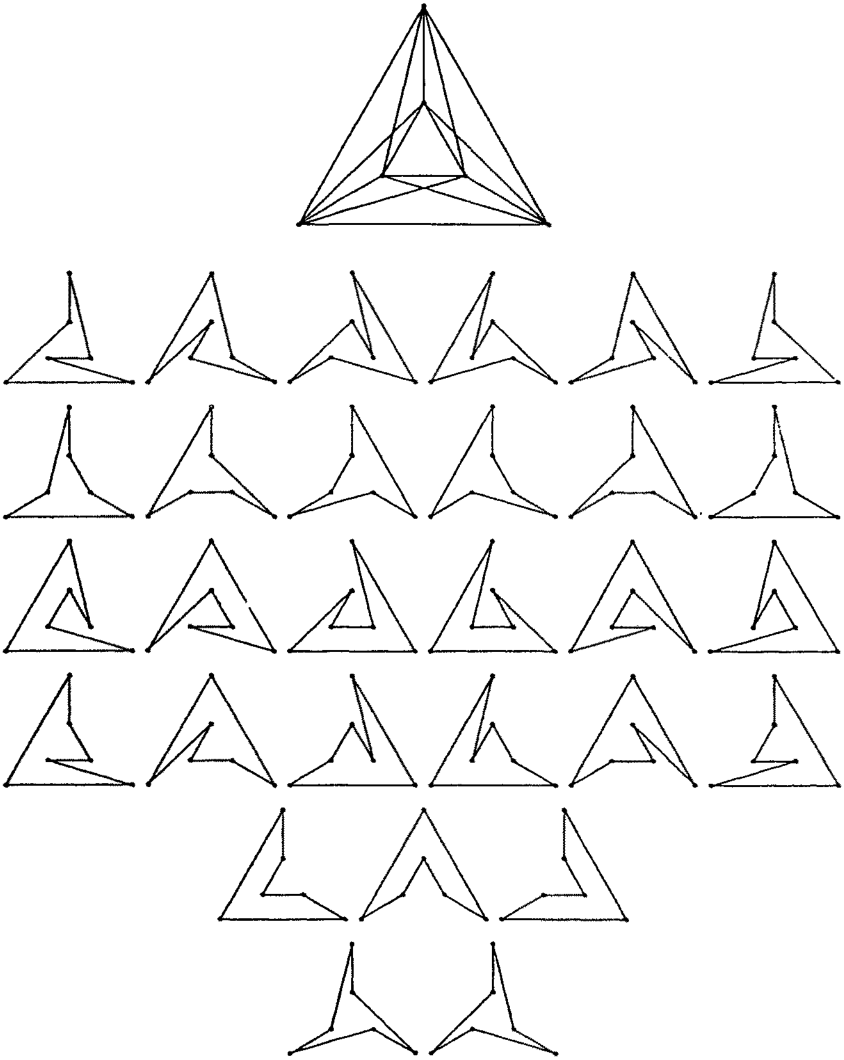

Fig. 1. A drawing of K_6 with 29 cfhc's.

of K_n have $\Phi(n)$ (respectively $\bar{\Phi}(n)$) cfhc's. Such drawings will be referred to as *cfhc-optimal* (respectively *rectilinear cfhc-optimal*) drawings. Figure 1 shows a drawing of K_6 which is both cfhc-optimal and rectilinear cfhc-optimal. The drawing has 29 cfhc's; the other 91 Hamiltonian cycles all have at least one crossing.

Let $\nu(n)$ (respectively $\bar{\nu}(n)$) refer to the minimum number of crossings of any drawing (respectively rectilinear drawing) of K_n. The *optimal crossing problem*, also known as the crossing number problem, is to determine the values of $\nu(n)$ and $\bar{\nu}(n)$, and to find which drawings attain this number of crossings. The optimal crossing problem is related to the optimal cfhc problem in that, in general, drawings with fewer crossings have more cfhc's. However, this is not always the case (e.g., see [H]). Although the optimal crossing problem has been extensively studied (see [EG] or [G]), exact values for $\nu(n)$ and $\bar{\nu}(n)$ are not known for $n > 10$.

The optimal cfhc problem was first explored by Newborn and Moser [NM]. They were able to determine $\Phi(n)$ and $\bar{\Phi}(n)$ exactly for n from 3 to 6, and established lower bounds for other small values of n. Later we extended this list of lower bounds [H]. The following is a list of the current best lower bounds for $\Phi(n)$ and $\bar{\Phi}(n)$, for n up to 13 (values for n up to 8 were established in [NM], all others are taken from [H]):

Best known lower bounds

n	3	4	5	6	7	8	9	10	11	12	13
$\Phi(n)$	1	3	8	29	92	339	1252	4956	18 383	75 231	306 446
$\bar{\Phi}(n)$	1	3	8	29	96	399	1461	6354	24 687	110 162	446 798

The first bounds for $\Phi(n)$ or $\bar{\Phi}(n)$ for arbitrary n were established by Newborn and Moser, who showed that

$$\tfrac{3}{20} \times 10^{n/3} \leq \bar{\Phi}(n) \leq 2 \times 6^{n-2} \times \left[\frac{n}{2}\right]!, \qquad \text{where} \quad 10^{1/3} \doteq 2.1544.$$

The upper bound was substantially improved by Ajtai *et al.* [ACNS], who showed that every planar drawing of any graph with n vertices contains at most $10\,000\,000\,000\,000^n$ crossing-free subgraphs. Thus both $\Phi(n)$ and $\bar{\Phi}(n)$ are exponential in n.

