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Good Pass Moves in No-Draw HyperHex: Two Proverbs*

Ryan B. Hayward
Computing Science University of Alberta
hayward@ualberta.ca

Abstract

For a position and a player with a winning move, the *pass-value* is the largest number of free moves the player can allow the opponent so that after these move(s) the player still has a winning move. For a cell-coloring game such as Hex, the pass-value is equivalently the smallest number of cells the opponent needs to color in order to reach a position where the opponent has a winning move. A move is *good* if it increases the pass-value. HyperHex is the hypergraph generalization of Hex: each player has a list of winsets, and wins by coloring all cells of any of her winsets. No-draw HyperHex is the maker-breaker restriction of HyperHex: each player's winset list contains every minimal set that intersects all of the other player's winsets (so draws are not possible). For no-draw HyperHex, we consider two good-move proverbs: your opponent's good move is your good move, and it's never too late for a good move.

1 Introduction

At the 2011 Banff International Research Station Workshop on Combinatorial Game Theory, I asked professional 9-dan Ziang Zhujiu [Jujo] about the Go proverb *your opponent's good move is your good move* [14]. His instant response was "it's not always true", and of course he is right. For example,

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in Figure 1 each player's winning move (in the opponent's territory) is the opponent's losing move, so no move is good for both a player and their opponent.

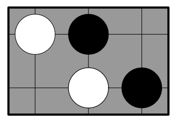

Figure 1: A Go position with no move that wins for both players.

This paper is about good moves in no-draw HyperHex, a game that generalizes Hex. We consider the Go proverb above, and the general proverb *it's never too late*, which when applied to games could be expanded as *it's never too late for a good move.*

2 No-draw HyperHex

In 1942 Piet Hein invented Polygon, the classic alternate-turn two-player connection game now known as Hex [5]. John Nash independently invented the same game, and in 1949 described it to David Gale, who built a board that was soon in frequent use in Princeton's Fine Hall [8, 7]. Later John Milnor and independently Claude Shannon [8] and independently Charles Titus [12, 13] invented Y, and Shannon created his eponymous (switching) game [6]. Y and the Shannon game each generalize Hex. For more on these games, see [2] or [4].

HyperHex is a hypergraph generalization of Hex, Y and the Shannon game. The board is a finite set of cells; each player has a collection of winning cell sets, or *winsets*; on each turn, a player colors an uncolored cell; a player wins by coloring all cells of any one of their winsets. Draws are possible; for example, if the board is {1,2,3,4} and black and white have respective winsets { {1,2}, {4} } and { {1,3} }, then the game move sequence (black:1, white:2, black:3, white:4) fills the board and yields a draw, since neither player wins.

No-draw HyperHex is the maker-breaker restriction of HyperHex. In a *maker-breaker* game, one player (maker) tries to establish some property — say, form a certain connection — and the other player (breaker) tries to prevent it [3]. Maker-breaker games apparently evolved from the Shannon game, in which maker tries to connect two terminal nodes, and breaker tries to thwart this connection.

For a HyperHex game not to end in a draw, it suffices that each possible coloring of all the board cells yields a colored winset for exactly one player, namely that the game is maker-breaker. To satisfy this condition, call the two players M and B, and select any non-empty list of non-empty winsets for M. Remove from M's list any winset that properly contains some other winset (such containing winsets are redundant), and let B's list consist of every minimal cell subset that intersect all of M's winsets. For example, if the board is {1,2,3,4} and M's list is { {1,2}, {4} } then B's list is { {1,4}, {2,4} }.

A hypergraph is a set of hyperedges (subsets) of a ground set, so in hypergraph terms the winset lists are hypergraphs defined on the same ground set (the set of board cells), M's hypergraph is a *clutter* (no hyperedge properly contains another), and B's hypergraph is the *blocker*, or *transversal* (the set of all hyperedges that intersect all of the other hypergraph's hyperedges) of M's hypergraph. The blocker of a blocker of a clutter is the original clutter (see Corollary 2 of Chapter 2 in *Hypergraphs* by Claude Berge [1]), so in no-draw HyperHex the roles of maker and breaker are interchangeable. So no-draw HyperHex is HyperHex on a clutter and its blocker [9].

