Perfectly Orderable P_4 Composition

Ryan B. Hayward, [a] William J. Lenhart, [b]

[a] *Computing Science, University of Alberta, Edmonton AB Canada T6G 2E8*

[b] *Computing Science, Williams College, Williamstown MA 01267 USA*

Abstract

A P_4 composition scheme is as follows: start with two graphs with vertex sets of different colours, say black $\bullet$ and white $\circ$; select a set of allowable four-vertex colour sequences, for example $\{\bullet\bullet\bullet\bullet, \circ\circ\circ\circ, \bullet\circ\circ\bullet, \circ\bullet\bullet\circ\}$; add edges between the graphs so that in the composed graph each four-vertex induced path is coloured with an allowable sequence. Answering a question of Chvátal, we determine all such schemes which preserve perfect orderability.

Key words: P_4, graph composition, perfect order, perfectly orderable

Given that determining the chromatic number of a graph is computationally infeasible [K], restricted approaches are often considered for vertex colouring. One approach is due to Chvátal [C1], who conceived of a class of graphs for which a certain efficient algorithm always yields an optimal colouring. He defined as *perfectly orderable* those graphs for which there exists a *perfect order*, namely, an ordering of the vertices and an ordering of colours such that for every (vertex induced) subgraph, the so-called "greedy" colouring algorithm (proceed through the vertices in vertex order, assigning to each vertex the first colour in colour order which has not been assigned to any neighbour) yields an optimal colouring. He also established a characterization which allows efficient verification [C1]: *a linear order $<$ of the vertices of a graph is a perfect order if and only if there is no induced path $(abcd)$ with $a < b$ and $d < c$.*

Since vertex colouring is easy for perfectly orderable graphs, it is interesting to ask "which graphs are perfectly orderable?". Middendorf and Pfeiffer showed that there is probably no "good" answer to this question, as recognition is NP-complete [MP]. However, many partial answers have been established by researchers who have shown that various classes of graphs are perfectly orderable. Here we give a partial answer of a different form, by considering certain composition schemes which preserve perfect orderability.

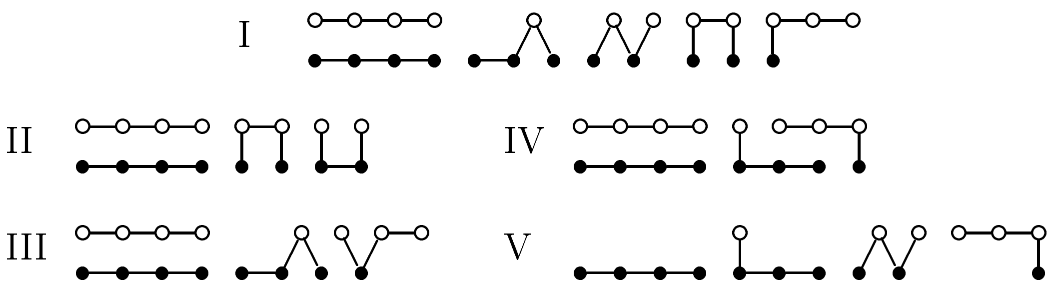

Fig. 1. Composition schemes which preserve perfect orderability.

The composition schemes we consider are: start with two graphs, and add edges between the graphs to form a new graph. We are interested in preserving perfect orderability: the composed graph should be perfectly orderable if the starting graphs are perfectly orderable. A P_4 in a graph is an induced path with exactly four vertices.[1] In light of Chvátal's characterization, it is reasonable to consider composition schemes defined in terms of P_4's. If we distinguish the starting graphs by giving the two vertex sets different colours, the colour patterns on the composed graph's P_4's might determine whether perfect orderability is preserved.

The *pattern* of a P_4 whose vertices are coloured black or white is the corresponding sequence of colours. For example, the pattern of a P_4 $(wxyz)$ whose vertices w, x, z are black and y is white is ●●○●. A *pattern set* is the set of patterns of all P_4's of some graph whose vertices are coloured black or white. Since the reversal of a P_4 is a P_4, pattern sets are closed under reversal. We append '*' to a pattern to indicate either the pattern or its reverse; for example, {●●○●*} is the same as {●●○●, ●○●●}. A P_4 *composition scheme* is a two-colour (one for each starting graph) composition scheme in which the pattern sets of composed graphs are a subset of a specified pattern set. These notions were introduced by Chvátal, who asked "*which P_4 composition schemes preserve perfect orderability?*" and who gave the following initial partial answer [C2]: *the P_4 composition scheme with the following pattern set preserves perfect orderability:* {●●●●*, ●●○●*, ●○●○*, ●○○●, ●○○○*, ○○○○*}.