The lower bound was first improved by Akl [A], who showed that $d_n < \bar{\Phi}(n)$, where d_n is asymptotically $k \times (5 + 3 \times \sqrt{5})^{n/3}$, with k a constant and $(5 + 3 \times \sqrt{5})^{1/3} \doteq 2.2707$.

In this paper, generalizing Akl's approach, we show how the lower bound can be substantially improved by counting a subset of the cfhc's of a certain drawing TS_n of K_n. We prove that $f_n < \bar{\Phi}(n)$, where f_n is asymptotically $k \times 3.2684^n$.

2. An Improved Lower Bound for $\bar{\Phi}(n)$

In this section we describe a certain rectilinear drawing TS_n of K_n, and then count a subset of its cfhc's. This gives a new lower bound for $\bar{\Phi}(n)$.

2.1. *A Description of the Drawing* TS_n

The "TS" in TS_n is mnemonic for "trilateral spiral". Roughly speaking, the vertices of TS_n can be thought of as resting on three gently spiralled arcs emanating from the origin.

More precisely, let arc A be the arc of the circle centered at the point in the plane with Cartesian coordinates $(x, 2)$ and joining (in clockwise order) the points $(0, 1)$ and $(0, 3)$, where $x \geq 7/\sqrt{3}$. Arcs B and C are formed by rotating arc A respectively 120 and 240° clockwise about the origin, namely the point $(0, 0)$. Place vertices $1, 4, 7, \ldots$ on arc A, vertices $2, 5, 8, \ldots$ on arc B, and vertices $3, 6, 9, \ldots$ on arc C, so that if v and w are on the same arc, and $v < w$, then v is closer to the origin than w (see Fig. 2). Figures 1, 3, and 4 show drawings of TS_6, TS_9, and TS_{12}, respectively.

The reason for choosing x as described above is to ensure that the line segment joining the far end of arc A to the near end of arc C does not intersect arc A in any other point. In fact, the arcs are constructed so that any line segment joining points on two different arcs intersects each of the two arcs in exactly one point, and does not intersect the third arc.

Let

$$a = \left[\frac{n+2}{3}\right], \qquad b = \left[\frac{n+1}{3}\right], \qquad c = \left[\frac{n}{3}\right],$$

and relabel vertices $1, 4, \ldots, 3a-2$ as $A_1, A_2, \ldots, A_a$, vertices $2, 5, \ldots, 3b-1$ as $B_1, B_2, \ldots, B_b$, and vertices $3, 6, \ldots, 3c$ as $C_1, C_2, \ldots, C_c$. Then the following

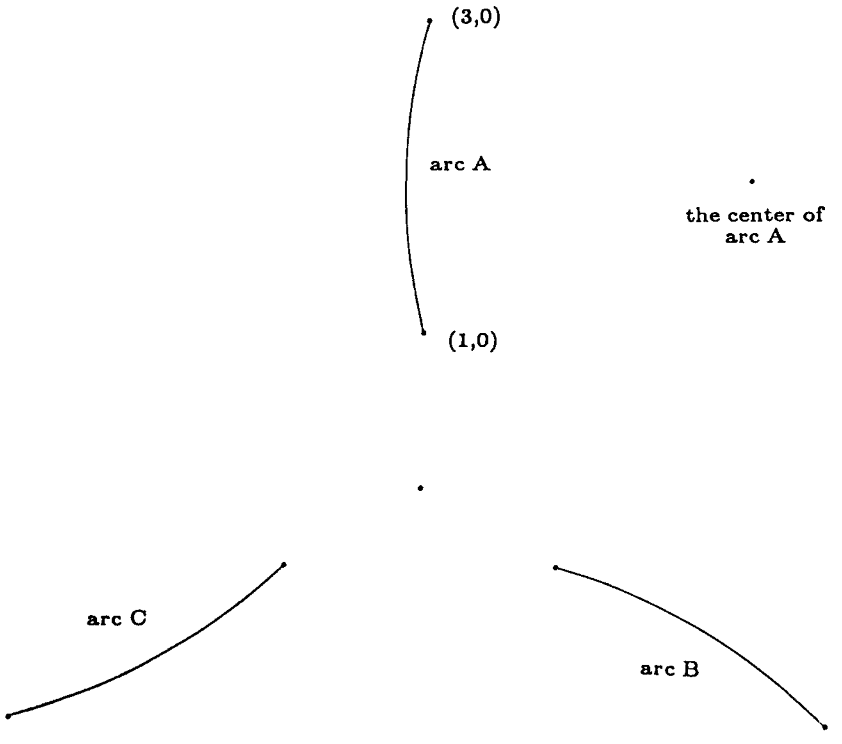

Fig. 2. Template arcs for TS_n.