Here are some no-draw HyperHex examples. Suppose the cell set is {1,2,3,4} and black's winset collection is { {1,3}, {2,3}, {2,4} }. Then white's winset collection is { {1,2}, {2,3}, {3,4} }. In Hex, a player wins by connecting their two borders. This HyperHex example is equivalent to Hex on the board in Figure 2.

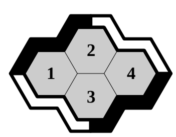

Figure 2: A Hex board.

Or suppose the cell set is {1,2,3} and black's winset collection is { {1,2}, {1,3}, {2,3} }. Then white's is the same. In Y, a player wins by connecting all three borders. This HyperHex example is equivalent to Y on the board in Figure 3.

Or suppose the cell set is {1,2,3,4,5,6} and black's winset collection is { {1,4}, {1,5}, {2,5} {2,6}, {3,4}, {3,6} }. Then white's is { {1,2,3}, {4,5,6}, {1,2,4,6}, {1,3,5,6}, {2,3,4,5} }. In the Shannon game, black wins by joining two specified terminal nodes, otherwise white wins. This HyperHex example

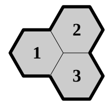

Figure 3: A Y board.

is equivalent to the Shannon game on the network in Figure 4.

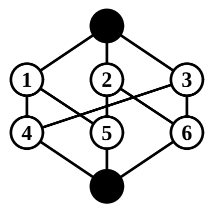

Figure 4: A Shannon board.

3 Pass-value and good moves

For a player P and a HyperHex position H, (H, P) is the game state starting from H in which P moves next. For P and an H with an empty cell c, $H + P(c)$ is the position obtained from H by P-coloring c. $\overline{P}$ is the opponent of P.

In HyperHex, P-coloring a cell — or uncoloring a $\overline{P}$-colored cell — is never disadvantageous for P, since a winning P-strategy can always be modified to accomodate such a change:

Observation 1 *For a player P and a HyperHex position H with empty cell c, and for $X = P$ or $\overline{P}$, if P wins (H, X) then P wins $(H + P(c), X)$.*

By Observation 1, if P has a second-player win strategy for H then P has a first-player win strategy for H, so each no-draw HyperHex position has one of three outcome-values: *neutral* if each player has a first-player win, P-*win* if P has a second-player win, and $\overline{P}$-*win* if $\overline{P}$ has a second-player win. See Figure 5.

We want a notion of "good move" that is more general than "winning move", so we introduce the notion of pass-value. A similar concept has been studied in Go [10].

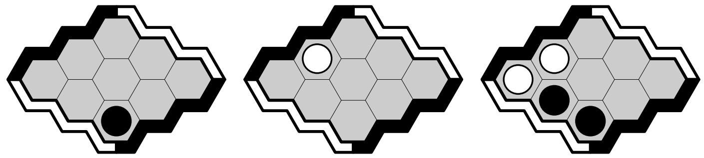

Figure 5: Hex positions with outcome-values BLACK, WHITE, and neutral.

Informally, the pass-value of a position is the number of times a player can *pass* — make no move but allow the opponent a move — and still leave a winning position. Formally, for a player P and a no-draw HyperHex position H, define the *pass-value* $\nu_P(H)$ as ∞ if a P-winset is already P-colored; otherwise, if P has a winning first-player strategy then $\nu_P(H)$ is the largest number of empty cells t such that P still has a first-player winning strategy after any t cells have been $\overline{P}$-colored; otherwise $\nu_P(H)$ is $-\,\nu_{\overline{P}}(H)$. Notice that, for all P and H, $\nu_P(H) = -\,\nu_{\overline{P}}(H)$.

For example, the respective black pass-values of the positions in Figure 5 are (from left) 1, -1, and 0. For a neutral position, each player's pass-value is 0; for a P-win position, P's pass-value is at least 1.

A move is *good* if it increases the player's pass-value, and *wasted* if it does not change it. By Observation 1, every no-draw HyperHex move is good or wasted. A move can increase the pass-value by more than one. For example, see Figure 6. Generalizing this example, it is easy to see that for each $n \geq 4$ there is an $n \times n$ position in which a move changes the pass-value from 0 to $n - 1$.

