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A Winning Strategy for $3 \times n$ Cylindrical Hex

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Abstract

For Cylindrical Hex on a board with circumference 3, we give a winning strategy for the end-to-end player. This is the first known winning strategy for odd circumference at least 3, answering a question of David Gale.

Keywords Hex; Cylindrical Hex; Annular Hex; Steven Alpern; Anatole Beck; David Gale

1 Introduction

The game of Hex, invented in 1942 by Danish polymath Piet Hein and independently in 1948 by American mathematician John Nash, has commanded the interests of the mathematically-minded for more than half a century [2]. The rules are simple. The board — after which the game is named — is a hexagonally-tiled parallelogram with m columns and n rows, where usually $m = n$. One player takes the color white and two opposing sides of the board; the other player takes black and the other two opposing sides. Players alternate moves. At her move, a player places a stone of her color on an unoccupied hexagonal tile (or colors an uncolored tile with her color). The game proceeds thus until one player has formed a continuous unicolored chain of tiles connecting their sides.

Yet despite its simple rules, Hex is not simple. There is always a winner [6], and — as Nash showed via a strategy-stealing argument — on $n \times n$ boards there exists a winning strategy for the first player [4]. Nonetheless, in over fifty years no such strategy has been found that holds for all n, not even a first winning move for the first player is known with mathematical certainty.[2]

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[2] To date, strategies have been found for board sizes up to 10×10 [5].

More recently, Anatole Beck and Steven Alpern created a variant of the game known as Cylindrical Hex (or Annular Hex), in which the usual $m \times n$ Hex board is wrapped about a cylinder (and, if a 2-dimensional board is desired, flattened into an annulus). Alpern's and Beck's motivation for considering Cylindrical Hex was their search for (and discovery of) a fixed-point theorem for annulus regions of the plane, similar to Gale's use of Hex to produce a proof of Brouwer's fixed-point Theorem [1, 2]. Here one player — White, or End — must connect the top and bottom ends of the cylinder (or inside and outside of the annulus), whereas the other — Black, or Around — must make a circuit about the cylinder (or around the annulus centre). We refer to the n rows of the original $m \times n$ board as *rings* of the cylinder or annulus. See Figure 1.

In 1991 Alpern and Beck demonstrated that White can win when the circumference m is even, specifying a pairing strategy to guarantee such a result [1]. In this paper we give a winning Cylindrical Hex strategy for White when the circumference $m = 3$. Alpern and Beck believe White can win on any board with odd circumference, but have been unable to prove this [1]. Among others who have considered the case of circumference three is David Gale. In a letter to Steven Alpern in 1987, he wrote, "It seems to me your cylinder game is a win for player I regardless of whether m is odd or even. Have you looked at the $3 \times n$ game?" [3]

While a student at the London School of Economics, the first author developed a winning strategy, working with Steven Alpern and the third author. In order to limit the number of cases, the third author had tried to develop a strategy symmetric with respect to the top and bottom of the cylinder, but he failed to do so and no such strategy is known. When submitted to this journal, one of the referees (the second author) improved upon this original strategy and was invited to join this paper. As has been observed by many players of Cylindrical Hex, White seems to win easily when n is odd and small. We are thus pleased to assert the following:

Theorem 1.1 (Gale's Conjecture) *For Cylindrical Hex on a board with circumference $m = 3$, White — the end-to-end player — has a winning strategy.*

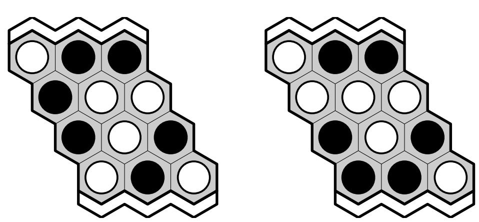

Figure 1: The board after two 3×4 Cylindrical Hex games. White wins on the left (the leftmost cell in a row touches the rightmost cells in the same row and in the row below), Black wins on the right (the four black tiles in the bottom two rows form a circuit).