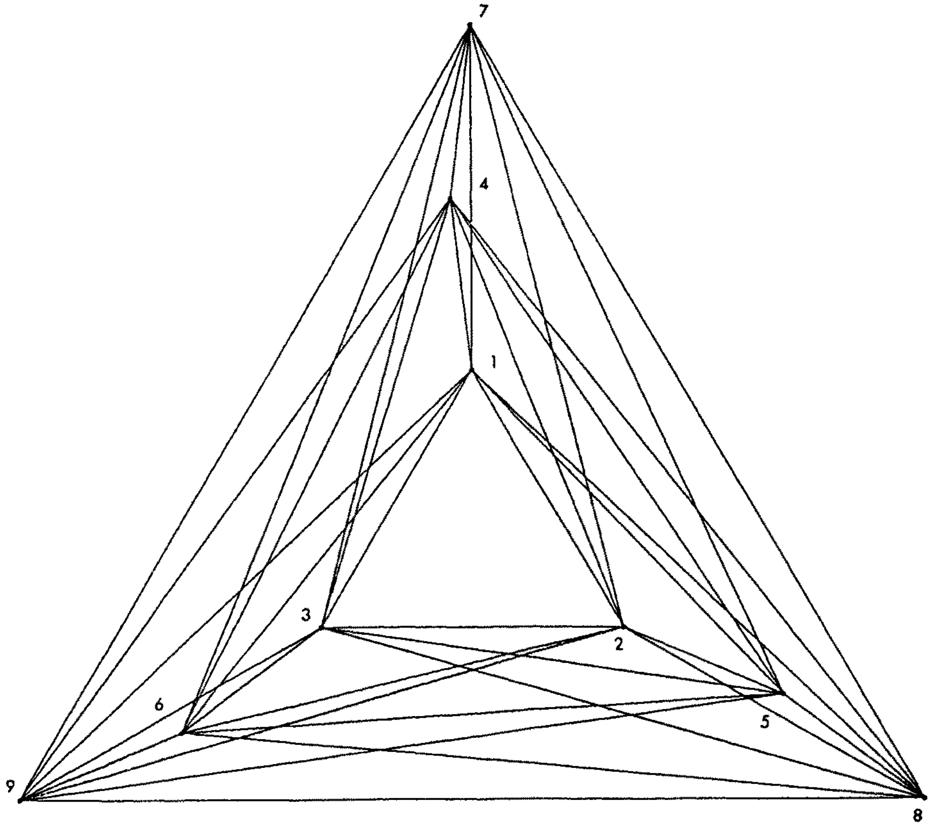

Fig. 3. The drawing TS_9.

is a description of all crossings of TS_n:

(1) $(A_i, A_k)(A_j, A_m)$ for $1 \le i < j < k < m \le a$,

(2) $(B_i, B_k)(B_j, B_m)$ for $1 \le i < j < k < m \le b$,

(3) $(C_i, C_k)(C_j, C_m)$ for $1 \le i < j < k < m \le c$,

(4) $(A_i, B_k)(A_j, B_m)$ for $1 \le i < j \le a$, $1 \le k < m \le b$,

(5) $(B_i, C_k)(B_j, C_m)$ for $1 \le i < j \le b$, $1 \le k < m \le c$,

(6) $(C_i, A_k)(C_j, A_m)$ for $1 \le i < j \le c$, $1 \le k < m \le a$,

(7) $(A_i, A_k)(A_j, B_m)$ for $1 \le i < j < k \le a$, $1 \le m \le b$,

(8) $(B_i, B_k)(B_j, C_m)$ for $1 \le i < j < k \le b$, $1 \le m \le c$,

(9) $(C_i, C_k)(C_j, A_m)$ for $1 \le i < j < k \le c$, $1 \le m \le a$.

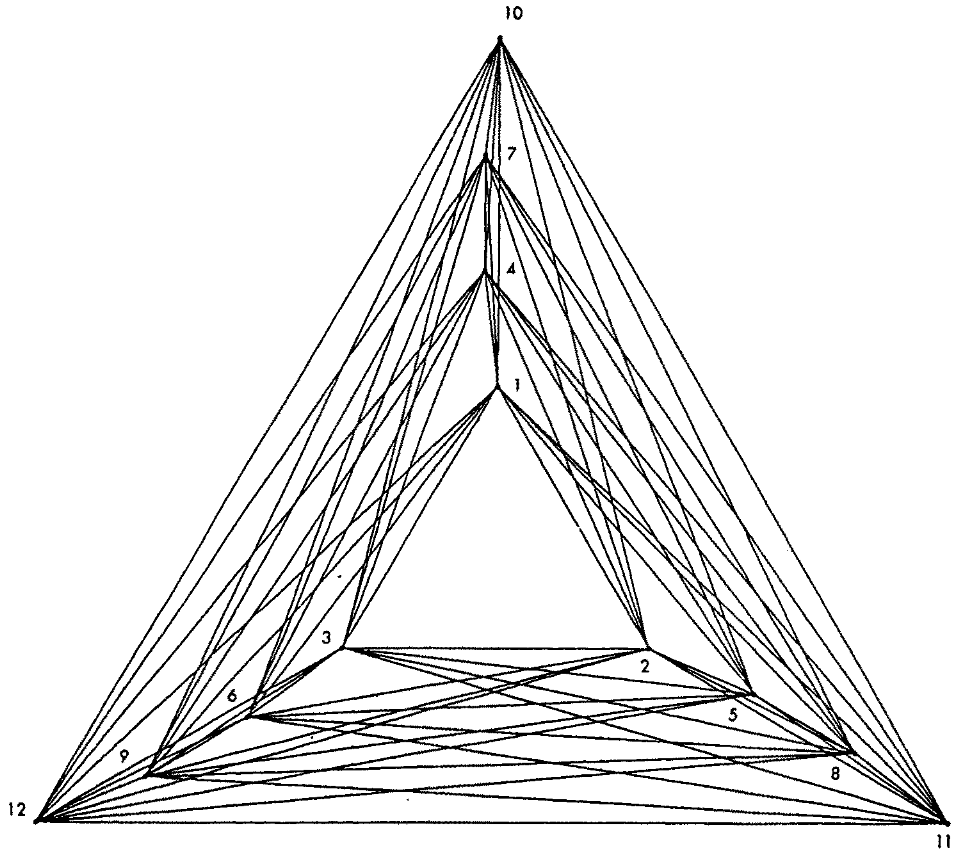

Fig. 4. The drawing TS_{12}.