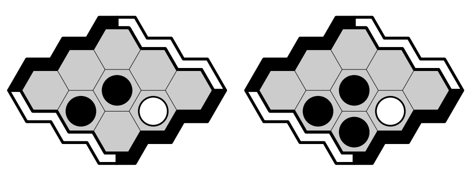

Figure 6: Left: black pass-value 0. Right: black pass-value 2.

4 It's never too late for a good move

In no-draw HyperHex, for a position with finite pass-value, is a good move always available?

If a player is ahead — has positive pass-value, i.e. a 2nd-player win — then sometimes but not always. For example, in Figure 7 each position has black pass-value 1. In the left position, black has good moves. But consider the right position. If black passes twice, white can color the top two marked cells and then win by playing in either of the other two other marked cells. A similar strategy holds for the four unmarked cells. Thus each black move from this position leaves one of these two white strategies intact, and so leaves the black pass-value at 1. So black has no good move.

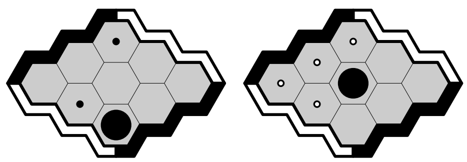

Figure 7: Left: two black-good moves. Right: no black-good moves.

But if the player is not ahead, then *it's never too late for a good move.* This follows almost immediately from the definition of pass-value.

Observation 2 *For a player P and any no-draw HyperHex position H with $\nu_P(H) = t \leq 0$, P has a good move.*

Proof. For $k \geq 1$, a *k-strategy* is a first-player strategy in which k cells are colored on the first move and one cell is colored on each successive move.

Assume P and H are as stated. Thus P has a winning $(1-t)$-strategy S whose first move is to a set C of $1-t \geq 1$ cells. Thus, for any cell c in C and the position $H' = H+P(c)$, the $-t$-strategy obtained from S by removing the cell c from the first move is a winning strategy, so $\nu_P(H') \geq t+1 = \nu_P(H)+1$, and we are done. □

By Observation 2, white has at least one good move in each position of Figure 7. We leave it as an exercise to the reader to find all such moves.

5 Your opponent's good move is your good move

In Hex as in Go, it's not always true that *your opponent's good move is your good move.* In Figure 8 both players have five good moves and three moves are good for both players, but in Figure 9 — found by Jonatan Rydh [11] — both players have two good moves but no move is good for both players. The next theorem gives some conditions where some move is good for both players.

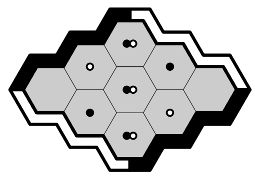

Figure 8: Dots show winning moves. Three moves win for both players.

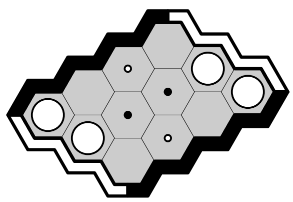

Figure 9: No move wins for both players.

Theorem 1 *A neutral no-draw HyperHex position H with t empty cells has a move that is good for both players if*

- *either player has only one winning move, or*
- $1 \leq t \leq 5$*, or*
- $1 \leq t \leq 7$ *and each player has only two winning moves.*

Proof. In a neutral position, a move is good if and only if it is winning, so it suffices to find moves that are winning for both players.

Let $c_1, \ldots, c_t$ be the empty cells of H. Let W_P and $W_{\overline{P}}$ be the respective sets of winning moves for P and $\overline{P}$.

Consider the first part of the theorem. Suppose P has only one winning move, say to cell c_1. Then P has no winning moves in $H + \overline{P}(c_1)$, so c_1 wins for $\overline{P}$, and we are done.

Consider the second part of the theorem. P is neutral so $t \geq 1$. Assume $t \leq 5$. We are done by the first part if W_P or $W_{\overline{P}}$ has size one, so assume each set has size at least two, so $t \geq 4$. Since $t \leq 5$, one set — say W_P — has size two. Relabel cells if necessary so that $W_P = \{c_1, c_2\}$.

Argue by contradiction: suppose $W_{\overline{P}}$ contains no cell of W_P. Then c_1 does not win for $\overline{P}$, so P has a winning move x in $H + \overline{P}(c_1)$. By Observation 1, x is also P-winning in H, so $x = c_2$. Similarly, c_1 is P's unique winning move in $H + \overline{P}(c_2)$.