2 Cylindrical Hex

Alpern and Beck's Hex variant, named Cylindrical Hex or Annular Hex, is played on a graph with vertices (x, y) with $x \in \{1, ..., m\}$ and $y \in \{1, ..., n\}$ arranged in annular fashion. A vertex (x, y) is adjacent to:

1. $(x - 1 \bmod m, y)$ and $(x + 1 \bmod m, y)$,
2. if $y < n$, $(x, y + 1)$ and $(x - 1 \bmod m, y + 1)$, and
3. if $y > 1$, $(x, y - 1)$ and $(x + 1 \bmod m, y - 1)$.

We refer to these vertices as the *cells* of the board. We say that adjacent cells *touch*. *White* owns the two ends and wins by connecting them, while *Black* wins by coloring a continuous circuit around the cylinder, i.e. by encircling the inside of the annulus. Alpern and Beck proved in 1991 that any game of Cylindrical Hex has a unique winner and that White can win whenever the circumference n is even. We present their strategy here. For a proof of correctness, see their paper [1].

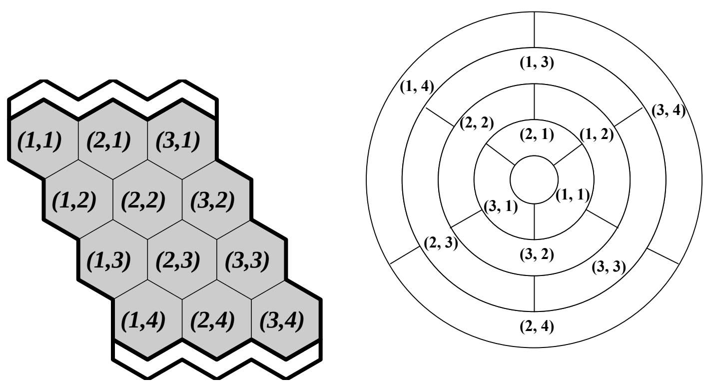

Figure 2: Board indices for rectangular and annular layouts.

Theorem 2.1 (Alpern and Beck, 1991) *For a game of Cylindrical Hex with circumference $m = 2t$, $t \in \mathbb{N}$, the following strategy yields a win for White:*

1. If Black just colored vertex (j, i), then color $\pi((j, i))$, where $\pi((j, i)) = ((j + t) \bmod m, i)$, the 180 degrees rotation map.

2. If this is not possible — either because Black has not yet moved or $\pi((j, i))$ is already occupied — then color any uncolored vertex.

Notice that the symmetry which underpins this strategy does not extend to odd circumference.

3 Strategy

We begin with the algorithm that defines White's strategy. A cell is *white* (resp. *black*, *uncolored*) if it is White-occupied (Black-occupied, neither-player-occupied). A set of cells is *black* (*white*, *uncolored*) if all cells in the set are black (white, uncolored).

Algorithm 3.1 ***(White's Strategy)***
Assume there exist rings 0 *and* $n+1$*, and that both are white.*[3]
Follow the first applicable rule:

I. If Black has not played, then play anywhere.

II. If Black's previous move is at cell (j, i)*, play as follows:*

1. in one of rings $i-1$*,* i*,* $i+1$ *so that there is then a white cell in ring* i *touching a white cell in ring* $i-1$ *and a white cell in ring* $i+1$*,*

2. in ring i *or* $i+1$ *so that there is then a white cell in ring* i *touching a white cell in ring* $i+1$*,*

3. in ring i *or* $i-1$ *so that there is then a white cell in ring* i *touching a white cell in ring* $i-1$*,*

4. in ring i*,*

5. anywhere.