The number of crossings is

$$\binom{a}{4}+\binom{b}{4}+\binom{c}{4}+\binom{a}{2}\binom{b}{2}+\binom{b}{2}\binom{c}{2}+\binom{c}{2}\binom{a}{2}+\binom{a}{3}b+\binom{b}{3}c+\binom{c}{3}a,$$

where $\binom{x}{y}$ is defined as 0 if $x<y$. Thus, the total number of crossings of TS_n is

$$\frac{11n^4-90n^3+225n^2-162n}{648}$$

$$\frac{11n^4-90n^3+249n^2-290n+120}{648}$$

$$\frac{11n^4-90n^3+249n^2-250n+48}{648}$$

for n congruent to 0, 1, 2 (mod 3), respectively.

2.2. *Counting cfhc's of* TS_n

Let cfhc (TS_n) represent the number of cfhc's of TS_n. We are unable to determine cfhc (TS_n) explicitly for arbitrary n. However, by counting a proper subset of the cfhc's of TS_n, we have established a lower bound for cfhc (TS_n), which gives an improved lower bound for $\bar{\Phi}(n)$.

Our counting argument is inductive, and relies on the fact that in any drawing of TS_n, any consecutive set of r vertices induces a drawing isomorphic to TS_r (two drawings of K_n are isomorphic if the vertices of one can be relabeled so that both drawings have the same set of crossings). Thus it follows that in a drawing of TS_{n+k}, the drawing induced by vertices 1 to n is isomorphic to TS_n. We will count cfhc's of TS_{n+k} by counting cfhc's of TS_n, and then enumerating various ways in which cfhc's of TS_n give rise to cfhc's of TS_{n+k}.

We classify each cfhc of TS_n according to which of the three outermost (convex hull) edges and which of the three innermost edges the cfhc contains. In TS_n, X, Y, and Z will represent, respectively, the edges $(n-2, n-1)$, $(n-1, n)$, and $(n-2, n)$ and x, y, and z will represent the edges $(1, 2)$, $(2, 3)$, and $(1, 3)$. We will use γ to represent cfhc's. Thus a $\gamma(X, n)$ will represent a cfhc of TS_n that includes the edge $(n-2, n-1)$ but neither edge $(n-1, n)$ nor $(n-2, n)$. A $\gamma(yz, n)$ will represent a cfhc of TS_n that includes the edges $(2, 3)$ and $(1, 3)$ but not edge $(1, 2)$. We will ignore cfhc's which contain all or none of either the outermost or innermost edges.

We create cfhc's of TS_{n+k} by starting with a cfhc of TS_n on vertices 1 to n, removing either one or two of its outermost edges, and then joining the resulting crossing-free path to vertices $n+1$ to $n+k$. For $k=1$ and 2 we enumerate by hand all the possible ways in which this can be done. For $k \geq 3$ we show how this can be done in a more systematic way (and in a way which allows for computer enumeration).

2.2.1. Case $k=1$: creating cfhc's of TS_{n+1} from cfhc's of TS_n. Figure 5 shows all nine ways in which cfhc's of TS_n give rise to cfhc's of TS_{n+1} upon removal of an outermost edge. (Only vertices $n-2$ to $n+1$ of TS_{n+1} are shown in Fig. 5. The dashed line in the figures represents that part of the cfhc which visits vertices 1 to $n-3$.) In particular,

each $\gamma(X, n)$ gives rise to a $\gamma(Z, n+1)$,
each $\gamma(Y, n)$ gives rise to a $\gamma(YZ, n+1)$,
each $\gamma(Z, n)$ gives rise to a $\gamma(Y, n+1)$,
each $\gamma(YZ, n)$ gives rise to a $\gamma(YZ, n+1)$ and a $\gamma(XY, n+1)$,
each $\gamma(ZX, n)$ gives rise to a $\gamma(Y, n+1)$ and a $\gamma(Z, n+1)$,
each $\gamma(XY, n)$ gives rise to a $\gamma(YZ, n+1)$ and a $\gamma(ZX, n+1)$.

For $\Omega = X, Y, Z, YZ, ZX, XY$, let cfhc (Ω, n) represent the number of $\gamma(\Omega, n)$, and let t_n be the six element vector whose components are cfhc (Ω, n). Then we have shown that