First assume $t = 4$. P has a winning reply for any $\overline{P}$-move in $H + \overline{P}(c_1) + P(c_2)$, so $\{c_2, c_3\}$ and $\{c_2, c_4\}$ are P-winsets. Similarly, $\{c_1, c_3\}$ and $\{c_1, c_4\}$ are P-winsets. Thus c_3 wins for P in H, (on the next move P can color one of c_1, c_2), contradiction ($W_P = \{c_1, c_2\}$).

Next assume $t = 5$. Again, P has a winning reply for any $\overline{P}$-move in $H + \overline{P}(c_1) + P(c_2)$; relabel c_3, c_4, c_5 if necessary so that $\{c_2, c_3\}$ and $\{c_2, c_4\}$ are P-winsets. Similarly, for at least two cells j, k in c_3, c_4, c_5, $\{c_1, j\}$ and $\{c_1, k\}$ are P-winsets. So, for some z in $\{j, k\}$, $\{c_1, z\}$ and $\{c_2, z\}$ are P-winsets, so z wins for P in H, contradiction. Thus the second part of the theorem holds.

Next consider the third part. Again, P is neutral so $t \geq 1$. If $t \leq 5$ we are done, so $t = 6$ or 7. Argue by contradiction: assume that W_P and $W_{\overline{P}}$ have no cell in common, say $W_P = \{c_1, c_2\}$ and $W_{\overline{P}} = \{c_3, c_4\}$.

If P first colors c_1 and then $\overline{P}$ colors c_3 then P has a winning move; similarly, if $\overline{P}$ first colors c_3 and then P colors c_1 then $\overline{P}$ has a winning move. So $H^* = H + P(c_1) + \overline{P}(c_3)$ is neutral and has at most 5 empty cells, so — by the second part — some empty c_w wins H^* for both players.

Notice that $c_w \neq c_2$: otherwise, c_2 wins for $\overline{P}$ in H^*, but also c_1 wins for P in $H' = H + \overline{P}(c_2)$, so c_3 does not win for $\overline{P}$ in $H' + P(c_1)$, so P wins $H^* + \overline{P}(c_2)$, contradiction. Similarly, $c_w \neq c_4$.

Similarly, some c_x wins for both players in $H + P(c_2) + \overline{P}(c_3)$, some c_y wins for both players in $H + P(c_1) + \overline{P}(c_4)$, some c_z wins for both players in $H + P(c_2) + \overline{P}(c_4)$, and none of c_w, c_x, c_y, c_z are in $\{c_1, c_2, c_3, c_4\}$.

Since $t \leq 7$, at least two of c_w, c_x, c_y, c_z are equal, say $c_w = c_x$. Then P has no winning move in $H + P(c_1) + \overline{P}(c_3) + \overline{P}(c_w)$, and no winning move in $H + P(c_2) + \overline{P}(c_3) + \overline{P}(c_x = c_w)$, so no winning move in $H + \overline{P}(c_w)$ (the only possible winning replies would be c_1 or c_2, but in each case $\overline{P}$ counters with c_3), so c_w wins for $\overline{P}$ in H, so c_w (not in $\{c_1, \ldots, c_4\}$) is in $W_{\overline{P}}$, contradiction, and we are done. $\square$

Jonatan Rydh's example in Figure 9 shows that the third part of the theorem cannot be strengthened in terms of t. Here is a 6-cell no-draw

HyperHex example that shows that the second part of the theorem cannot be strengthened: black's winsets are {1,3,4}, {1,3,6}, {1,4,5}, {1,5,6}, {2,3,4}, {2,3,5}, {2,4,6}, {2,5,6}; white's winsets are {1,2}, {1,3,6}, {1,4,5}, {2,3,5}, {2,4,6}, {3,4,5}, {3,4,6}, {3,5,6}, {4,5,6}. We leave it as an exercise for the reader to verify that the sets of winning opening moves for black and white are {1,2} and {3,4,5,6} respectively.

We close with an open problem. Among all neutral no-draw HyperHex positions with t empty cells, w_1 that win for one player, w_2 that win for the other, and no cell that wins for both, what is the smallest possible value of t? We have shown that there is no such t if $w_1 = 1$ or $w_2 = 1$, and that $t = 6$ if $w_1 = w_2 = 2$.

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