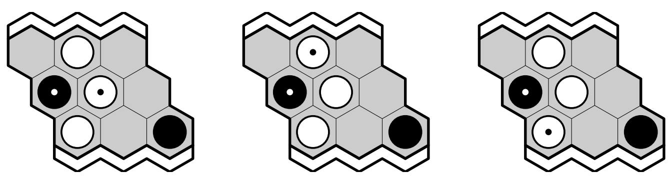

Figure 3: Example of play according to rule 1 of White's strategy.

We shall show this algorithm yields a win for White in any game of $3 \times n$ Cylindrical Hex, thereby proving Gale's Conjecture. We first note that the above strategy is well-defined: if no rule applies, then the board is filled and the game is over.

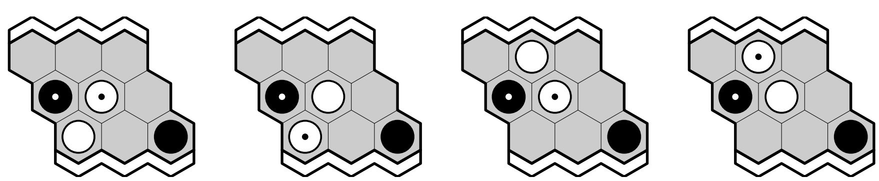

Figure 4: Play according to rules 2 (left) and 3 (right) of White's strategy.

Proof of Theorem 1.1 (Gale's Conjecture). Assume that White plays a game of $3 \times n$ Cylindrical Hex according to Algorithm 3.1. We wish to show that White wins. Alpern and Beck proved that draws cannot occur in Cylindrical Hex [1], so it suffices to show that Black does not win.

[3]This allows the strategy to function on rings 1 and n.

Observe that for $3 \times n$ Cylindrical Hex there exist exactly two kinds of minimal cell set that, if black, yield a Black win: a ring, or for some ring index i and some column index j — with column indices reduced mod 3 — the set

$$C = \{ (j+1, i+1),\ (j+2, i),\ (j+3, i),\ (j+3, i+1) \} .$$

Thus to prove the theorem it suffices to show that no minimal black winning set ends black. No ring ends black: by rules 1-4, whenever Black plays into an uncolored ring, White replies in that ring. So it remains only to show that no set C ends black. Consider an arbitrary set C. By, if necessary, relabelling column indices, we may assume that $j = 0$, so $C = \{ (1, i+1),\ (2, i),\ (3, i),\ (3, i+1) \}$. There are five cases.

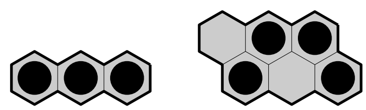

Figure 5: The two minimal black winning patterns.

Case 0. Any of the first four moves into C are by White. Thus C ends not black and we are done.

Case 1. Black moves first into C, at $(1, i+1)$.

By rules 1-4, after White's reply to this Black move, at least one of $(2, i+1), (3, i+1)$ is white. In the latter case C ends not black and we are done. So consider the former case: $(2, i+1)$ is white and $(3, i+1)$ is uncolored.

a. Black moves second into C, at $(2, i)$. Now $(3, i)$ is the only non-black cell in ring i touching a white cell in ring $i+1$. So by rules 1 (if it applies) or 2 (if 1 does not apply) White replies at $(3, i)$ and C ends not black.

b. Black moves second into C, at $(3, i)$. Similar to the previous case.

c. Black moves second into C, at $(3, i+1)$. Now $(2, i)$ and $(3, i)$ are the only cells in ring i touching a white cell in ring $i+1$, so if White follows rule 1 or 3, then White replies in one of these two cells and we are done. Therefore, assume White replies elsewhere. Hence, it must be that White follows rule 2, and so White plays at one of $(1, i+2), (2, i+2)$.

i. Black moves third into C, at $(2, i)$. Now $(3, i)$ is the only non-black cell in ring i touching a white cell in ring $i+1$. So White follows rule 1 or 2 and plays at $(3, i)$. C ends not all black.

ii. Black moves third into C, at $(3, i)$. Similar to the previous case.