$$t_{n+1} \geq N_1 \times t_n, \tag{1}$$

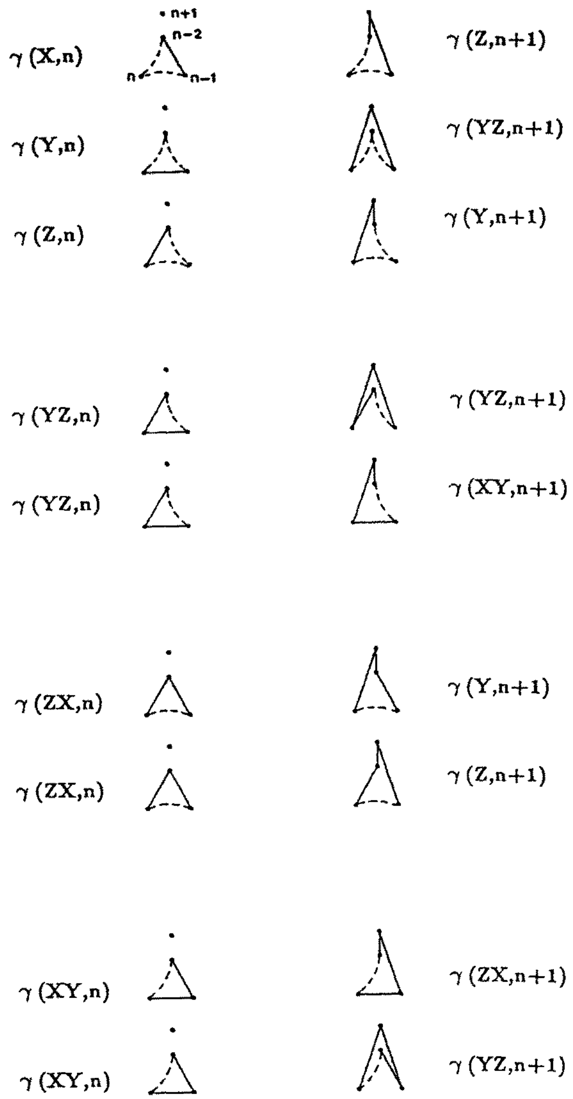

Fig. 5. Creating cfhc's of TS_{n+1} from cfhc's of TS_n.

where

$$N_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

From (1) it follows that cfhc (TS_n) is asymptotically at least $c \times r_1^n$, where c is some constant and r_1 is the dominant eigenvalue of N_1, namely (to four decimal places) 1.8124.

2.2.2. Case $k=2$: creating cfhc's of TS_{n+2} *from cfhc's of* TS_n. Figure 6 shows all ways in which cfhc's of TS_{n+2} are created from cfhc's of TS_n. (Only vertices

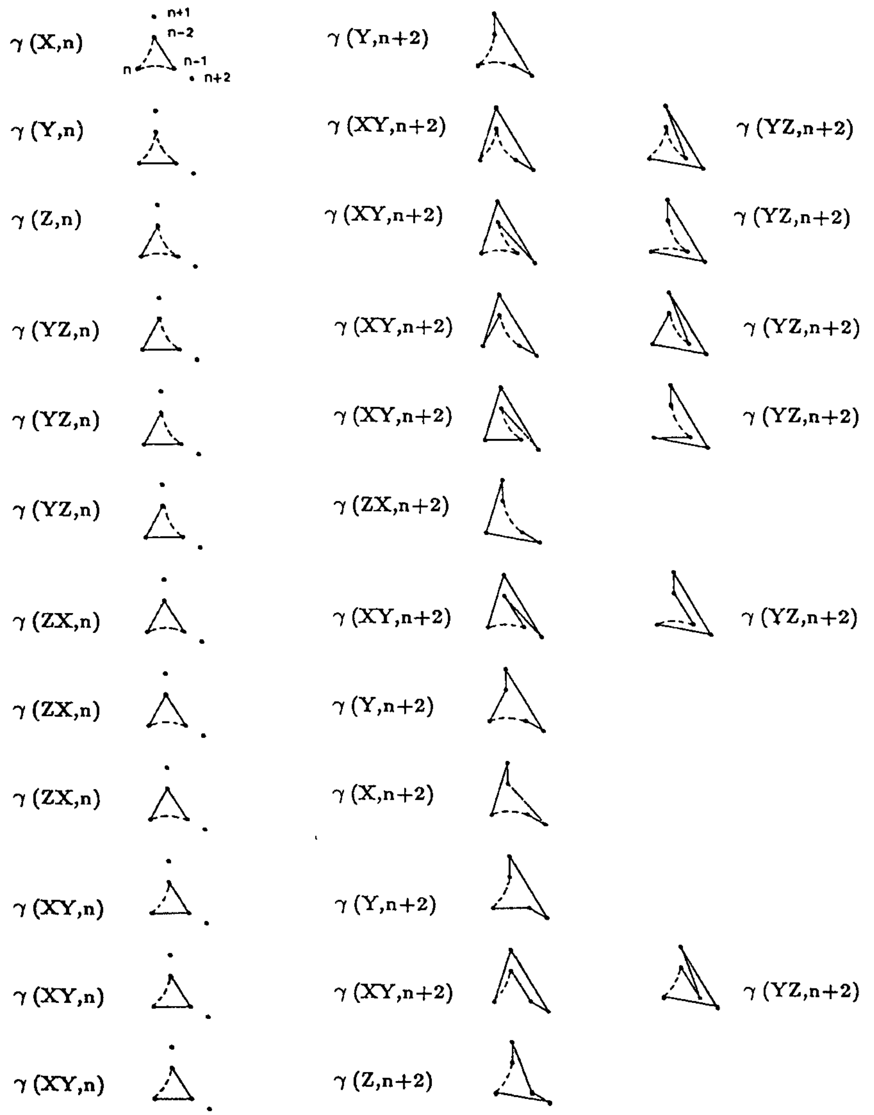

Fig. 6. Creating cfhc's of TS_{n+2} from cfhc's of TS_n.

$n-2$ to $n+2$ of TS_{n+2} are shown in Fig. 6.) In particular,

each $\gamma(X, n)$ yields a $\gamma(Y, n+2)$,
each $\gamma(Y, n)$ yields a $\gamma(YZ, n+2)$ and a $\gamma(XY, n+2)$,
each $\gamma(Z, n)$ yields a $\gamma(YZ, n+2)$ and a $\gamma(XY, n+2)$,
each $\gamma(YZ, n)$ yields two $\gamma(YZ, n+2)$, a $\gamma(ZX, n+2)$, and two $\gamma(XY, n+2)$,
each $\gamma(ZX, n)$ yields a $\gamma(X, n+2)$, a $\gamma(Y, n+2)$, a $\gamma(YZ, n+2)$, and a $\gamma(XY, n+2)$,
each $\gamma(XY, n)$ yields a $\gamma(Y, n+2)$, a $\gamma(Z, n+2)$, a $\gamma(YZ, n+2)$, and a $\gamma(XY, n+2)$.