Our main result is the theorem below, which completely answers Chvátal's question. The five maximal composition schemes which preserve perfect orderability are illustrated in Figure 1.

Theorem. *A P_4 composition scheme preserves perfect orderability if and only if its associated pattern set (or its colour exchange equivalent) is a reversal-closed subset of one of the following:*

[1] This differs from the usual definition of a P_4 as a *set* of four vertices which induce a path; in order to define the *pattern* of a P_4, we need to know the order of vertices in the associated path.

I {●●●●, ●●○●*, ●○●○*, ●○○●, ●○○○*, ○○○○} ,
II {●●●●, ●○○●, ○●●○, ○○○○} ,
III {●●●●, ●●○●*, ○●○○*, ○○○○} ,
IV {●●●●, ●●●○*, ●○○○*, ○○○○} ,
V {●●●●, ●●●○*, ●○●○*, ●○○○*} .

Substitution is the following composition process: given a vertex v in a graph G and a graph H, form a new graph $S(v, G, H)$ by replacing v with H, making every neighbour of v in G adjacent to every vertex of H. A key lemma in proving the above theorem is that substitution preserves perfect orderability: if G and H are perfectly orderable, then so is $S(v, G, H)$.

The details of the proof of our theorem are omitted in this extended abstract. A substantial part of the proof consists of describing the structure of graphs which admit two-colourings with the given pattern sets. For example, the graphs which can be formed by scheme *II* are exactly the graphs formed by starting with a split graph (a graph whose vertex set can be partitioned into a clique and an independent set) and repeatedly substituting an arbitrary (not necessarily perfectly orderable) graph for any vertex. The analogous result holds for scheme *III* (respectively *IV*) by starting with a graph in which some vertex is not the end (respectively middle) vertex of any P_4. In some cases we also characterize the class of (perfectly orderable) graphs formed if all substituted graphs are perfectly orderable. For example, we call the class of graphs formed by scheme *II* in this way *strongly split-substitute.* We can show that a graph is strongly split-substitute if and only if it contains none of the five graphs listed below.

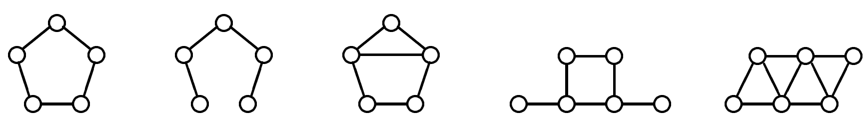

Fig. 2. All minimal forbidden strongly split-substitute graphs.

Another part of the proof consists of showing that the schemes described are best possible. This follows by considering the two-coloured graphs shown in Figure 3. Any scheme whose pattern set strictly contains a pattern set of the theorem is represented by at least one of the graphs in the figure; however, while each monochromatic induced subgraph of each graph is perfectly orderable, none of the graphs themselves is perfectly orderable.

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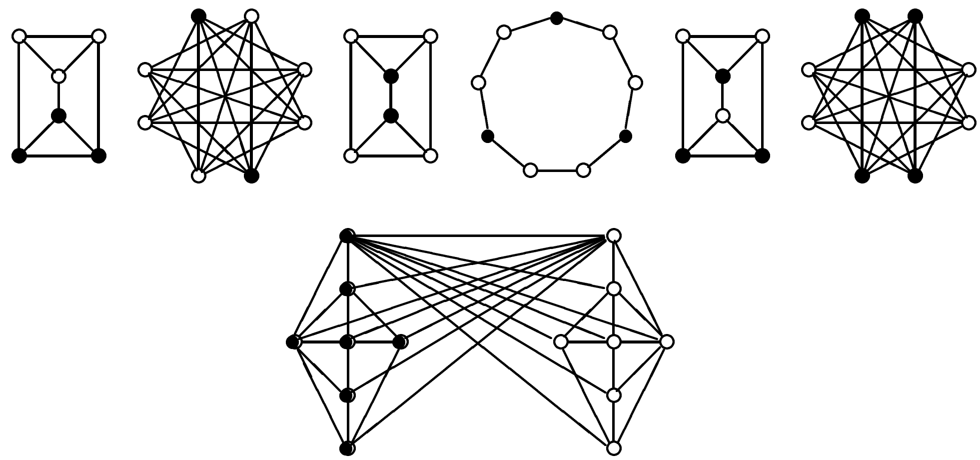

Fig. 3. Our schemes are best possible.

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