Case 2. Black moves first into C, at $(3, i+1)$. By relabelling columns this reduces to Case 1.

Case 3. Black moves first into C, at $(2, i)$. By rules 1-4, White replies in $(1, i)$ or $(3, i)$. In the latter case C ends not black and we are done. Thus consider the former case: White replies at $(1, i)$ and $(3, i)$ is uncolored.

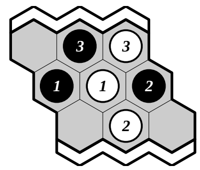

Figure 6: Case 1.c.i.

a. Black moves second into C, at $(3, i)$. Now $(1, i)$ is the only white cell in ring i, and the two cells in ring $i + 1$ that touch this white cell — $(1, i + 1), (3, i + 1)$ — are both uncolored. So by rule 1 or 2, White replies at one of these two cells, and C ends not black.

b. Black moves second into C, at $(1, i + 1)$. Notice that $x = (3, i + 1)$ is the only cell in ring $i + 1$ touching a white cell in ring i. If White follows rule 1 or 3 then White replies at x, and we are done. Thus, assume White does not follow rule 1 or 3. As rule 1 does not apply, no white cell in ring $(i + 2)$ touches $(3, i + 1)$. Hence, it must be that White follows rule 2. After White's reply, a white cell in ring $i + 1$, namely $(2, i + 1)$, touches a white cell in ring $i + 2$.

i. Black moves third into C, at $(3, i)$. Now ring i is full, and the only cell in ring $i + 1$ touching a white cell in ring i is $x = (3, i + 1)$. White replies at x by rule 1 or 2, and C ends not black.

ii. Black moves third into C, at $(3, i + 1)$. Now ring $i + 1$ is full. The only cell in ring i touching a white cell in ring $i + 1$ is $y = (3, i)$. The white cell in $i + 1$ in turn touches a white cell in ring $i + 2$. Thus, White replies at y by rule 1. C ends not black.

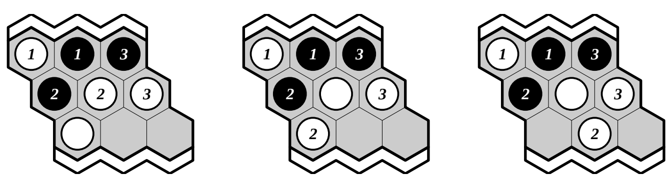

Figure 7: Case 3.b.i.

c. Black moves second into C, at $(3, i + 1)$. If White's reply is at $z = (1, i + 1)$ we are done. Thus, assume White replies according to rule 2 so that a white cell in ring $i + 1$, namely $(2, i + 1)$, touches a white cell in ring $i + 2$.

i. Black moves third into C, at $(3, i)$. Then, by rule 1 or 2, White replies at $(1, i + 1)$, so C ends not black.

ii. Black moves third into C, at $(1, i + 1)$. Then, by rule 1, White replies at $(3, i)$, so C ends not black.

Case 4. Black moves first into C, at $(3, i)$. By relabelling columns, this reduces to Case 3.

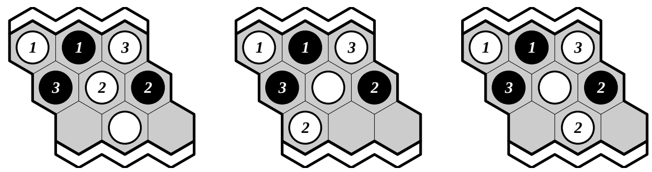

Figure 8: Case 3.c.ii.

This covers all possible winning circuits for Black.
Black does not win.
White to win! □

4 Conclusion

It remains unknown who wins $m \times n$ Cylindrical Hex for odd m at least five. The complexity of winning strategies may increase as m gets large, similar to the case for winning strategies for $n \times n$ Hex [5, 7]. Future work may seek a generalized proof that White wins Cylindrical Hex for all odd m, perhaps by extending the above strategy.

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