This information is summarized in matrix form as

$$t_{n+2} \geq N_2 \times t_n, \tag{2.1}$$

where

$$N_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 \end{bmatrix}.$$

Finally, note that if we count cfhc's of TS_{n+2} created by either adding two vertices to a cfhc of TS_n (as just described), or by twice adding a single vertex (as described in Case $k=1$ above), then we have the following improvement:

$$t_{n+2} \geq M_1 \times t_{n+1} + M_2 \times t_n \tag{2.2}$$

where $M_1 = N_1$, and $M_2 = N_2 - (N_1 \times M_1)$. The matrices M_1 and M_2 are listed in the Appendix. The asymptotic rate of growth of the lower bound for cfhc (TS_n) as determined by (2.2) is $c \times r_2^n$, where r_2 is the dominant eigenvalue of the 12 by 12 matrix P_2, where

$$P_2 = \begin{bmatrix} M_1 & M_2 \\ I_6 & 0 \end{bmatrix},$$

and where I_6 is the 6 by 6 identity matrix. The value of r_2 is (to four decimal places) 2.1215.

2.2.3. Case $k \geq 3$: creating cfhc's of TS_{n+k} *from cfhc's of* TS_n. We now show how to enumerate the ways in which cfhc's of TS_{n+k} can be created from cfhc's of TS_n, without having to draw figures corresponding to those shown in Figs. 5 and 6.

A cfhc of TS_{n+k} is created by taking a drawing of TS_{n+k}, placing a cfhc of TS_n on vertices 1 to n, removing one or two of its outermost edges, placing a cfhc of TS_{3+k} on vertices $n-2$ to $n+k$, and then removing one or two of the latter cfhc's innermost edges, so that each of the edges $(n-2, n-1)$, $(n-1, n)$, and $(n-2)$ will have been removed from either the former or latter cfhc. For example, Fig. 7 shows how a cfhc of TS_{11} is created by drawing a $\gamma(X, 8)$ on vertices 1 to 8, removing the edge $X=(6,7)$, drawing a $\gamma(yz, 6)$ on vertices 6 to 11, and removing the edges $y=(7,8)$ and $z=(6,8)$. The following is a summary of all ways in which cfhc's of TS_{n+k} can be thus created:

Cfhc on vertices 1 to n	Edge(s) removed from cfhc on $1 \ldots n$	Cfhc on vertices $n-2$ to $n+k$	Edge(s) removed from cfhc on $n-2 \ldots n+k$
$\gamma(X, n)$	X	$\gamma(yz, 3+k)$	y, z
$\gamma(Y, n)$	Y	$\gamma(zx, 3+k)$	z, x
$\gamma(Z, n)$	Z	$\gamma(xy, 3+k)$	x, y
$\gamma(YZ, n)$	Y	$\gamma(zx, 3+k)$	z, x
$\gamma(YZ, n)$	Z	$\gamma(xy, 3+k)$	x, y
$\gamma(YZ, n)$	Y, Z	$\gamma(x, 3+k)$	x
$\gamma(ZX, n)$	Z	$\gamma(xy, 3+k)$	x, y
$\gamma(ZX, n)$	X	$\gamma(yz, 3+k)$	y, z
$\gamma(ZX, n)$	Z, X	$\gamma(y, 3+k)$	y
$\gamma(XY, n)$	X	$\gamma(yz, 3+k)$	y, z
$\gamma(XY, n)$	Y	$\gamma(zx, 3+k)$	z, x
$\gamma(XY, n)$	X, Y	$\gamma(z, 3+k)$	z

For $\Omega = X, Y, Z, YZ, ZX, XY$ and $\alpha = x, y, z, yz, zx, xy$, let $T_n(\Omega, \alpha)$ be the number of $\gamma(\Omega, \alpha)$ of TS_n. Then the following inequality follows from the above summary:

$$\begin{aligned} t_{n+k}(\Omega) \geq\ & T_{3+k}(\Omega, yz) \times t_n(X) \\ & + T_{3+k}(\Omega, zx) \times t_n(Y) \\ & + T_{3+k}(\Omega, xy) \times t_n(Z) \\ & + (T_{3+k}(\Omega, zx) + T_{3+k}(\Omega, xy) + T_{3+k}(\Omega, x)) \times t_n(YZ) \\ & + (T_{3+k}(\Omega, xy) + T_{3+k}(\Omega, yz) + T_{3+k}(\Omega, y)) \times t_n(ZX) \\ & + (T_{3+k}(\Omega, yz) + T_{3+k}(\Omega, zx) + T_{3+k}(\Omega, z)) \times t_n(XY). \end{aligned}$$

Let T_n be the 6 by 6 matrix whose entries are $T_n(\Omega, \alpha)$, and let

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

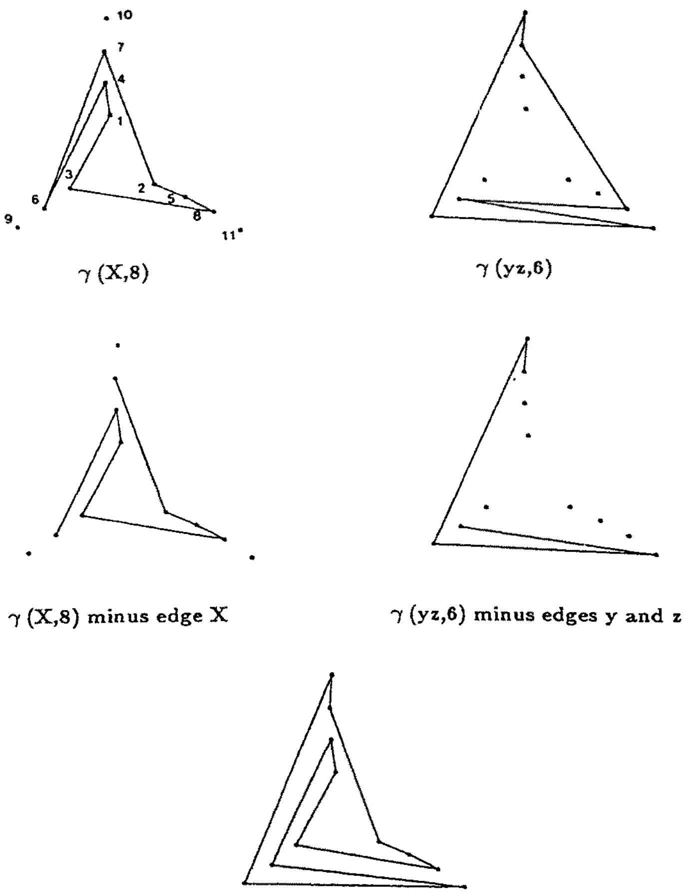

Fig. 7. Creating a cfhc of TS_{11} from a cfhc of TS_8 and a cfhc of TS_6.

Then the preceding inequality can be written in matrix form, namely

$$t_{n+k} \geq N_k \times t_n, \qquad \text{where} \quad N_k = T_{3+k} \times Q. \tag{k.1}$$

As before, we can improve slightly on this inequality by creating cfhc's of TS_{n+k} from cfhc's of TS_n, $TS_{n+1}, \ldots, TS_{n+k-1}$ (i.e., not just from TS_n). This yields the following:

$$t_{n+k} \geq M_1 \times t_{n+k-1} + M_2 \times t_{n+k-2} + \cdots + M_k \times t_n, \tag{k.2}$$

where

$$M_k = N_k - (N_1 \times M_{k-1} + N_2 \times M_{k-2} + \cdots + N_{k-1} \times M_1).$$

The matrices T_{3+k} were determined by computer enumeration, for $k = 3$ to 11.

Programs were written in C and run on a Vax 11-750 with operating system Unix 4.2. Time constraints prevented further computations. For successive values of k, the amount of c.p.u. time used increased by a factor of about 6. As approximately 50 hours of c.p.u. time were required for $k=11$, about 300 hours might be necessary for the case $k=12$.

As was the case with (2.2), the inequality (k.2) can be written as an inequality involving the single $6k$ by $6k$ matrix P_k, where

$$P_k = \begin{bmatrix} M_1 & M_2 & \cdots & M_{k-1} & M_k \\ I_6 & 0 & \cdots & 0 & 0 \\ 0 & I_6 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & I_6 & 0 \end{bmatrix}.$$

The dominant eigenvalue r_k of P_k gives the asymptotic rate of growth of the lower bound for cfhc (TS_n) as determined by (k.2). The matrices T_{3+k} and the eigenvalues r_k are all given in the Appendix. The best lower bound (to four decimal places), achieved with $k=11$, is cfhc $(\mathrm{TS}_n) \geq c \times 3.2684^n$. Thus it follows immediately that $\bar{\Phi}(n) \geq c \times 3.2684^n$.

3. Open Problems

We have established an improved lower bound for $\bar{\Phi}(n)$, namely

$$c \times 3.2684^n \leq \bar{\Phi}(n), \qquad \text{for some constant } c,$$

by counting only a proper subset of the cfhc's of TS_n. Thus determining cfhc (TS_n) explicitly or even asymptotically is still open. Extrapolating the values r_k (see the Appendix) suggests that cfhc(TS_n) might be something near $c \times 3.5^n$, for some constant c.

There are several rectilinear drawings of K_n that have fewer crossings than TS_n, and almost certainly have more cfhc's (see [H]). TS_n was selected for analysis of its number of cfhc's because its symmetries allow for a recursive counting argument. Crucial to our argument is the fact that any k consecutive vertices of TS_n induce a drawing isomorphic to TS_k; we know of no drawing of K_n with fewer crossings than TS_n which has this property.

For all values of n for which cfhc's have been explicitly counted, no drawing of TS_n has more cfhc's than a certain non-rectilinear drawing BK_n (see [H]); all values of lower bounds for $\Phi(n)$ which appear in the table in Section 1, correspond to the number of cfhc's of BK_n. We conjecture that the number of cfhc's of BK_n serve as an upper bound for both $\bar{\Phi}(n)$ and $\Phi(n)$. From the values of cfhc (BK_n) that appear in this table, we conjecture that cfhc (BK_n) is asymptotically $c \times r^n$, where $4.3 < r < 4.5$. Finally, we conjecture that both $\bar{\Phi}(n)$ and $\Phi(n)$ are less than $c \times 4.5^n$, for some constant c.

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Appendix

This Appendix contains

the matrices T_6 to T_{14} (augmented),
the dominant eigenvalues of matrices P_1 to P_{11}.

Recall that in a cfhc of the drawing TS_n,

X, Y, and Z represent, respectively, the edges $(n-2, n-1)$, $(n-1, n)$, and $(n-2, n)$,
x, y, and z represent, respectively, the edges $(1,2)$, $(2,3)$, and $(1,3)$.

Recall that the entry $t_n(\Omega, \alpha)$ of matrix T_n is the number of cfhc's of the drawing TS_n with outermost edge set Ω and innermost edge set α, where Ω takes on the values X, Y, X, YZ, ZX, XY and α takes on the values x, y, z, yz, zx, xy. For example, the entry in row 5, column 2 of matrix T_7 is the number of cfhc's of TS_7 with outermost edge set $\{Z, X\}$ and innermost edge set $\{y\}$, i.e., the number of cfhc's of TS_7 that contain edges $(5,7)$ and $(5,6)$ but not edge $(6,7)$, and that contain edge $(2,3)$ but not edges $(1,2)$ and $(1,3)$. For the sake of completeness, the matrices T_n have been augmented in this Appendix to include a seventh row, corresponding to those cfhc's containing none of the edges X, Y, and Z, and a seventh column, corresponding to those cfhc's containing none of the edges x, y, and z. Thus, for $n > 3$, the sum of all entries of the augmented matrix T_n gives the total number of cfhc's of the drawing TS_n.

Recall that the number of different cfhc's of the drawing TS_{n+k} that can be created by adding exactly k vertices outside a drawing of TS_n is given by the equation

$$t_{n+k} \geq N_k \times t_n, \tag{k.1}$$

where $N_k = T_{3+k} \times Q$, and where the entries of the column vector t_n are the number of cfhc's of TS_n with outermost edge set (respectively) X, Y, Z, YZ, ZX, XY. The matrix Q is shown below.

Recall that the number of different cfhc's of the drawing TS_{n+k} that can be created by adding $1, 2, \ldots,$ or k vertices to the drawings TS_{n+k-1}, $TS_{n+k-2}, \ldots, TS_n$, respectively, is given by the equation

$$t_{n+k} \geq M_1 \times t_{n+k-1} + M_2 \times t_{n+k-2} + \cdots + M_k \times t_n, \tag{k.2}$$

where $M_k = N_k - (N_1 \times M_{k-1} + N_2 \times M_{k-2} + \cdots + N_{k-1} \times M_1)$.

Recall that the asymptotic rate of growth of the number of cfhc's of TS_n determined by equation (k.2) is equal to $c \times r_k^n$, where r_k is the dominant eigenvalue of the $6k$ by $6k$ matrix P_k, shown below.

Recall that the dominant eigenvalue of P_k is the largest real root of the characteristic polynomial of P_k.

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}. \qquad P_k = \begin{bmatrix} M_1 & M_2 & \cdots & M_{k-1} & M_k \\ I_6 & 0 & \cdots & 0 & 0 \\ 0 & I_6 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & I_6 & 0 \end{bmatrix}.$$

Matrices T_6 to T_{14} (augmented)

2	1	1	0	0	0	0
1	2	1	0	0	0	0
1	1	2	0	0	0	0
0	0	0	1	2	2	0
0	0	0	2	1	2	0
0	0	0	2	2	1	0
0	0	0	0	0	0	2
2	4	2	1	2	1	1
3	2	4	2	1	2	1
4	2	3	2	1	2	2
1	2	1	3	4	3	0
2	1	2	3	3	2	0
2	1	2	1	3	3	0
2	1	1	0	0	0	4
10	7	12	5	5	7	5
14	10	10	7	6	6	6
10	10	7	5	7	5	7
6	5	7	6	8	8	1
7	5	5	3	7	6	1
6	7	5	7	8	8	2
6	7	5	1	2	1	12
38	36	36	26	22	22	24
36	38	36	22	26	22	24

Matrices T_6 to T_{14} (augmented)—*continued*

36	36	38	22	22	26	24
26	22	22	20	22	22	9
22	26	22	22	20	22	9
22	22	26	22	22	20	9
24	24	24	9	9	9	48
130	118	121	68	82	70	96
145	121	141	75	84	86	94
137	130	145	90	84	79	99
84	82	84	64	70	64	42
79	70	86	58	64	53	41
90	68	75	47	64	58	42
99	96	94	42	42	41	160
526	465	489	257	282	300	393
558	534	526	321	315	319	394
534	441	465	245	300	271	393
319	271	300	186	218	208	178
321	245	257	139	196	186	175
315	300	282	196	212	218	182
394	393	393	175	182	178	580
1990	2077	2077	1192	1108	1105	1650
2077	1990	2077	1105	1192	1108	1650
2077	2077	1990	1108	1105	1192	1650
1192	1105	1108	703	745	745	754
1108	1192	1105	745	703	745	754
1105	1108	1192	745	745	703	754
1650	1650	1650	754	754	754	2232
7785	6920	7480	3658	4203	3771	6581
8271	7480	7858	3992	4377	4289	6736
7818	7785	8271	4396	4374	4081	6739
4374	4203	4377	2553	2682	2556	3158
4081	3771	4289	2365	2556	2252	3071
4396	3658	3992	2154	2553	2365	3037
6739	6581	6736	3037	3158	3071	8412
31762	29365	29190	14537	15759	16227	27613
32690	32079	31762	16754	16937	17041	28313
32079	27782	29365	13976	16273	15341	27313
17041	15341	16227	8699	9612	9210	12931
16754	13976	14537	7418	8889	8699	12474
16937	16273	15759	8889	9314	9612	12968
28313	27313	27613	12474	12968	12931	33265

Dominant eigenvalue of matrices P_1 to P_{11}

1.8124	2.9551
2.1215	3.0457
2.4992	3.1410
2.6004	3.2039
2.7273	3.2684
2.8726	

Number of cfhc's of TS_3 to TS_{14}

With at least one innermost edge and at least one outermost edge	Total number
—	1
—	3.
—	8
27	29
79	91
257	313
942	1 188
3 166	4 154
11 517	15 527
45 441	62 097
165 986	233 042
642 106	918 595

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